Honam Mathematical J. **39** (2017), No. 2, pp. 175–185
https://doi.org/10.5831/HMJ.2017.39.2.175

FREDHOLM TOEPLITZ OPERATORS ON THE PLURIHARMONIC DIRICHLET SPACE

Young Joo Lee*

Abstract. In this paper we consider Toeplitz operators on the pluriharmonic Dirichlet space of the unit ball in the n-dimensional complex space. We then characterize Fredholm Toeplitz operators and describe the essential spectrum of a Toeplitz operator as a consequence.

1. Introduction

Let B be the unit ball in the complex n-space $\mathbb{C}^n$ and V be the Lebesgue volume measure on $\mathbb{C}^n$ normalized so that $V(B) = 1$. The Sobolev space $\mathscr{S}$ is the completion of the space of all smooth functions f on B for which

$$\|f\| = \left\{ \left| \int_B f\, dV \right|^2 + \int_B \left(|\mathcal{R}f|^2 + |\widetilde{\mathcal{R}}f|^2 \right) dV \right\}^{1/2} < \infty$$

where

$$\mathcal{R}f(z) = \sum_{i=1}^n z_i \frac{\partial f}{\partial z_i}(z), \quad \widetilde{\mathcal{R}}f(z) = \sum_{i=1}^n \overline{z_i} \frac{\partial f}{\partial \overline{z_i}}(z)$$

for $z = (z_1, \cdots, z_n) \in B$. Then the Sobolev space $\mathscr{S}$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_B f\, dV \int_B \bar{g}\, dV + \int_B \left(\mathcal{R}f\overline{\mathcal{R}g} + \widetilde{\mathcal{R}}f\overline{\widetilde{\mathcal{R}}g} \right) dV.$$

The pluriharmonic Dirichlet space $\mathscr{D}_{ph}$ is a subspace of $\mathscr{S}$ consisting of all pluriharmonic functions on B. Recall that a twice continuously

Received November 21, 2016. Accepted April 6, 2017.
2010 Mathematics Subject Classification. 47B35, 32A37.
Key words and phrases. Toeplitz operator, pluriharmonic Dirichlet space, Fredholm operator.
*Corresponding author

differentiable function u on B is said to be pluriharmonic if the one-variable function $\lambda \mapsto u(a+\lambda b)$, defined for $\lambda \in \mathbb{C}$ such that $a+\lambda b \in B$, is harmonic for each $a \in B$ and $b \in \mathbb{C}^n$. Then one can see that the pluriharmonic Dirichlet space $\mathscr{D}_{ph}$ is closed in $\mathscr{S}$. We let Q be the Hilbert space orthogonal projection from $\mathscr{S}$ onto $\mathscr{D}_{ph}$. Put

$$\mathscr{L}^{1,\infty} = \left\{ \varphi \in \mathscr{S} \ : \varphi, \frac{\partial \varphi}{\partial z_j}, \frac{\partial \varphi}{\partial \bar{z}_j} \in L^\infty, j = 1, \cdots, n \right\}$$

where the derivatives are taken in the sense of distribution. By Sobolev's embedding theorem([1, Theorem 5.4]), we can see that each function in $\mathscr{L}^{1,\infty}$ can be extended to a continuous function on the closed unit ball $\bar{B}$. We will use the same notation between a function in $\mathscr{L}^{1,\infty}$ and its continuous extension to $\bar{B}$. For $\varphi \in \mathscr{L}^{1,\infty}$, we note that $\mathcal{R}\varphi, \widetilde{\mathcal{R}}\varphi \in L^\infty$.

Given a function $u \in \mathscr{L}^{1,\infty}$, the *Toeplitz operator* T_u with symbol u is defined on $\mathscr{D}_{ph}$ by

$$T_u f = Q(uf)$$

for functions $f \in \mathscr{D}_{ph}$. Then T_u is a bounded linear operator on $\mathscr{D}_{ph}$; see Proposition 1 of Section 3.

