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BESSEL MULTIPLIERS AND APPROXIMATE DUALS IN HILBERT C^*-MODULES

Morteza Mirzaee Azandaryani

Abstract. Two standard Bessel sequences in a Hilbert C^*-module are approximately duals if the distance (with respect to the norm) between the identity operator on the Hilbert C^*-module and the operator constructed by the composition of the synthesis and analysis operators of these Bessel sequences is strictly less than one. In this paper, we introduce (a, m)-approximate duality using the distance between the identity operator and the operator defined by multiplying the Bessel multiplier with symbol m by an element a in the center of the C^*-algebra. We show that approximate duals are special cases of (a, m)-approximate duals and we generalize some of the important results obtained for approximate duals to (a, m)-approximate duals. Especially we study perturbations of (a, m)-approximate duals and (a, m)-approximate duals of modular Riesz bases.

1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [12] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [11]. Various generalizations of frames have been introduced, e.g. g-frames [37].

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. In [15] Frank and Larson presented a general approach to the frame theory in Hilbert C^*-modules. Also, g-frames have been introduced in Hilbert C^*-modules (see [20]).

Bessel multipliers in Hilbert spaces were introduced by Balazs in [2]. As we know in frame theory, the composition of the synthesis and analysis operators of a frame is called the frame operator. We see in [2] that a multiplier for two Bessel sequences is an operator that combines the analysis operator, a multiplication pattern with a fixed sequence, called the symbol, and the synthesis operator. Bessel multipliers have useful applications, for example they are used

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for solving approximation problems and they have applications as time-variant filters in acoustical signal processing, for more information see [3, 35]. Multipliers have been investigated for Bessel fusion sequences in Hilbert spaces [27] (called Bessel fusion multipliers) and for generalized Bessel sequences in Hilbert spaces [32] (called g-Bessel multipliers). Also multipliers were introduced for p-Bessel sequences in Banach spaces [33] and for continuous frames [5]. Recently the present author and A. Khosravi generalized Bessel multipliers, g-Bessel multipliers and Bessel fusion multipliers to Hilbert C^*-modules and it was shown that they share many useful properties with their corresponding notions in Hilbert and Banach spaces (see [24]).

Approximate duals in frame theory have important applications, especially are used for the reconstruction of signals when it is difficult to find alternate duals. Approximate duals are useful for wavelets [9, 16], [18, Section 2.13], Gabor systems [6, 38], [14, Sections 3,4], in coorbit theory [13] and in sensor modeling [28]. Approximate duality of frames in Hilbert spaces was recently investigated in [10] and some interesting applications of approximate duals were obtained. For example, it was shown how approximate duals can be obtained via perturbation theory and some applications of approximate duals to Gabor frames especially Gabor frames generated by the Gaussian were presented. In Section 6 in [10], some numerical approaches to construct approximate duals have been stated. Also, approximate duality for g-frames has been introduced in [23] and it was shown in [23] that approximate duals are stable under small perturbations and they are useful for erasures. Moreover approximate duals of frames and g-frames have been generalized to Hilbert C^*-modules [29].

Let $\mathcal{F}$ be a Bessel sequence in a Hilbert space H with the synthesis operator $T_{\mathcal{F}}$. Then the Bessel sequence $\mathcal{G}$ is called an approximate dual of $\mathcal{F}$ if there exists $0 \leq K < 1$ with

$$\|f - T_{\mathcal{G}} T_{\mathcal{F}}^* f\| \leq K \|f\|, \quad \forall f \in H.$$

By Neumann algorithm $T_{\mathcal{G}} T_{\mathcal{F}}^*$ is invertible and the inverse of this operator is used in the process of the reconstruction.

In this paper, we introduce (a, m)-approximate duals in Hilbert C^*-modules which generalize approximate duals of frames and g-frames. First in the following section, we recall the definitions of frames and g-frames in Hilbert C^*-modules.

2. Frames and g-frames in Hilbert C^*-modules

Let $\mathfrak{A}$ be a unital C^*-algebra and suppose that E is a left $\mathfrak{A}$-module such that the linear structures of $\mathfrak{A}$ and E are compatible. Then E is called a pre-Hilbert $\mathfrak{A}$-module if E is equipped with an $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \longrightarrow \mathfrak{A}$, such that

(i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for each $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in E$;
(ii) $\langle ax, y \rangle = a \langle x, y \rangle$ for each $a \in \mathfrak{A}$ and $x, y \in E$;

(iii) $\langle x,y\rangle = \langle y,x\rangle^*$ for each $x,y \in E$;

(iv) $\langle x,x\rangle \geq 0$ for each $x \in E$ and if $\langle x,x\rangle = 0$, then $x = 0$.

For each $x \in E$, we define $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$ and $|x| = \langle x,x\rangle^{\frac{1}{2}}$. If E is complete with $\|\cdot\|$, it is called a *Hilbert $\mathfrak{A}$-module* or a *Hilbert C^*-module* over $\mathfrak{A}$. We call $\mathcal{Z}(\mathfrak{A}) = \{a \in \mathfrak{A} : ab = ba, \forall b \in \mathfrak{A}\}$, the *center* of $\mathfrak{A}$. Note that if $a \in \mathcal{Z}(\mathfrak{A})$, then $a^* \in \mathcal{Z}(\mathfrak{A})$, and if a is an invertible element of $\mathcal{Z}(\mathfrak{A})$, then $a^{-1} \in \mathcal{Z}(\mathfrak{A})$, also if a is a positive element of $\mathcal{Z}(\mathfrak{A})$, since $a^{\frac{1}{2}}$ is in the closure of the set of polynomials in a, we have $a^{\frac{1}{2}} \in \mathcal{Z}(\mathfrak{A})$. Let E and F be Hilbert $\mathfrak{A}$-modules. An operator $T : E \longrightarrow F$ is called *adjointable* if there exists an operator $T^* : F \longrightarrow E$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for each $x \in E$ and $y \in F$. Every adjointable operator T is bounded and $\mathfrak{A}$-linear (that is, $T(ax) = aT(x)$ for each $x \in E$ and $a \in \mathfrak{A}$). We denote the set of all adjointable operators from E into F by $\mathfrak{L}(E,F)$. Note that $\mathfrak{L}(E,E)$ is a C^*-algebra and it is denoted by $\mathfrak{L}(E)$. A Hilbert $\mathfrak{A}$-module E is *finitely generated* if there exists a finite set $\{x_1, \ldots, x_n\} \subseteq E$ such that every element $x \in E$ can be expressed as an $\mathfrak{A}$-linear combination $x = \sum_{i=1}^n a_i x_i, a_i \in \mathfrak{A}$. A Hilbert $\mathfrak{A}$-module E is *countably generated* if there exists a countable set $\{x_i\}_{i\in I} \subseteq E$ such that E equals the norm-closure of the $\mathfrak{A}$-linear hull of $\{x_i\}_{i\in I}$. For more details about Hilbert C^*-modules, see [26].

Definition 2.1. Let E be a Hilbert $\mathfrak{A}$-module. A family $\{f_i\}_{i\in I} \subseteq E$ is a *frame* for E, if there exist real constants $0 < A \leq B < \infty$, such that for each $x \in E$,

$$A\langle x,x\rangle \leq \sum_{i\in I}\langle x,f_i\rangle\langle f_i,x\rangle \leq B\langle x,x\rangle. \tag{1}$$

The numbers A and B are called the *lower* and *upper bound* of the frame, respectively. In this case we call it an (A,B) *frame*. If only the second inequality is required, we call it a *Bessel sequence*. If the sum in (1) converges in norm, the frame is called *standard*.

Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ be standard Bessel sequences in E. Then we say that $\mathcal{G}$ (resp. $\mathcal{F}$) is an *alternate dual* or a *dual* of $\mathcal{F}$ (resp. $\mathcal{G}$), if $x = \sum_{i\in I}\langle x,f_i\rangle g_i$ or equivalently $x = \sum_{i\in I}\langle x,g_i\rangle f_i$ for each $x \in E$ (see [17, Proposition 3.8]).