We let $\mathcal{B}$ denote the C^*-algebra consisting of all bounded linear operators on $\mathscr{D}_{ph}$. Also, let $\mathcal{K}$ be the algebra of all compact operators on $\mathscr{D}_{ph}$. An operator $L \in \mathcal{B}$ is said to be Fredholm if $L + \mathcal{K}$ is invertible in the quotient algebra $\mathcal{B}/\mathcal{K}$. Note that $L \in \mathcal{B}$ is Fredholm if and only if there exist $L_1, L_2 \in \mathcal{B}$ such that $L_1 L - I, LL_2 - I \in \mathcal{K}$. See Chapter 6 of [3] for details.

In this paper we study the problem of when a Toeplitz operator is Fredholm. For Toeplitz operators acting on the Hardy space, Bergman space or holomorphic Dirichlet space, such problems have been well studied as in [4], [6], [2] and [5].

We in this paper continue to study the same problem for Toeplitz operators on the pluriharmonic Dirichlet space $\mathscr{D}_{ph}$. We consider general symbols in $\mathscr{L}^{1,\infty}$ and characterize Fredholm Toeplitz operators. The following is the our main theorem.

Main theorem. *Let $u \in \mathscr{L}^{1,\infty}$. Then T_u is Fredholm on $\mathscr{D}_{ph}$ if and only if u has no zero on the boundary of B.*

In Section 2, we collect some preliminary results. In Section 3, we prove the main theorem and describe the essential spectrum of a Toeplitz operator as an immediate consequence.

2. Preliminaries

The Dirichlet space $\mathscr{D}$ is a closed subspace of $\mathscr{S}$ consisting of all holomorphic functions in $\mathscr{S}$. We let P be the Hilbert space orthogonal projection from $\mathscr{S}$ onto $\mathscr{D}$. Each point evaluation is easily verified to be bounded linear functionals on both $\mathscr{D}$ and $\mathscr{D}_{ph}$. Hence, for each $z \in B$, there exist functions $K_z \in \mathscr{D}$ and $R_z \in \mathscr{D}_{ph}$ which have the following reproducing properties:

$$f(z) = \langle f, K_z \rangle, \qquad u(z) = \langle u, R_z \rangle$$

for functions $f \in \mathscr{D}$ and $u \in \mathscr{D}_{ph}$. As is well known, a real-valued function on B is pluriharmonic if and only if it is the real part of a holomorphic function on B. Hence every pluriharmonic function on B can be expressed, uniquely up to an additive constant, as a sum of a holomorphic function and an antiholomorphic function; see Chapter 4 of [7]. Using this fact, we can see that $\mathscr{D}_{ph} = \mathscr{D} + \overline{\mathscr{D}}$. Thus there is a useful relation between R_z and K_z:

$$R_z = K_z + \overline{K_z} - 1$$

Since $P\varphi = \langle \varphi, K_z \rangle$ for $z \in B$, the formula above leads us to the following useful connection between P and Q:

$$Q(\varphi) = P(\varphi) + \overline{P(\overline{\varphi})} - P(\varphi)(0) \tag{1}$$

for functions $\varphi \in \mathscr{S}$.

We let $L^2 = L^2(B, V)$ be the usual Lebesgue space and A^2 be the well known Bergman space consisting of all holomorphic functions in L^2. Let Φ be the Bergman projection which is the orthogonal projection from L^2 onto A^2 whose its explicit formula can be written as

$$\Phi\psi(z) = \int_B \psi(w)\overline{B_z(w)}\, dV(w), \qquad z \in B$$

for functions $\psi \in L^2$. Here B_z is the well known Bergman kernel given by

$$B_z(w) = \frac{1}{(1 - w \cdot \overline{z})^{n+1}}, \qquad w \in B$$

where $w \cdot \bar{z} = w_1\overline{z}_1 + \cdots + w_n\overline{z}_n$ is the Hermitian inner product for points $z, w \in \mathbb{C}^n$. For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ where each α_k is a nonnegative integer, we will write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. We will also write

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

for $z = (z_1, \cdots, z_n) \in B$. Note that $\mathcal{R} z^\alpha = |\alpha| z^\alpha$ for every multi-index α. Since

$$B_z(w) = \sum_{|\alpha| \geq 0} \frac{(n+|\alpha|)!}{n!\alpha!} \overline{z^\alpha} w^\alpha, \qquad z, w \in B,$$

we have

$$\Phi\psi(z) = \sum_{|\alpha| \geq 0} \frac{(n+|\alpha|)!}{n!\alpha!} z^\alpha \int_B \overline{w^\alpha} \psi(w)\, dV(w), \qquad z \in B \tag{2}$$

for functions $\psi \in L^2$; see Chapter 2 of [8] for details and related facts.