Let $\{E_i\}_{i\in I}$ be a sequence of finitely or countably generated Hilbert C^*-modules over a unital C^*-algebra $\mathfrak{A}$. A sequence $\Lambda = \{\Lambda_i \in \mathfrak{L}(E,E_i) : i \in I\}$ is called a *g-frame* for E with respect to $\{E_i : i \in I\}$ if there exist real constants $A, B > 0$ such that

$$A\langle x,x\rangle \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leq B\langle x,x\rangle$$

for each $x \in E$. In this case we call it an (A,B) *g-frame*. If only the second-hand inequality is required, then Λ is called a *g-Bessel sequence*. Note that *standard* g-frames are defined similar to the standard frames.

Recall that if $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ are standard g-Bessel sequences such that $\sum_{i\in I} \Gamma_i^*\Lambda_i x = x$ or equivalently $\sum_{i\in I} \Lambda_i^*\Gamma_i x = x$ for each $x \in E$, then Γ (resp. Λ) is called a *g-dual* of Λ (resp. Γ).

For more results about frames and g-frames in Hilbert C^*-modules, see [1, 15, 20, 39].

In this paper all C^*-algebras are assumed to be unital and all Hilbert C^*-modules are assumed to be finitely or countably generated. All frames, g-frames and Bessel sequences are assumed to be standard and all index sets are assumed to be finite or countable subsets of $\mathbb{N}$.

3. (a, m)-approximate duality of frames and g-frames

Recall that $\ell^\infty(I, \mathfrak{A})$ is $\left\{\{a_i\}_{i\in I} \subseteq \mathfrak{A} : \|\{a_i\}\|_\infty = \sup\{\|a_i\| : i \in I\} < \infty\right\}$. In this note m is always a sequence $\{m_i\}_{i\in I} \in \ell^\infty(I, \mathfrak{A})$ with $m_i \in \mathcal{Z}(\mathfrak{A})$, for each $i \in I$. Each sequence with these properties is called a *symbol*.

Let E_1 and E_2 be Hilbert $\mathfrak{A}$-modules, and let $\mathcal{F} = \{f_i\}_{i\in I} \subseteq E_1$ and $\mathcal{G} = \{g_i\}_{i\in I} \subseteq E_2$ be standard Bessel sequences. It was proved in [24] that the operator $M_{m,\mathcal{G},\mathcal{F}} : E_1 \longrightarrow E_2$ which is defined by $M_{m,\mathcal{G},\mathcal{F}}(x) = \sum_{i\in I} m_i\langle x, f_i\rangle g_i$, is adjointable.

Definition 3.1. $M_{m,\mathcal{G},\mathcal{F}}$ is called the *Bessel multiplier* for the Bessel sequences $\mathcal{F}$ and $\mathcal{G}$ with symbol m. If $m_i = 1_{\mathfrak{A}}$ for each $i \in I$, then we denote $M_{m,\mathcal{G},\mathcal{F}}$ by $M_{\mathcal{G}\mathcal{F}}$.

In this paper $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ are standard Bessel sequences in a Hilbert C^*-module E, so $M_{m,\mathcal{G},\mathcal{F}} \in \mathfrak{L}(E)$.

Let $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ be standard g-Bessel sequences for E with respect to $\{E_i\}_{i\in I}$. Then it was shown in [24] that the operator $M_{m,\Gamma,\Lambda} : E \longrightarrow E$ which is defined by $M_{m,\Gamma,\Lambda}(x) = \sum_{i\in I} m_i\Gamma_i^*\Lambda_i x$ is adjointable.

Definition 3.2. $M_{m,\Gamma,\Lambda}$ is called the *g-Bessel multiplier* for the g-Bessel sequences Λ and Γ with symbol m. If $m_i = 1_{\mathfrak{A}}$ for each $i \in I$, then $M_{m,\Gamma,\Lambda}$ is denoted by $M_{\Gamma\Lambda}$.

We recall the definitions of approximate duals and approximate g-duals in Hilbert C^*-modules from [29].

Definition 3.3.
- (i) Two standard g-Bessel sequences Λ and Γ are *approximately dual g-frames* if $\|Id_E - M_{\Gamma\Lambda}\| < 1$. In this case, we say that Γ is an *approximate g-dual* of Λ.
- (ii) Two standard Bessel sequences $\mathcal{F}$ and $\mathcal{G}$ are *approximately dual* frames if $\|Id_E - M_{\mathcal{G}\mathcal{F}}\| < 1$. In this case, we say that $\mathcal{G}$ is an *approximate dual* of $\mathcal{F}$.

Note that if $a \in \mathcal{Z}(\mathfrak{A})$ and $T \in \mathfrak{L}(E)$, then the operator $aT : E \longrightarrow E$ which is defined by $(aT)(x) = aT(x)$ is adjointable with $(aT)^* = a^*T^*$.

Definition 3.4. Let m be a symbol and $a \in \mathcal{Z}(\mathfrak{A})$.

(i) Let Λ and Γ be standard g-Bessel sequences. Then we say that Γ is an (a, m)-approximate g-dual (resp. (a, m)-g-dual) of Λ if $\|Id_E - aM_{m,\Gamma,\Lambda}\| < 1$ (resp. $Id_E = aM_{m,\Gamma,\Lambda}$).

(ii) Let $\mathcal{F}$ and $\mathcal{G}$ be standard Bessel sequences. Then we say that $\mathcal{G}$ is an (a, m)-approximate dual (resp. (a, m)-dual) of $\mathcal{F}$ if $\|Id_E - aM_{m,\mathcal{G},\mathcal{F}}\| < 1$ (resp. $Id_E = aM_{m,\mathcal{G},\mathcal{F}}$).

Note that if $a = 1_{\mathfrak{A}}$, $m_i = 1_{\mathfrak{A}}$ for each $i \in I$, then (a, m)-approximate duality coincides with the concept of approximate duality stated in Definition 3.3.

The following result is a generalization of parts (i) of Proposition 2.3 in [23] and Theorem 3.2 in [29] to (a, m)-approximate g-duals:

Theorem 3.5. *Let Γ be an (a, m)-approximate g-dual of Λ. Then*

(i) *Λ and Γ are standard g-frames.*

(ii) *Λ is an (a^*, m^*)-approximate g-dual of Γ, where $m^* = \{m_i^*\}_{i \in I}$.*

Proof. (i) Since $\|Id_E - aM_{m,\Gamma,\Lambda}\| < 1$, $aM_{m,\Gamma,\Lambda}$ is invertible. Let T be the inverse of $aM_{m,\Gamma,\Lambda}$. Then $(aT)M_{m,\Gamma,\Lambda} = T(aM_{m,\Gamma,\Lambda}) = Id_E$ and $M_{m,\Gamma,\Lambda}(aT) = (aM_{m,\Gamma,\Lambda})T = Id_E$. Hence $M_{m,\Gamma,\Lambda}$ is invertible. It was shown in [24] that $M^*_{m,\Gamma,\Lambda} = M_{m^*,\Lambda,\Gamma}$, so $M_{m,\Gamma,\Lambda}$ and $M_{m^*,\Lambda,\Gamma}$ are invertible and consequently they are bounded below. Now by Proposition 3.9 in [24], Λ and Γ are standard g-frames.

(ii) The result follows from the equality

$$\|Id_E - aM_{m,\Gamma,\Lambda}\| = \|(Id_E - aM_{m,\Gamma,\Lambda})^*\| = \|Id_E - a^* M_{m^*,\Lambda,\Gamma}\|. \qquad \square$$

Remark 3.6. Let $\mathcal{F} = \{f_i\}_{i \in I}$ and $\mathcal{G} = \{g_i\}_{i \in I} \subseteq E$ be standard Bessel sequences. It was shown in Example 3.1 in [20] that if $\phi_i, \psi_i : E \longrightarrow \mathfrak{A}$ are defined by $\phi_i(x) = \langle x, f_i \rangle$, $\psi_i(x) = \langle x, g_i \rangle$, then $\Phi = \{\phi_i\}_{i \in I}$ and $\Psi = \{\psi_i\}_{i \in I}$ are standard g-Bessel sequences and in this case $M_{m,\Psi,\Phi} = M_{m,\mathcal{G},\mathcal{F}}$ ([24, Remark 3.6]). Thus if $\mathcal{G}$ is an (a, m)-approximate dual (resp. (a, m)-dual) of $\mathcal{F}$, then Ψ is an (a, m)-approximate g-dual (resp. (a, m)-g-dual) of Φ.