Using Lemma 1.11 of [8], we see that

$$||z^\alpha||^2 = |\alpha|^2 \int_B |z^\alpha|^2 \, dV(z) = \frac{n!|\alpha|^2 \alpha!}{(n+|\alpha|)!}$$

for each multi-index α with $|\alpha| > 0$. Note that the set $\{z^\alpha : |\alpha| \geq 0\}$ spans a dense subset of $\mathscr{D}$. Thus it can be easily seen that the kernel function K_z on $\mathscr{D}$ has the following explicit formula

$$K_z(w) = 1 + \sum_{|\alpha| > 0} \frac{(n+|\alpha|)!}{n!|\alpha|^2 \alpha!} \overline{z^\alpha} w^\alpha$$

for $z, w \in B$. Since $K_z(0) = 1$ for all $z \in B$, it follows from the reproducing property that

$$P\psi(z) = \int_B \psi \, dV + \sum_{|\alpha| > 0} \frac{(n+|\alpha|)!}{n!|\alpha|\alpha!} z^\alpha \int_B \overline{w^\alpha} \mathcal{R}\psi(w)\, dV(w) \tag{3}$$

for functions $\psi \in \mathscr{S}$ and points $z \in B$. Combining the above with (2), we can see

$$\begin{aligned} \mathcal{R}(P\psi)(z) &= \sum_{|\alpha| > 0} \frac{(n+|\alpha|)!}{n!\alpha!} z^\alpha \int_B \overline{w^\alpha} \mathcal{R}\psi(w)\, dV(w) \\ &= \Phi\left(\mathcal{R}\psi\right)(z) - \Phi\left(\mathcal{R}\psi\right)(0), \qquad z \in B \end{aligned} \tag{4}$$

for functions $\psi \in \mathscr{S}$.

In the following, we use the notation

$$||f||_2 = \left(\int_B |f|^2 \, dV \right)^{\frac{1}{2}}$$

for functions $f \in L^2$. Note that $|f(0)| \leq ||f||_2$ for all $f \in A^2$. Also, one can see that

$$\|f\|_2 \leq \|\mathcal{R} f\|_2 \leq \|f\| \tag{5}$$

for every $f \in \mathscr{D}$. See Chapter 2 of [8] for details.

We begin with the boundedness of Toeplitz operators.

Proposition 1. *For $u \in \mathscr{L}^{1,\infty}$, the Toeplitz operator T_u is bounded on $\mathscr{D}_{ph}$.*

Proof. Let $\varphi \in \mathscr{D}_{ph}$ be arbitrary and write $\varphi = f+\bar{g}$ for some $f, g \in \mathscr{D}$ with $f(0) = 0$. Note $u\varphi \in \mathscr{S}$ and $||\varphi||^2 = ||f||^2 + ||g||^2$. Hence, by (5) we see

$$\|\varphi\|_2 \leq ||f||_2 + ||g||_2 \leq ||f|| + ||g|| \leq 2\|\varphi\|. \tag{6}$$

Hence, by (3),

$$|P(\bar{u}\bar{\varphi})(0)| = \left|\int_B u\varphi \, dV\right| \leq ||u||_\infty ||\varphi||_2 \leq 2||u||_\infty ||\varphi||. \tag{7}$$

Also, by (4) and the L^2-boundedness of the Bergman projection Φ, we see

$$\begin{aligned} ||P(u\varphi) - P(u\varphi)(0)|| &= \|\mathcal{R}(P(u\varphi))\|_2 \\ &\leq \|\Phi(\mathcal{R}(u\varphi))\|_2 + |\Phi\left(\mathcal{R}(u\varphi)\right)(0)| \\ &\leq 2\|\Phi(\mathcal{R}(u\varphi))\|_2 \\ &\leq 2\|\mathcal{R}(u\varphi)\|_2 \\ &= 2\|\varphi\mathcal{R}u + u\mathcal{R}\varphi\|_2 \\ &\leq 4\left(\|\mathcal{R}u\|_\infty + \|u\|_\infty\right)\|\varphi\| \end{aligned} \tag{8}$$

and similarly

$$\|\mathcal{R}(P(\bar{u}\bar{\varphi}))\|_2 \leq 4\left(\|\mathcal{R}\bar{u}\|_\infty + \|u\|_\infty\right)\|\varphi\|.$$