Using the above theorem and remark, we get the following result which is a generalization of part (i) of Corollary 3.3 in [29] to (a, m)-approximate duals:

Corollary 3.7. *Let $\mathcal{G}$ be an (a, m)-approximate dual of $\mathcal{F}$. Then*

(i) *$\mathcal{F}$ and $\mathcal{G}$ are standard frames.*

(ii) *$\mathcal{F}$ is an (a^*, m^*)-approximate dual of $\mathcal{G}$.*

It follows from the proof of Theorem 3.5 that if T is the inverse of $aM_{m,\Gamma,\Lambda}$, then aT is the inverse of $M_{m,\Gamma,\Lambda}$. Hence if Γ is an (a, m)-approximate g-dual of Λ, then using Neumann series, we get $M^{-1}_{m,\Gamma,\Lambda} = a \sum_{n=0}^{\infty} (Id_E - aM_{m,\Gamma,\Lambda})^n$, and for each $x \in E$, we have the following reconstruction formula:

$$x = M_{m,\Gamma,\Lambda} M^{-1}_{m,\Gamma,\Lambda} x = a \sum_{n=0}^{\infty} M_{m,\Gamma,\Lambda} (Id_E - aM_{m,\Gamma,\Lambda})^n x.$$

The following remark shows that (a,m)-approximate duals generate dual frames (we state the following remark for Hilbert spaces because of more applications).

Remark 3.8. Suppose that H is a Hilbert space, $\{H_i\}_{i\in I}$ is a sequence of Hilbert spaces, $\Lambda=\{\Lambda_i\in L(H,H_i): i\in I\}$ and $\Gamma=\{\Gamma_i\in L(H,H_i): i\in I\}$, where $L(H,H_i)$ is the set of all linear and bounded operators from H into H_i. Also assume that $\alpha\in\mathbb{C}$ and $m=\{m_i\}_{i\in I}\subseteq\mathbb{C}$. If Γ is an (α,m)-approximate g-dual of Λ, then for each $x\in H$, we have

$$\begin{aligned} x &= M_{m,\Gamma,\Lambda}M^{-1}_{m,\Gamma,\Lambda}x \\ &= \sum_{i\in I} m_i\Gamma_i^*\Lambda_i M^{-1}_{m,\Gamma,\Lambda}x = \sum_{i\in I}\Gamma_i^*(\alpha m_i\Lambda_i\sum_{n=0}^{\infty}(Id_H-\alpha M_{m,\Gamma,\Lambda})^n x). \end{aligned}$$

Therefore $\{\alpha m_i\Lambda_i\sum_{n=0}^{\infty}(Id_H-\alpha M_{m,\Gamma,\Lambda})^n\}_{i\in I}$ is a g-dual of Γ.

Using the above remark and Remark 3.6, we can construct duals for Bessel sequences in Hilbert spaces using Bessel multipliers, i.e., if $\mathcal{F}=\{f_i\}_{i\in I},\mathcal{G}=\{g_i\}_{i\in I}\subseteq H$ such that $\mathcal{G}$ is an (α,m)-approximate dual of $\mathcal{F}$, then

$$\{\overline{\alpha m_i}\sum_{n=0}^{\infty}(Id_H-\overline{\alpha}M_{\overline{m},\mathcal{F},\mathcal{G}})^n f_i\}_{i\in I}$$

is a dual of $\mathcal{G}$, where $\overline{m}=\{\overline{m_i}\}_{i\in I}$.

We recall the following definition and theorem from [4]:

Definition 3.9. Let H be a Hilbert space, $\mathcal{F}=\{f_i\}_{i\in I}\subseteq H$ and $\omega=\{\omega_i : i\in I\}\subseteq\mathbb{R}^+$($\omega$ is a sequence of positive weights). Then $(\omega,\mathcal{F})$ is called a ω-frame if there exist constants $0<A,B<\infty$ such that

$$A\|f\|^2\le\sum_{i\in I}\omega_i|\langle f,f_i\rangle|^2\le B\|f\|^2.$$

In this case $\mathcal{F}$ is called a weighted frame.

A sequence $\{c_i\}_{i\in I}$ is called *semi-normalized* if there are constants $b\ge a>0$ such that $a\le|c_i|\le b$ for each $i\in I$.

Theorem 3.10. *Let $\mathcal{F}=\{f_i\}_{i\in I}$ be a sequence in H. Let $\{m_i\}_{i\in I}$ be a positive, semi-normalized sequence. Then the following are equivalent.*

(i) *$\mathcal{F}$ is a frame.*
(ii) *$M_{m,\mathcal{F},\mathcal{F}}$ is a positive and invertible operator.*
(iii) *There are constants $R,r\ge 0$ such that*
$$r\|f\|^2\le\sum_{i\in I}m_i|\langle f,f_i\rangle|^2\le R\|f\|^2$$
for each $f\in H$.
(iv) *$\{\sqrt{m_i}f_i\}_{i\in I}$ is a frame.*
(v) *$M_{m',\mathcal{F},\mathcal{F}}$ is a positive and invertible operator for each positive, semi-normalized sequence $m'=\{m'_i\}_{i\in I}$.*

The above theorem makes a relationship between weighted frames and Bessel multipliers, so (a,m)-approximte duals can be useful for weighted frames.

Recall from [25] that a frame $\{f_i\}_{i\in I}$ for a Hilbert space H is called *scalable* if there exist scalars $c_i \geq 0, i \in I$ such that $\{c_i f_i\}_{i\in I}$ is a Parseval frame (the frame bounds are equal to 1). If, in addition, $c_i > 0$ for each $i \in I$, then $\{f_i\}_{i\in I}$ is called positively scalable. It is clear that $\mathcal{F}$ is a weighted frame if and only if $\{\omega_i^{\frac{1}{2}} f_i\}_{i\in I}$ is a frame. Therefore a positively scalable frame can be assumed as some kind of weighted frame (with $\omega = \{c_i^2\}_{i\in I}$), so (a,m)-approximate duals have applications for scalable frames.

Proposition 3.11. *Suppose that $m = \{m_i\}_{i\in I}$ is a symbol such that there exists a positive number α with $\alpha 1_{\mathfrak{A}} \leq m_i$ for each $i \in I$. If Γ is an (a,m)-approximate g-dual of Λ for some positive, invertible element $a \in \mathcal{Z}(\mathfrak{A})$, then $\Lambda + \Gamma = \{\Lambda_i + \Gamma_i\}_{i\in I}$ is a standard g-frame.*

Proof. Because Γ is an (a,m)-approximate g-dual of Λ, by Theorem 3.5, both of them are standard g-frames and Λ is an (a,m)-approximate g-dual of Γ, so Proposition 3.7 in [24] yields that $M_{m,\Lambda,\Lambda}$ and $M_{m,\Gamma,\Gamma}$ are positive and invertible operator and we have $\|Id_E - aM_{m,\Gamma,\Lambda}\| < 1$ and $\|Id_E - aM_{m,\Lambda,\Gamma}\| < 1$. Hence $\|2Id_E - (aM_{m,\Gamma,\Lambda} + aM_{m,\Lambda,\Gamma})\| < 2$ and since $(aM_{m,\Gamma,\Lambda} + aM_{m,\Lambda,\Gamma})$ is self-adjoint, Lemma 2.2.2 in [31] implies that $(aM_{m,\Gamma,\Lambda} + aM_{m,\Lambda,\Gamma})$ is a positive operator and using the invertibility of a, we get $M_{m,\Gamma,\Lambda} + M_{m,\Lambda,\Gamma}$ is also positive. Now it is easy to see that

$$M_{m,(\Lambda+\Gamma),(\Lambda+\Gamma)} = M_{m,\Lambda,\Lambda} + M_{m,\Lambda,\Gamma} + M_{m,\Gamma,\Lambda} + M_{m,\Gamma,\Gamma}.$$