Combining the above with (7), one obtains

$$||P(\bar{u}\bar{\varphi})||^2 = |P(\bar{u}\bar{\varphi})(0)|^2 + \|\mathcal{R}(P(\bar{u}\bar{\varphi}))\|_2^2 \leq C_1||\varphi||^2$$

for some constant C_1 depending only on u. It follows from (1) and (8) that

$$\begin{aligned} \|T_u\varphi\|^2 &= \|Q(u\varphi)\|^2 \\ &= ||P(u\varphi) - P(u\varphi)(0)||^2 + ||P(\bar{u}\bar{\varphi})||^2 \\ &\leq C\|\varphi\|^2 \end{aligned}$$

for some constant C depending only on u, which implies the boundedness of T_u as desired. The proof is complete. $\square$

3. Fredholm Toeplitz operators

In this section, we prove the main theorem and compute the essential spectrum of a Toeplitz operator as an immediate consequence.

We let $\mathscr{D}_0$ be the space of all $f \in \mathscr{D}$ such that $f(0) = 0$. Note that $\mathscr{D}_{ph} = \mathscr{D}_0 \oplus \overline{\mathscr{D}}$.

Proposition 2. *If a sequence $u_j = f_j + \overline{g_j} \in \mathscr{D}_0 + \overline{\mathscr{D}}$ converges to 0 weakly in $\mathscr{D}_{ph}$, then f_j and g_j converge to 0 weakly in $\mathscr{D}$. Also, if a sequence $h_j \in \mathscr{D}$ converges to 0 weakly in $\mathscr{D}$, then h_j and $\bar{h}_j$ converge to 0 weakly in $\mathscr{D}_{ph}$.*

Proof. Let $\varphi \in \mathscr{D}$. Since $f_j(0) = 0$, we first have $\overline{g_j(0)} = u_j(0) = \langle u_j, 1\rangle$ and hence

$$\langle f_j, \varphi\rangle = \langle u_j - \overline{g_j}, \varphi\rangle = \langle u_j, \varphi\rangle - \langle u_j, 1\rangle\, \overline{\varphi(0)}$$

for each j. So, if $u_j \to 0$ weakly in $\mathscr{D}_{ph}$, we have $\langle u_j, \varphi\rangle$ and $\langle u_j, 1\rangle$ converge to 0 as $n \to \infty$. Hence f_j converges to 0 weakly in $\mathscr{D}$. Similarly, since

$$\langle g_j, \varphi\rangle = \langle \overline{u_j} - \overline{f_j}, \varphi\rangle = \langle \overline{u_j}, \varphi\rangle - \overline{f_j}(0)\overline{\varphi(0)} = \overline{\langle u_j, \bar{\varphi}\rangle} \to 0$$

as $j \to \infty$, we see g_j converges to 0 weakly in $\mathscr{D}$.

To prove the remaining part, let $a + \bar{b} \in \mathscr{D}_{ph} = \mathscr{D}_0 + \overline{\mathscr{D}}$. Then

$$\langle h_j, a + \bar{b}\rangle = \langle h_j, a\rangle + h_j(0)b(0) = \langle h_j, a\rangle + \langle h_j, 1\rangle b(0)$$

for each j. Note $a, 1 \in \mathscr{D}$. So, if h_j converges to 0 weakly in $\mathscr{D}$, we see $\langle h_j, a + \bar{b}\rangle \to 0$ as $j \to \infty$. Hence $h_j \to 0$ weakly and then $\overline{h_j} \to 0$ weakly in $\mathscr{D}_{ph}$. This completes the proof. □

We let b^2 be the pluriharmonic Bergman space consisting of all pluriharmonic functions in L^2. By (6), we see that the identity operator from $\mathscr{D}_{ph}$ into b^2 is bounded. The following lemma shows that it is in fact compact. Recall that the identity operator from $\mathscr{D}$ into A^2 is compact; see [5] for example.