Since $M_{m,\Gamma,\Gamma}$, $M_{m,\Lambda,\Gamma} + M_{m,\Gamma,\Lambda}$ are positive operators and $M_{m,\Lambda,\Lambda}$ is invertible, we have $M_{m,(\Lambda+\Gamma),(\Lambda+\Gamma)} \geq \|M^{-1}_{m,\Lambda,\Lambda}\|^{-1} Id_E$, so $M_{m,(\Lambda+\Gamma),(\Lambda+\Gamma)}$ is a positive and invertible operator. Now Proposition 3.7 in [24] implies that $\Lambda + \Gamma$ is a standard g-frame. □

Corollary 3.12. *Suppose that $m = \{m_i\}_{i\in I}$ is a symbol such that there exists a positive number α with $\alpha 1_{\mathfrak{A}} \leq m_i$ for each $i \in I$. If $\mathcal{G}$ is an (a,m)-approximate dual of $\mathcal{F}$ for some positive, invertible element $a \in \mathcal{Z}(\mathfrak{A})$, then $\mathcal{F} + \mathcal{G} = \{f_i + g_i\}_{i\in I}$ is a standard frame.*

Note that by considering $a = 1_{\mathfrak{A}}$ and $m_i = 1_{\mathfrak{A}}$ for each $i \in I$ in the above proposition and corollary and using the fact that Hilbert spaces are special cases of Hilbert C^*-modules, we conclude that Proposition 2.6 and Corollary 2.7 in [23] are special cases of Proposition 3.11 and Corollary 3.12, respectively.

The following result is a generalization of [23, Proposition 2.3(ii)] and [29, Theorem 3.2(ii), (iii)] to (a,m)-approximate g-duals:

Proposition 3.13. *Let Γ be an (a,m)-approximate g-dual of Λ. Then*

(i) $\Psi = \{\sum_{n=0}^{\infty} \Gamma_i (Id_E - a^* M_{m^*,\Lambda,\Gamma})^n\}_{i\in I}$ *is an (a,m)-g-dual of Λ.*

(ii) *For each* $N \in \mathbb{N}$, *define* $\psi_i^{(N)} = \sum_{n=0}^{N} \Gamma_i(Id_E - a^* M_{m^*,\Lambda,\Gamma})^n$. *Then* $\Psi_N = \{\psi_i^{(N)}\}_{i\in I}$ *is an* (a,m)*-approximate* g*-dual of* Λ *with* $\|Id_E - aM_{m,\Psi_N,\Lambda}\| \leq \|Id_E - aM_{m,\Gamma,\Lambda}\|^{N+1} < 1$ *and* $Id_E = \lim_{N\to\infty} aM_{m,\Psi_N,\Lambda}$.

Proof. (i) Let $T = (a^* M_{m^*,\Lambda,\Gamma})^{-1}$. Then $(M_{m^*,\Lambda,\Gamma})^{-1} = a^* T$, so $M_{m,\Gamma,\Lambda}^{-1} = aT^*$. Now since $T = \sum_{n=0}^{\infty}(Id_E - a^* M_{m^*,\Lambda,\Gamma})^n$, it is easy to see that $\Psi = \{\Gamma_i T\}_{i\in I}$ is a standard g-Bessel sequence and

$$aM_{m,\Psi,\Lambda}x = a\sum_{i\in I} m_i T^* \Gamma_i^* \Lambda_i x = aT^*(M_{m,\Gamma,\Lambda}x) = x$$

for each $x \in E$. This means that Ψ is an (a,m)-g-dual of Λ.

(ii) Define $T_N : E \longrightarrow E$ by $T_N = \sum_{n=0}^{N}(Id_E - a^* M_{m^*,\Lambda,\Gamma})^n$. Then $\Psi_N = \{\Gamma_i T_N\}_{i\in I}$, so Ψ_N is a standard g-Bessel sequence and for each $x \in E$, we have

$$M_{m^*,\Lambda,\Gamma} T_N x = \sum_{i\in I} m_i^* \Lambda_i^* \Gamma_i T_N(x) = \sum_{i\in I} m_i^* \Lambda_i^* \psi_i^{(N)}(x) = M_{m^*,\Lambda,\Psi_N} x.$$

Thus

$$\begin{aligned} a^* M_{m^*,\Lambda,\Psi_N} &= a^* M_{m^*,\Lambda,\Gamma} T_N \\ &= \Big[Id_E - (Id_E - a^* M_{m^*,\Lambda,\Gamma})\Big] \sum_{n=0}^{N} (Id_E - a^* M_{m^*,\Lambda,\Gamma})^n \\ &= Id_E - (Id_E - a^* M_{m^*,\Lambda,\Gamma})^{N+1}. \end{aligned}$$

This yields that

$$\|Id_E - a^* M_{m^*,\Lambda,\Psi_N}\| = \|(Id_E - a^* M_{m^*,\Lambda,\Gamma})^{N+1}\| \leq \|Id_E - a^* M_{m^*,\Lambda,\Gamma}\|^{N+1} < 1.$$

Therefore Λ is an (a^*, m^*)-approximate g-dual of Ψ_N and consequently Ψ_N is an (a,m)-approximate g-dual of Λ. Since $\lim_{N\to\infty} \|Id_E - a^* M_{m^*,\Lambda,\Gamma}\|^{N+1} = 0$, we have

$$\lim_{N\to\infty} \|Id_E - aM_{m,\Psi_N,\Lambda}\| = \lim_{N\to\infty} \|Id_E - a^* M_{m^*,\Lambda,\Psi_N}\| = 0.$$

This completes the proof. □

The next corollary is a generalization of [10, Proposition 3.2] and [29, Corollary 3.3(ii), (iii)] to (a,m)-approximate duals:

Corollary 3.14. *Let* $\mathcal{G}$ *be an* (a,m)*-approximate dual of* $\mathcal{F}$. *Then*

(i) $\{\sum_{n=0}^{\infty}(Id_E - aM_{m,\mathcal{G},\mathcal{F}})^n g_i\}_{i\in I}$ *is an* (a,m)*-dual of* $\mathcal{F}$.

(ii) *For each* $N \in \mathbb{N}$, *define* $h_i^{(N)} = \sum_{n=0}^{N}(Id_E - aM_{m,\mathcal{G},\mathcal{F}})^n g_i$. *Then* $h_N = \{h_i^{(N)}\}_{i\in I}$ *is an* (a,m)*-approximate dual of* $\mathcal{F}$ *with* $\|Id_E - aM_{m,h_N,\mathcal{F}}\| \leq \|Id_E - aM_{m,\mathcal{G},\mathcal{F}}\|^{N+1} < 1$ *and* $Id_E = \lim_{N\to\infty} aM_{m,h_N,\mathcal{F}}$.

Now we recall tensor products of C^*-algebras and Hilbert C^*-modules from [31] and [26], respectively.

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two C^*-algebras. Then $\mathfrak{A} \otimes \mathfrak{A}'$ is a C^*-algebra with the spatial norm and for each $a \in \mathfrak{A}$ and $a' \in \mathfrak{A}'$, we have $\|a \otimes a'\| = \|a\|\|a'\|$. The multiplication and involution on simple tensors are defined by $(a \otimes a')(b \otimes b') = ab \otimes a'b'$ and $(a \otimes a')^* = a^* \otimes a'^*$, respectively.

Now let E be a Hilbert $\mathfrak{A}$-module and E' be a Hilbert $\mathfrak{A}'$-module. Then the tensor product $E \otimes E'$ is a Hilbert $\mathfrak{A} \otimes \mathfrak{A}'$-module. The module action and inner product for simple tensors are defined by $(a \otimes a')(x \otimes x') = (ax) \otimes (a'x')$ and $\langle x \otimes x', y \otimes y'\rangle = \langle x, y\rangle \otimes \langle x', y'\rangle$, respectively. Let U and U' be adjointable operators on E and E', respectively. Then the tensor product $U \otimes U'$ is an adjointable operator on $E \otimes E'$. Also $(U \otimes U')^* = U^* \otimes U'^*$ and $\|U \otimes U'\| = \|U\|\|U'\|$. For more results about tensor products of C^*-algebras and Hilbert C^*-modules, see [26, 31].

Tensor products of frames and g-frames have been studied by some authors, see [8, 19, 20, 22].

In the following theorem and proposition $\Lambda' = \{\Lambda'_j \in \mathfrak{L}(E', E'_j) : j \in J\}$, $\Gamma' = \{\Gamma'_j \in \mathfrak{L}(E', E'_j) : j \in J\}$, $\mathcal{F}' = \{f'_j\}_{j\in J}$ and $\mathcal{G}' = \{g'_j\}_{j\in J} \subseteq E'$, where E' and E'_j's are Hilbert $\mathfrak{A}'$-modules and $m' = \{m'_j\}_{j\in J} \subseteq \mathfrak{A}'$, $a' \in \mathfrak{A}'$. Also $\Lambda \otimes \Lambda' = \{\Lambda_i \otimes \Lambda'_j\}_{i\in I, j\in J}$, $m \otimes m' = \{m_i \otimes m'_j\}_{i\in I, j\in J}$ and $\mathcal{F} \otimes \mathcal{F}' = \{f_i \otimes f'_j\}_{i\in I, j\in J}$.

Theorem 3.15 ([30]). (i) *If Λ and Λ' are standard g-Bessel sequences, then $\Lambda \otimes \Lambda'$ is a standard g-Bessel sequence. Moreover Λ and Λ' are standard g-frames if and only if $\Lambda \otimes \Lambda'$ is a standard g-frame.*

(ii) *Let m and m' be two symbols. If Λ and Γ are standard g-Bessel sequences for E with respect to $\{E_i\}_{i\in I}$ and Λ' and Γ' are standard g-Bessel sequences for E' with respect to $\{E'_j\}_{j\in J}$, then $m \otimes m'$ is a symbol and $M_{(m\otimes m'),(\Gamma\otimes\Gamma'),(\Lambda\otimes\Lambda')} = M_{m,\Gamma,\Lambda} \otimes M_{m',\Gamma',\Lambda'}$.*

The next result is a generalization of [29, Proposition 3.6] to (a, m)-approximate duality.

Proposition 3.16. (i) *Let Γ be an (a, m)-approximate g-dual (resp. (a, m)-g-dual) of Λ. If Γ' is an (a', m')-g-dual of Λ', then $\Gamma \otimes \Gamma'$ is an $(a \otimes a', m \otimes m')$-approximate g-dual (resp. $(a \otimes a', m \otimes m')$-g-dual) of $\Lambda \otimes \Lambda'$.*

(ii) *Let $\mathcal{G}$ be an (a, m)-approximate dual (resp. (a, m)-dual) of $\mathcal{F}$. If $\mathcal{G}'$ is an (a', m')-dual of $\mathcal{F}'$, then $\mathcal{G} \otimes \mathcal{G}'$ is an $(a \otimes a', m \otimes m')$-approximate dual (resp. $(a \otimes a', m \otimes m')$-dual) of $\mathcal{F} \otimes \mathcal{F}'$.*

Proof. (i) It follows from Theorem 3.15 that $\Lambda\otimes\Lambda'$, $\Gamma\otimes\Gamma'$ are standard g-Bessel sequences, $m\otimes m'$ is a symbol and $M_{(m\otimes m'),(\Gamma\otimes\Gamma'),(\Lambda\otimes\Lambda')} = M_{m,\Gamma,\Lambda} \otimes M_{m',\Gamma',\Lambda'}$. Now we get

$$\begin{aligned}
&\|(a \otimes a')M_{(m\otimes m'),(\Gamma\otimes\Gamma'),(\Lambda\otimes\Lambda')} - Id_{(E\otimes E')}\| \\
&= \|aM_{m,\Gamma,\Lambda} \otimes a'M_{m',\Gamma',\Lambda'} - Id_E \otimes Id_{E'}\| \\
&= \|(aM_{m,\Gamma,\Lambda} - Id_E) \otimes Id_{E'}\| = \|aM_{m,\Gamma,\Lambda} - Id_E\| < 1.
\end{aligned}$$

For (a,m)-g-duality, we have $aM_{m,\Gamma,\Lambda} = Id_E$ and $a'M_{m',\Gamma',\Lambda'} = Id_{E'}$, so

$$(a\otimes a')M_{(m\otimes m'),(\Gamma\otimes\Gamma'),(\Lambda\otimes\Lambda')} = aM_{m,\Gamma,\Lambda}\otimes a'M_{m',\Gamma',\Lambda'} = Id_{E\otimes E'}.$$

(ii) The result follows from Remark 3.6 and part (i). □

4. (a, m)-approximate duals, modular Riesz bases and perturbations

In this section, we consider (a,m)-approximate duals of modular Riesz bases and perturbations of (a,m)-approximate duals.

Note that if $\{E_i : i\in I\}$ is a sequence of Hilbert $\mathfrak{A}$-modules, then $\oplus_{i\in I}E_i$, which is the set

$$\oplus_{i\in I}E_i = \Big\{\{x_i\}_{i\in I} : x_i\in E_i \text{ and } \sum_{i\in I}\langle x_i,x_i\rangle \text{ is norm convergent in } \mathfrak{A}\Big\},$$

is a Hilbert $\mathfrak{A}$-module with pointwise operations and $\mathfrak{A}$-valued inner product $\langle x,y\rangle = \sum_{i\in I}\langle x_i,y_i\rangle$, where $x=\{x_i\}_{i\in I}$ and $y=\{y_i\}_{i\in I}$. Recall that for a standard g-Bessel sequence Λ, the operator $T_\Lambda : \oplus_{i\in I}E_i \longrightarrow E$ defined by $T_\Lambda(\{x_i\}_{i\in I}) = \sum_{i\in I}\Lambda_i^*(x_i)$ is called the *synthesis operator* of Λ. T_Λ is adjointable and $T_\Lambda^*(x) = \{\Lambda_i x\}_{i\in I}$. Now we define the operator S_Λ on E by $S_\Lambda x = T_\Lambda T_\Lambda^*(x) = \sum_{i\in I}\Lambda_i^*\Lambda_i(x)$. If Λ is a standard (A,B) g-frame, then $A\ Id_E \le S_\Lambda \le B\ Id_E$.

Note that if $\mathcal{F}=\{f_i\}_{i\in I}$ is a standard Bessel sequence (resp. frame), then $\Lambda_{\mathcal{F}} = \{\Lambda_{f_i}\}_{i\in I}$ is a standard g-Bessel sequence (resp. g-frame), where $\Lambda_{f_i}(x) = \langle x, f_i\rangle$, for each $x\in E$. We denote $S_{\Lambda_{\mathcal{F}}}$ by $S_{\mathcal{F}}$.

Let $\Lambda=\{\Lambda_i\}_{i\in I}$ be an (A,B) standard g-frame. We call $\widetilde{\Lambda} = \{\widetilde{\Lambda_i}\}$, where $\widetilde{\Lambda_i} = \Lambda_i S_\Lambda^{-1}$ the *canonical g-dual* of Λ which is an $(\frac{1}{B},\frac{1}{A})$ standard g-frame. We denote the *canonical dual* of a standard frame $\mathcal{F}=\{f_i\}_{i\in I}$ by $\widetilde{\mathcal{F}} = \{\widetilde{f}_i\}_{i\in I}$, where $\widetilde{f}_i = S_{\mathcal{F}}^{-1}f_i$.