Lemma 3. *The identity operator from $\mathscr{D}_{ph}$ into b^2 is compact.*

Proof. Let u_j be a sequence converging weakly to 0 in $\mathscr{D}_{ph}$ and write $u_j = f_j + \overline{g_j} \in \mathscr{D}_0 + \overline{\mathscr{D}}$. To prove the result, we need to show that $||u_j||_2 \to 0$ as $j \to \infty$. By Proposition 2, f_n and g_n converge weakly to 0 in $\mathscr{D}$. Since the identity operator from $\mathscr{D}$ into A^2 is compact as mentioned before, we see that $||f_j||_2$ and $||g_j||_2$ converge to 0 as $j \to \infty$. It follows that

$$||u_j||_2 \le ||f_j||_2 + ||g_j||_2 \to 0$$

as $j \to \infty$. The proof is complete. $\square$

Given $u \in \mathscr{L}^{1,\infty}$, the *(little) Hankel operator* $h_u : \mathscr{D} \to \overline{\mathscr{D}}$ with symbol u is defined by

$$h_u(f) = \overline{P(u\bar{f})}$$

for functions $f \in \mathscr{D}$.

The following shows that the Hankel operator is always compact.

Proposition 4. *For $u \in \mathscr{L}^{1,\infty}$, the Hankel operator h_u is compact on $\mathscr{D}$.*

Proof. Let f_j be a sequence converging weakly to 0 on $\mathscr{D}$ as $j \to \infty$. Since the identity operator from $\mathscr{D}$ into A^2 is compact, we have

$$\lim_{j\to\infty} \int_B |f_j|^2 \, dV = 0. \tag{9}$$

Note that

$$|P(u\bar{f}_j)(0)| = \left| \int_B u\bar{f}_j \, dV \right| \le ||u||_\infty ||f_j||_2$$

for each j. It follows from (4) and the L^2-boundedness of the Bergman projection Φ that

$$\begin{aligned} ||h_u f_j||^2 &= ||P(u\bar{f}_j)||^2 \\ &= |P(u\bar{f}_j)(0)|^2 + ||\mathcal{R}[P(u\bar{f}_j)]||_2^2 \\ &= ||u||_\infty^2 ||f_j||_2^2 + ||\Phi[(\mathcal{R}u)\bar{f}_j] - \Phi[(\mathcal{R}u)\bar{f}_j](0)||_2^2 \\ &\le ||u||_\infty^2 ||f_j||_2^2 + 4||\Phi[(\mathcal{R}u)\bar{f}_j]||_2^2 \\ &\le ||u||_\infty^2 ||f_j||_2^2 + 4||(\mathcal{R}u)\bar{f}_j||_2^2 \\ &\le ||u||_\infty^2 ||f_j||_2^2 + 4||\mathcal{R}u||_\infty^2 ||f_j||_2^2 \end{aligned}$$

for each j. It follows that

$$||h_u f_j||^2 \le \left(||u||_\infty^2 + 4||\mathcal{R}u||_\infty^2 \right) \int_B |f_j|^2 \, dV$$

for each j. Combining the above with (9), we see $||h_u f_j|| \to 0$ as $j \to \infty$, which implies the compactness of h_u as desired. The proof is complete. $\square$

Given $u \in \mathscr{L}^{1,\infty}$, the *(Dirichlet space) Toeplitz operator* t_u with symbol u is defined on $\mathscr{D}$ by $t_u f = P(uf)$ for functions $f \in \mathscr{D}$. Then t_u is a bounded linear operator on $\mathscr{D}$; see [5] for example.

The following lemma shows that there are useful relations between Toeplitz operators and Hankel operators. The notation L^* denotes the adjoint operator of a bounded operator L.

Lemma 5. *For $u \in \mathscr{L}^{1,\infty}$, the following statements hold for every $f \in \mathscr{D}$.*

(a) $||T_u f||^2 = ||t_u f||^2 - |\langle f, t_u^* 1\rangle|^2 + ||h_{\bar{u}} f||^2$.
(b) $||T_u \bar{f}||^2 = ||h_u f||^2 + ||t_{\bar{u}} f||^2 - |\langle f, t_{\bar{u}}^* 1\rangle|^2$.
(c) $||T_u^* f||^2 = ||t_u^* f||^2 - |\langle f, t_u 1\rangle|^2 + ||h_u^* \bar{f}||^2$.

Proof. Given $F + \bar{G} \in \mathscr{D}_{ph} = \mathscr{D}_0 + \overline{\mathscr{D}}$, we first note that

$$||F + \bar{G}||^2 = ||F||^2 + ||G||^2$$

and $||G - G(0)||^2 = ||G||^2 - |G(0)|^2$. Fix a function $f \in \mathscr{D}$. Since

$$T_u f = P(uf) - P(uf)(0) + \overline{P(\overline{uf})} = t_u f - t_u f(0) + h_{\bar{u}} f$$

by (1), it follows that

$$||T_u f||^2 = ||t_u f||^2 - |t_u f(0)|^2 + ||h_{\bar{u}} f||^2.$$

Now, (a) follows from the fact that

$$t_u f(0) = \langle t_u f, 1\rangle = \langle f, t_u^* 1\rangle.$$

By the similar argument, we can prove (b). To prove (c), we first note that

$$\langle T_u^* f, a + \bar{b}\rangle = \langle t_u^* f, a\rangle + \langle b, h_u^* \overline{f}\rangle$$

for every $a + \bar{b} \in \mathscr{D}_0 + \overline{\mathscr{D}}$. It follows that

$$\begin{aligned} T_u^* f(z) &= \langle T_u^* f, R_z\rangle \\ &= \langle T_u^* f, K_z - 1 + \overline{K_z}\rangle \\ &= \langle t_u^* f, K_z - 1\rangle + \langle K_z, h_u^* \overline{f}\rangle \\ &= t_u^* f(z) - t_u^* f(0) + \overline{h_u^*(\bar{f})(z)} \end{aligned}$$

for every $z \in B$. Then, (c) follows from the similar argument as in the proof of (a). This completes the proof. □

For $u \in \mathscr{L}^{1,\infty}$, we note from the reproducing property

$$P(uF)(0) = \langle t_u F, K_0\rangle = \langle t_u F, 1\rangle = \langle F, t_u^* 1\rangle$$

and hence

$$P(u\bar{F})(0) = \overline{P(\bar{u}F)(0)} = \overline{\langle F, t_{\bar{u}}^* 1\rangle}$$

for every $F \in \mathscr{D}$. Thus, by (1), we have

$$\begin{aligned} T_u\varphi &= P(uf) + \overline{P(\bar{u}g)} + h_{\bar{u}}f + \overline{h_u g} - \langle f, t_u^* 1\rangle - \overline{\langle g, t_{\bar{u}}^* 1\rangle} \\ &= t_u f + \overline{t_{\bar{u}} g} + h_{\bar{u}}f + \overline{h_u g} - \langle f, t_u^* 1\rangle - \overline{\langle g, t_{\bar{u}}^* 1\rangle} \\ &= A_u\varphi + B_u\varphi + C_u\varphi \end{aligned}$$

for functions $\varphi = f + \bar{g} \in \mathscr{D}_0 \oplus \overline{\mathscr{D}}$. Here $A_u, B_u, C_u : \mathscr{D}_0 \oplus \overline{\mathscr{D}} \to \mathscr{D}_{ph}$ are bounded linear operators defined by

$$\begin{aligned} A_u(f + \bar{g}) &= t_u f + \overline{t_{\bar{u}} g} \\ B_u(f + \bar{g}) &= h_{\bar{u}} f + \overline{h_u g} \\ C_u(f + \bar{g}) &= -\langle f, t_u^* 1\rangle - \overline{\langle g, t_{\bar{u}}^* 1\rangle} \end{aligned}$$

respectively. Thus we have the following decomposition for T_u:

$$T_u = A_u + B_u + C_u. \tag{10}$$

The following lemma shows that the Fredholm properties of T_u and A_u are equivalent.