Now we recall the following definition from [21]:

Definition 4.1. (i) A standard g-frame Λ is a *modular g-Riesz basis* if it has the following property:
if $\sum_{i\in\Omega}\Lambda_i^* g_i = 0$, where $g_i\in E_i$ and $\Omega\subseteq I$, then $g_i=0$ for each $i\in\Omega$.

(ii) A standard frame $\{f_i\}_{i\in I}$ for E is a *modular Riesz basis* if it has the following property:
if an $\mathfrak{A}$-linear combination $\sum_{i\in\Omega} a_i f_i$ with coefficients $\{a_i : i\in\Omega\}\subseteq\mathfrak{A}$ and $\Omega\subseteq I$ is equal to zero, then $a_i = 0$ for each $i\in\Omega$.

As we know the canonical dual is the unique dual of a Riesz basis in a Hilbert space. A similar result holds for modular Riesz bases in Hilbert C^*-modules, see [21]. Now we have the following proposition and corollary for (a,m)-duals and approximate duals of modular g-Riesz bases and modular Riesz bases which generalize parts (iv) of Theorem 3.2 and Corollary 3.3 in [29], respectively.

Proposition 4.2. *Let a and m_i's be invertible elements of $\mathcal{Z}(\mathfrak{A})$ and let Λ be a modular g-Riesz basis. Then*

(i) $\{(a^*m_i^*)^{-1}\widetilde{\Lambda_i}\}_{i\in I}$ *is the unique (a,m)-g-dual of Λ.*

(ii) *If Γ is an (a,m)-approximate g-dual of Λ, then*

$$\widetilde{\Lambda_i} = (a^*m_i^*)\sum_{n=0}^{\infty}\Gamma_i(Id_E - a^*M_{m^*,\Lambda,\Gamma})^n.$$

Proof. (i) It is easy to see that $\{(a^*m_i^*)^{-1}\widetilde{\Lambda_i}\}_{i\in I}$ is a standard g-Bessel sequence. Let $x \in E$. Then

$$aM_{m,\{(a^*m_i^*)^{-1}\widetilde{\Lambda_i}\}_{i\in I},\Lambda}x = \sum_{i\in I} am_i[(a^*m_i^*)^*]^{-1}\widetilde{\Lambda_i}^*\Lambda_i(x) = \sum_{i\in I}\widetilde{\Lambda_i}^*\Lambda_i x = x.$$

This shows that $\{(a^*m_i^*)^{-1}\widetilde{\Lambda_i}\}_{i\in I}$ is an (a,m)-g-dual of Λ. Now let $\Gamma = \{\Gamma_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be (a,m)-g-duals of Λ. Then for each $x \in E$

$$\sum_{i\in I} a^*m_i^*\Lambda_i^*\Gamma_i x = x = \sum_{i\in I} a^*m_i^*\Lambda_i^*\psi_i x,$$

so $\sum_{i\in I}\Lambda_i^*(a^*m_i^*(\Gamma_i - \psi_i)x) = 0$. Since Λ is a modular g-Riesz basis, $a^*m_i^*(\Gamma_i - \psi_i)x = 0$ for each $i \in I$. Because $a^*m_i^*$ is invertible, we have $(\Gamma_i - \psi_i)x = 0$, so $\Gamma_i = \psi_i$ for each $i \in I$.

(ii) The result follows from (i) and Proposition 3.13. $\square$

Corollary 4.3. *Let a and m_i's be invertible elements of $\mathcal{Z}(\mathfrak{A})$ and let $\mathcal{F}$ be a modular Riesz basis. Then*

(i) $\{(am_i)^{-1}\widetilde{f_i}\}_{i\in I}$ *is the unique (a,m)-dual of $\mathcal{F}$.*

(ii) *If $\mathcal{G}$ is an (a,m)-approximate dual of $\mathcal{F}$, then*

$$\widetilde{f_i} = (am_i)\sum_{n=0}^{\infty}(Id_E - aM_{m,\mathcal{G},\mathcal{F}})^n g_i.$$

Recall from [37] that $\{\Lambda_i \in L(H,H_i) : i \in I\}$ is *g-complete* if $\{f : \Lambda_i f = 0, \forall i \in I\} = \{0\}$, and we call it a *g-Riesz basis* for H, if it is g-complete and there exist two constants $0 < A \le B < \infty$, such that for each finite subset $F \subseteq I$ and $f_i \in H_i, i \in F$,

$$A\sum_{i\in F}\|f_i\|^2 \le \left\|\sum_{i\in F}\Lambda_i^* f_i\right\|^2 \le B\sum_{i\in F}\|f_i\|^2.$$

A family $\{f_i\}_{i\in I} \subseteq H$ is *complete* if the span of $\{f_i\}_{i\in I}$ is dense in H. We say that $\{f_i\}_{i\in I}$ is a *Riesz basis* for H, if it is complete in H and there exist two constants $0 < A \le B < \infty$, such that

$$A\sum_{i\in F}|c_i|^2 \le \left\|\sum_{i\in F}c_i f_i\right\|^2 \le B\sum_{i\in F}|c_i|^2$$

for each sequence of scalars $\{c_i\}_{i\in F}$, where F is a finite subset of I.

As a consequence of Proposition 4.2 and Corollary 4.3, we get the following result.

Corollary 4.4. *Suppose that α, m_i's are nonzero elements of $\mathbb{C}$, $m = \{m_i\}_{i\in I}$ and $\overline{m} = \{\overline{m_i}\}_{i\in I}$.*

(i) *If $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ is a g-Riesz basis, then $\{\frac{1}{\overline{\alpha m_i}}\widetilde{\Lambda_i}\}_{i\in I}$ is the unique (α, m)-g-dual of Λ.*

(ii) *If $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ is an (α, m)-approximate g-dual of Λ (Λ is a g-Riesz basis), then*

$$\widetilde{\Lambda_i} = \sum_{n=0}^{\infty} \overline{\alpha m_i}\Gamma_i(Id_H - \overline{\alpha}M_{\overline{m},\Lambda,\Gamma})^n.$$

(iii) *If $\mathcal{F} = \{f_i\}_{i\in I} \subseteq H$ is a Riesz basis, then $\{\frac{1}{\alpha m_i}\widetilde{f_i}\}_{i\in I}$ is the unique (α, m)-dual of $\mathcal{F}$.*

(iv) *If $\mathcal{G} = \{g_i\}_{i\in I} \subseteq H$ is an (α, m)-approximate dual of $\mathcal{F}$ ($\mathcal{F}$ is a Riesz basis), then $\widetilde{f_i} = \sum_{n=0}^{\infty} \alpha m_i(Id_H - \alpha M_{m,\mathcal{G},\mathcal{F}})g_i$.*

The next theorem and corollary are generalizations of [23, Proposition 3.10] and [29, Proposition 3.8] to (a, m)-approximate duality of g-frames and frames, respectively.

Theorem 4.5. *Let $0 \le \lambda_1, \lambda_2 < 1$. Suppose that Λ is a standard g-Bessel sequence with upper bound B and $\Psi = \{\psi_i\}_{i\in I}$ is an (a, m)-g-dual of Λ with upper bound D. If $\{\Gamma_i\}_{i\in I}$ is a sequence satisfying*

$$(2) \qquad \Big\|\sum_{i\in\Omega}(\Lambda_i^* - \Gamma_i^*)f_i\Big\| \le \lambda_1\Big\|\sum_{i\in\Omega}\Lambda_i^* f_i\Big\| + \lambda_2\Big\|\sum_{i\in\Omega}\Gamma_i^* f_i\Big\| + \varepsilon\Big\|\sum_{i\in\Omega}|f_i|^2\Big\|^{\frac{1}{2}}$$

for each finite subset $\Omega \subseteq I$, $f_i \in E_i$ with

$$\lambda_1 + (\|a\|\sqrt{D}\|m\|_\infty)\Big[\varepsilon + \lambda_2\frac{[(1+\lambda_1)\sqrt{B}+\varepsilon]}{(1-\lambda_2)}\Big] < 1,$$

then Ψ is an (a, m)-approximate g-dual of Γ.