Lemma 6. *Let $u \in \mathscr{L}^{1,\infty}$. Then T_u is Fredholm on $\mathscr{D}_{ph}$ if and only if A_u is Fredholm on $\mathscr{D}_{ph}$.*

Proof. First we prove that the operators B_u and C_u are compact on $\mathscr{D}_{ph}$. To do this, let $\varphi_j = f_j + \overline{g_j} \in \mathscr{D}_0 + \overline{\mathscr{D}}$ be a sequence converging weakly to 0 on $\mathscr{D}_{ph}$. Then, by Proposition 2, f_j and g_j converge weakly to 0 on $\mathscr{D}$. Since the operators h_u and $h_{\bar{u}}$ are compact by Lemma 4, we have $||h_{\bar{u}} f_j|| \to 0$ and $||h_u g_j|| \to 0$ as $n \to \infty$. Thus, the operator B_u is compact on $\mathscr{D}_{ph}$. Also, the compactness of C_u follows from it's definition. Now, decomposition (10) gives the desired result. The proof is complete. □

We say that $L \in \mathcal{B}$ is left Fredholm if there exists $L_1 \in \mathcal{B}$ such that $L_1 L - I \in \mathcal{K}$. Also, $L \in \mathcal{B}$ is called right Fredholm if there exists $L_2 \in \mathcal{B}$ such that $LL_2 - I \in \mathcal{K}$. Thus, L is Fredholm if and only if L is left and right Fredholm. Also, it is known that L is not left(resp. right) Fredholm if and only if there exists a sequence $\{f_j\}$ of unit vectors for which $f_j \to 0$ weakly and $||Lf_j||$(resp. $||L^* f_j||$) converges to 0 as $j \to \infty$; see Chapter 6 of [3] for details and related facts.

Now we prove the our main theorem. In the course of the proof of the main theorem, we will use a characterization of Fredholm Toeplitz operators on the Dirichlet space $\mathscr{D}$. Given $u \in \mathscr{L}^{1,\infty}$, it turns out that

t_u is Fredholm on $\mathscr{D}$ if and only if u has no zero on ∂B, the boundary of B; see [5] for detail.

Proof of the main theorem. First assume T_u is Fredholm on $\mathscr{D}_{ph}$ and suppose u has a zero on ∂B. Then t_u is not Fredholm on $\mathscr{D}$. Suppose t_u is not left Fredholm on $\mathscr{D}$. Then, there is a sequence $\{f_j\} \in \mathscr{D}$ of unit vectors converging weakly to 0 on $\mathscr{D}$ for which $||t_u f_j|| \to 0$ as $j \to \infty$. Note $||h_{\bar{u}} f_j|| \to 0$ as $j \to \infty$ by Proposition 4. Since $\{f_j\}$ converges weakly to 0 in $\mathscr{D}$, it follows from Lemma 5(a) that

$$||T_u f_j||^2 = ||t_u f_j||^2 - |\langle f_j, t_u^* 1\rangle|^2 + ||h_{\bar{u}} f_j||^2 \to 0$$

as $j \to \infty$. Since $\{f_j\}$ converges weakly to 0 in $\mathscr{D}_{ph}$ by Proposition 2, we see T_u is not left Fredholm on $\mathscr{D}_h$, which is a contradiction. Now suppose t_u is not right Fredholm on $\mathscr{D}$. As before, there is a sequence $\{g_j\} \in \mathscr{D}$ of unit vectors converging weakly to 0 on $\mathscr{D}$ for which $||t_u^* g_j|| \to 0$ as $n \to \infty$. Note $\{\overline{g_j}\}$ converges weakly to 0 in $\bar{\mathscr{D}}$. By Lemma 5(c) and Lemma 4, we have

$$||T_u^* g_j||^2 = ||t_u^* g_j||^2 - |\langle g_j, t_u 1\rangle|^2 + ||h_u^* \bar{g}_j||^2 \to 0$$

as $j \to \infty$. Since $\{g_j\}$ also converges weakly to 0 in $\mathscr{D}_{ph}$ by Proposition 2, we see T_u is not right Fredholm on $\mathscr{D}_{ph}$, which is also a contradiction. Thus u has no zero on ∂B.