Proof. It follows from the first part of the proof of Proposition 3.8 in [29] that Γ is a standard g-Bessel sequence with upper bound $C = \frac{[(1+\lambda_1)\sqrt{B}+\varepsilon]^2}{(1-\lambda_2)^2}$. Since Λ and Γ are standard g-Bessel sequences, $\sum_{i\in I}\Lambda_i^* f_i$ and $\sum_{i\in I}\Gamma_i^* f_i$ are convergent for each $\{f_i\}_{i\in I} \in \oplus_{i\in I}E_i$ and by (2) we get

$$(3) \quad \Big\|\sum_{i\in I}\Lambda_i^* f_i - \sum_{i\in I}\Gamma_i^* f_i\Big\| \le \lambda_1\Big\|\sum_{i\in I}\Lambda_i^* f_i\Big\| + \lambda_2\Big\|\sum_{i\in I}\Gamma_i^* f_i\Big\| + \varepsilon\Big\|\sum_{i\in I}|f_i|^2\Big\|^{\frac{1}{2}}.$$

Let $x \in E$. For $f_i = a^* m_i^* \psi_i x$, we have

$$\sum_{i\in I}\langle f_i, f_i\rangle = a^* a\sum_{i\in I} m_i^* m_i\langle\psi_i x, \psi_i x\rangle \le \|a\|^2\|m\|_\infty^2\sum_{i\in I}\langle\psi_i x, \psi_i x\rangle.$$

Because Ψ is a standard g-Bessel sequence, we obtain that $\{f_i\}_{i\in I} \in \oplus_{i\in I} E_i$. Now (3) implies that

$$\begin{aligned}&\Big\|\sum_{i\in I} a^* m_i^* \Lambda_i^* \psi_i x - \sum_{i\in I} a^* m_i^* \Gamma_i^* \psi_i x\Big\| \\ &\leq \lambda_1 \Big\|\sum_{i\in I} a^* m_i^* \Lambda_i^* \psi_i x\Big\| + \lambda_2 \Big\|\sum_{i\in I} a^* m_i^* \Gamma_i^* \psi_i x\Big\| + \varepsilon \Big\|\sum_{i\in I} |a^* m_i^* \psi_i x|^2\Big\|^{\frac{1}{2}}.\end{aligned}$$

Since $a^* M_{m^*,\Lambda,\Psi} = (a M_{m,\Psi,\Lambda})^* = Id_E$ and $\|M_{m^*,\Gamma,\Psi}\| \leq \|m\|_\infty \sqrt{CD}$ (see [24]), we get

$$\begin{aligned}\|x - a^* M_{m^*,\Gamma,\Psi} x\| &\leq \lambda_1 \|x\| + \lambda_2 \|a^* M_{m^*,\Gamma,\Psi} x\| + \varepsilon \|a\| \|m\|_\infty \sqrt{D} \|x\| \\ &\leq \Big(\lambda_1 + (\|a\| \sqrt{D} \|m\|_\infty)[\varepsilon + \lambda_2 \sqrt{C}]\Big) \|x\|.\end{aligned}$$

This implies that

$$\|Id_E - a^* M_{m^*,\Gamma,\Psi}\| \leq \Big(\lambda_1 + (\|a\| \sqrt{D} \|m\|_\infty)[\varepsilon + \lambda_2 \sqrt{C}]\Big) < 1,$$

meaning that Γ is an (a^*, m^*)-approximate g-dual of Ψ equivalently (by Theorem 3.5) Ψ is an (a, m)-approximate g-dual of Γ. □

Corollary 4.6. *Let $0 \leq \lambda_1, \lambda_2 < 1$. Suppose that $\mathcal{F} = \{f_i\}_{i\in I}$ is a standard Bessel sequence with upper bound B and $\mathcal{N} = \{h_i\}_{i\in I}$ is an (a, m)-dual of $\mathcal{F}$ with upper bound D. If $\{g_i\}_{i\in I}$ is a sequence satisfying*

$$\Big\|\sum_{i\in\Omega} a_i f_i - \sum_{i\in\Omega} a_i g_i\Big\| \leq \lambda_1 \Big\|\sum_{i\in\Omega} a_i f_i\Big\| + \lambda_2 \Big\|\sum_{i\in\Omega} a_i g_i\Big\| + \varepsilon \Big\|\sum_{i\in\Omega} |a_i|^2\Big\|^{\frac{1}{2}}$$

for each finite subset $\Omega \subseteq I$, $\{a_i\}_{i\in\Omega} \subseteq \mathfrak{A}$ with $\lambda_1 + (\|a\| \sqrt{D} \|m\|_\infty)\Big[\varepsilon + \lambda_2 \frac{[(1+\lambda_1)\sqrt{B}+\varepsilon]}{(1-\lambda_2)}\Big] < 1$, then $\mathcal{N}$ is an (a, m)-approximate dual of $\mathcal{G}$.

For a standard g-frame Λ (resp. frame $\mathcal{F}$), we denote its lower bound and upper bound by A_Λ and B_Λ (resp. $A_\mathcal{F}$ and $B_\mathcal{F}$), respectively.

The following propositions and corollaries generalize Propositions 4.6, 4.10 and Corollary 4.7 in [35] to (a, m)-approximate duals in Hilbert C^*-modules.

Proposition 4.7. (i) *Let Γ be a g-dual of Λ, $0 \leq \lambda < \frac{1}{\sqrt{B_\Lambda B_\Gamma}}$ and let m be a symbol such that $(1-\lambda)1_\mathfrak{A} \leq m_i \leq (1+\lambda)1_\mathfrak{A}$ for each $i \in I$. Then Λ (resp. Γ) is an $(1_\mathfrak{A}, m)$-approximate g-dual of Γ (resp. Λ).*

(ii) *Let $\mathcal{G}$ be a dual of $\mathcal{F}$, $0 \leq \lambda < \frac{1}{\sqrt{B_\mathcal{F} B_\mathcal{G}}}$ and let m be a symbol such that $(1-\lambda)1_\mathfrak{A} \leq m_i \leq (1+\lambda)1_\mathfrak{A}$ for each $i \in I$. Then $\mathcal{F}$ (resp. $\mathcal{G}$) is an $(1_\mathfrak{A}, m)$-approximate dual of $\mathcal{G}$ (resp. $\mathcal{F}$).*

Proof. (i) Because $-\lambda 1_{\mathfrak{A}} \leq m_i - 1_{\mathfrak{A}} \leq \lambda 1_{\mathfrak{A}}$, we have $\|m_i - 1_{\mathfrak{A}}\| \leq \lambda$ for each $i \in I$ and this yields that $\|\{m_i - 1_{\mathfrak{A}}\}_{i\in I}\|_\infty \leq \lambda$. Let $x \in E$. Then

$$\begin{aligned}\|M_{m,\Gamma,\Lambda}x - x\| &= \left\| \sum_{i\in I} m_i \Gamma_i^* \Lambda_i x - \sum_{i\in I} \Gamma_i^* \Lambda_i x \right\| \\ &\leq \|\{m_i - 1_{\mathfrak{A}}\}\|_\infty \sqrt{B_\Gamma B_\Lambda}\|x\| \leq \lambda \sqrt{B_\Gamma B_\Lambda}\|x\|.\end{aligned}$$

This means that Γ is an $(1_{\mathfrak{A}}, m)$-approximate g-dual of Λ and since $1_{\mathfrak{A}}$ and m_i's are self-adjoint, by Theorem 3.5, Λ is an $(1_{\mathfrak{A}}, m)$-approximate g-dual of Γ.

(ii) The result follows from part (i) and Remark 3.6. □

As we know if Λ is a standard g-frame with lower bound A_Λ, then $\frac{1}{A_\Lambda}$ is an upper bound for $\widetilde{\Lambda}$.

Corollary 4.8. (i) *Let Λ be a standard g-frame. Suppose that $0 \leq \lambda < \sqrt{\frac{A_\Lambda}{B_\Lambda}}$ and m is a symbol such that $(1-\lambda)1_{\mathfrak{A}} \leq m_i \leq (1+\lambda)1_{\mathfrak{A}}$ for each $i \in I$. Then Λ (resp. $\widetilde{\Lambda}$) is an $(1_{\mathfrak{A}}, m)$-approximate g-dual of $\widetilde{\Lambda}$ (resp. Λ).*

(ii) *Let $\mathcal{F}$ be a standard frame. Suppose that $0 \leq \lambda < \sqrt{\frac{A_{\mathcal{F}}}{B_{\mathcal{F}}}}$ and m is a symbol such that $(1-\lambda)1_{\mathfrak{A}} \leq m_i \leq (1+\lambda)1_{\mathfrak{A}}$ for each $i \in I$. Then $\mathcal{F}$ (resp. $\widetilde{\mathcal{F}}$) is an $(1_{\mathfrak{A}}, m)$-approximate dual of $\widetilde{\mathcal{F}}$ (resp. $\mathcal{F}$).*

Proposition 4.9. *Let Γ be a g-dual of Λ. If $\Psi = \{\psi_i\}_{i\in I}$ is a g-Bessel sequence such that there exists $\lambda \in [0, \frac{1}{B_\Lambda})$ with*

$$\left\| \sum_{i\in I} \langle (m_i\psi_i - \Gamma_i)x, (m_i\psi_i - \Gamma_i)x\rangle \right\| \leq \lambda \|x\|^2$$

for each $x \in E$, then Λ (resp. Ψ) is an $(1_{\mathfrak{A}}, m)$ (resp. $(1_{\mathfrak{A}}, m^*)$-)approximate g-dual of Ψ (resp. Λ).

Proof. For each $x \in E$, we have

$$\left\| M_{m,\Lambda,\Psi}x - x \right\| = \left\| \sum_{i\in I} \Lambda_i^*(m_i\psi_i x) - \sum_{i\in I} \Lambda_i^* \Gamma_i x \right\| \leq \sqrt{B_\Lambda \lambda}\|x\|.$$

Thus $\|M_{m,\Lambda,\Psi} - Id_E\| < 1$. This means that Λ is an $(1_{\mathfrak{A}}, m)$-approximate g-dual of Ψ and consequently Ψ is an $(1_{\mathfrak{A}}, m^*)$-approximate g-dual of Λ. □

Corollary 4.10. *Let $\mathcal{G}$ be a dual of $\mathcal{F}$. If $\mathcal{N} = \{h_i\}_{i\in I}$ is a Bessel sequence such that there exists $\lambda \in [0, \frac{1}{B_{\mathcal{F}}})$ with $\left\| \sum_{i\in I} |\langle x, m_i h_i - g_i\rangle|^2 \right\| \leq \lambda \|x\|^2$ for each $x \in E$, then $\mathcal{N}$ (resp. $\mathcal{F}$) is an $(1_{\mathfrak{A}}, m)$ (resp. $(1_{\mathfrak{A}}, m^*)$-)approximate dual of $\mathcal{F}$ (resp. $\mathcal{N}$).*

Corollary 4.11. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ be a g-frame and let $m = \{m_i\}_{i\in I} \subseteq \mathbb{C}$.*

(i) *If* $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ *is a g-dual of* Λ, $0 \leq \lambda < \frac{1}{\sqrt{B_\Lambda B_\Gamma}}$ *and* $(1-\lambda) \leq m_i \leq (1+\lambda)$ *for each* $i \in I$, *then* $M_{m,\Lambda,\Gamma}$ *and* $M_{m,\Gamma,\Lambda}$ *are invertible with*

$$\frac{1}{1+\lambda\sqrt{B_\Lambda B_\Gamma}}\|x\| \leq \|Mx\| \leq \frac{1}{1-\lambda\sqrt{B_\Lambda B_\Gamma}}\|x\|$$

for each $x \in H$, *where* M *is* $M_{m,\Lambda,\Gamma}$ *or* $M_{m,\Gamma,\Lambda}$.

(ii) *If* $0 \leq \lambda < \sqrt{\frac{A_\Lambda}{B_\Lambda}}$ *and* $(1-\lambda) \leq m_i \leq (1+\lambda)$ *for each* $i \in I$, *then* $M_{m,\Lambda,\widetilde{\Lambda}}$ *and* $M_{m,\widetilde{\Lambda},\Lambda}$ *are invertible with*

$$\frac{1}{1+\lambda\sqrt{\frac{B_\Lambda}{A_\Lambda}}}\|x\| \leq \|Mx\| \leq \frac{1}{1-\lambda\sqrt{\frac{B_\Lambda}{A_\Lambda}}}$$

for each $x \in H$, *where* M *is* $M_{m,\Lambda,\widetilde{\Lambda}}$ *or* $M_{m,\widetilde{\Lambda},\Lambda}$.

(iii) *Let* Γ *be a g-dual of* Λ. *If* $\Psi = \{\psi_i \in L(H, H_i) : i \in I\}$ *is a g-Bessel sequence such that there exists* $\lambda \in [0, \frac{1}{B_\Lambda})$ *with*

$$\sum_{i\in I} |(m_i\psi_i - \Gamma_i)x|^2 \leq \lambda\|x\|^2$$

for each $x \in H$, *then* $M_{m,\Lambda,\Psi}$ *and* $M_{\overline{m},\Psi,\Lambda}$ *are invertible with*

$$\frac{1}{1+\sqrt{\lambda B_\Lambda}}\|x\| \leq \|Mx\| \leq \frac{1}{1-\sqrt{\lambda B_\Lambda}}\|x\|$$

for each $x \in H$, *where* M *is* $M_{m,\Lambda,\Psi}$ *or* $M_{\overline{m},\Psi,\Lambda}$.

Proof. (i) The result follows from Proposition 4.7. The relation

$$\frac{1}{1+\lambda\sqrt{B_\Lambda B_\Gamma}}\|x\| \leq \|Mx\| \leq \frac{1}{1-\lambda\sqrt{B_\Lambda B_\Gamma}}\|x\|,$$

can be obtained similar to the proof of Proposition 4.6 in [35].

(ii) The result is obtained from (i) using $\Gamma = \widetilde{\Lambda}$ and $B_\Gamma = \frac{1}{A_\Lambda}$.

(iii) We can get the result from Proposition 4.9 and a similar proof as Proposition 4.10 in [35]. □

Remark 4.12. Note that in the definition of (a, m)-approximate duals, we have assumed that $\mathcal{F}$ and $\mathcal{G}$ are Bessel sequences. But (as we see in [36] for Hilbert spaces) $M_{m,\mathcal{G},\mathcal{F}}$ can be well-defined and adjointable, although $\mathcal{F}$ or $\mathcal{G}$ is not a Bessel sequence, so the definition of (a, m)-approximate duals can be presented for general sequences. We recall from [34] that for a measure space (X, μ), the pair of mappings (Ψ, Φ), where $\Psi, \Phi : X \longrightarrow H$ are weakly measurable (H is a Hilbert space) is called a reproducing pair for H if the operator $S_{\Psi,\Phi} : H \longrightarrow H$ weakly defined by $S_{\Psi,\Phi}f = \int_X \langle f, \Psi(x)\rangle\Phi(x)d\mu(x)$ is an element of $GL(H)$ ($GL(H)$ is the set of bounded operators with bounded inverse on H). Now using the operator $S_{\Psi,\Phi}$ instead of $M_{m,\Gamma,\Lambda}$ in the definition of (a, m)-approximate duals, we can generalize the concept of (a, m)-approximate duality for reproducing pairs. We mention that some sequences related to frame theory have been classified in [7]. Similar to the above conclusions, we can generalize the notion of (a, m)-approximate duality to these sequences.

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Morteza Mirzaee Azandaryani
Department of Mathematics
University of Qom
Qom, Iran
E-mail address: morteza_ma62@yahoo.com; m.mirzaee@qom.ac.ir