Conversely, suppose u has no zero on ∂B and hence t_u is Fredholm on $\mathscr{D}$. Since t_u is left Fredholm in particular, there exists a bounded linear operator S_1 on $\mathscr{D}$ such that $S_1 t_u - I$ is compact on $\mathscr{D}$. Also, since $\bar{u}$ has no zero on ∂D, by the same reason, there exists a bounded linear operator S_2 on $\mathscr{D}$ such that $S_2 t_{\bar{u}} - I$ is compact on $\mathscr{D}$. With theses operators S_1 and S_2, let us define $T : \mathscr{D}_0 + \overline{\mathscr{D}} \to \mathscr{D}_{ph}$ by

$$T(f + \bar{g}) = S_1(f) + \overline{S_2(g)}$$

for functions $f + \bar{g} \in \mathscr{D}_0 + \overline{\mathscr{D}}$. Then, it is easy to check that T is well defined and linear. Also, using the boundedness of S_1, S_2 on $\mathscr{D}$, one can see that T is also bounded. Recall the operator A_u defined by

$$A_u(f + \bar{g}) = t_u f + \overline{t_{\bar{u}} g}$$

for functions $f + \bar{g} \in \mathscr{D}_0 + \overline{\mathscr{D}}$. Now, we show $TA_u - I$ is compact on $\mathscr{D}_{ph}$. Let $\varphi_j = f_j + \bar{g}_j \in \mathscr{D}_0 + \overline{\mathscr{D}}$ be a sequence converging weakly to 0 on $\mathscr{D}_{ph}$. By a simple manipulation, we can see

$$(TA_u - I)(\varphi_j) = [S_1 t_u - I](f_j) + \overline{[S_2 t_{\bar{u}} - I](g_j)} + t_u f_j(0)[\overline{S_2 1} - S_1 1]$$

for each j. By Proposition 2, we note that f_j and g_j converge weakly to 0 on $\mathscr{D}$. Since $S_1 t_u - I$ and $S_2 t_{\bar{u}} - I$ are compact on $\mathscr{D}$, we see that $[S_1 t_u - I](f_j)$ and $\overline{[S_2 t_{\bar{u}} - I](g_j)}$ converge to 0 in $\mathscr{D}$. Also, since $t_u f_j(0) = \langle f_j, t_u^* 1\rangle$ for each j, we have $t_u f_j(0) \to 0$ in $\mathscr{D}$. Hence A_u is left Fredholm on $\mathscr{D}_{ph}$. Also, by a similar argument, we can see that A_u is right Fredholm on $\mathscr{D}_{ph}$ and then A_u is Fredholm on $\mathscr{D}_h$. Now, by Lemma 6, we see T_u is Fredholm on $\mathscr{D}_{ph}$, as desired. The proof is complete. □

Recall that the essential spectrum $\sigma_e(L)$ of $L \in \mathcal{B}$ is defined to be the spectrum of $L + \mathcal{K}$ in $\mathcal{B}/\mathcal{K}$. As an immediate consequence of the main theorem, we describe the essential spectrum of a Toeplitz operator on the pluriharmonic Dirichlet space as shown in the following.

Corollary 7. *For $u \in \mathscr{L}^{1,\infty}$, we have $\sigma_e(T_u) = u(\partial B)$.*

References

[1] R. Adams, *Sobolev Spaces*, Academic press, New York, 1975.

[2] G. Cao, *Fredholm properties of Toeplitz operators on Dirichlet spaces,* Pacific J. Math. 188 (1999) 209-223.

[3] J. B. Conway, *A Course in Operator Theory,* Amer. Math. Soc. Providence, Rhode Island, 1999.

[4] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.

[5] Y. J. Lee, *Compact sums of Toeplitz products and Toeplitz algebra on the Dirichlet space*, Tohoku Math. J., 68 (2016) 253-271.

[6] G. McDonald, *Fredholm properties of a class of Toeplitz operators on the ball*, Indiana Univ. Math. J., 26 (1977) 567-576.

[7] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York, 1980.

[8] K. Zhu, *Space of Holomorphic Functions in the Unit Ball*, Springer-Verlag, New York, 2005.

Young Joo Lee
Department of Mathematics,
Chonnam National University,
Gwangju 61186, Korea.
E-mail:leeyj@chonnam.ac.kr