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TOEPLITZ OPERATORS ON HARDY AND BERGMAN SPACES OVER BOUNDED DOMAINS IN THE PLANE

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Abstract. In this paper, we show that algebraic properties of Toeplitz operators on both Bergman spaces and Hardy spaces on the unit disc essentially generalizes on arbitrary bounded domains in the plane. In particular, we obtain results for the uniqueness property and commuting problems of the Toeplitz operators on the Hardy and the Bergman spaces associated to bounded domains.

1. Introduction

Suppose that Ω is a bounded domain in the complex plane with C^∞ smooth boundary. For $\varphi \in L^\infty(b\Omega)$, the Toeplitz operator HT^Ω_φ with symbol φ on the Hardy space $H^2(b\Omega)$ is the bounded linear operator on $H^2(b\Omega)$ defined by

$$HT^\Omega_\varphi(f) = P_\Omega(\varphi f),\ f \in H^2(b\Omega)$$

where P_Ω is the Szegő projection associated to Ω. Toeplitz operators on Bergman spaces are similarly defined. For a bounded domain Ω(possibly without smoothness of the boundary) and for $\varphi \in L^\infty(\Omega)$, the Toeplitz operator BT^Ω_φ with symbol φ on the Bergman space $H^2(\Omega)$ is the operator on $H^2(\Omega)$ defined by

$$BT^\Omega_\varphi(f) = B_\Omega(\varphi f),\ f \in H^2(\Omega)$$

where B_Ω is the Bergman projection associated to Ω. Algebraic properties, for instance, commuting properties of the Toeplitz operators on both Hardy spaces and Bergman spaces have been studied mainly for the case of the unit disc until now. So it is natural to ask whether we can generalize the results proved in the case of the unit disc to general bounded domains. Simply connected bounded domains are conformally

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equivalent to the unit disc via the Riemann mapping function and similarly finitely connected bounded domains are proper holomorphically mapped onto the unit disc via the Ahlfors map. So in this direction of work, we need to study the boundary behavior of biholomorphic and proper holomorphic mappings.

This paper is outlined as follows. In §2 we introduce notations and notions used in the paper and list known results. In particular, we survey on important properties of the classical kernel functions and orthogonal projections to be used often in the paper. In addition we mention the (known) transformation formulas for the Bergman projections and the Szegő projections under proper holomorphic mappings between domains which are key elements for obtaining our results in this paper.

In §3 we find the transformation formulas for the Toeplitz operators under the Riemann map and the Ahlfors map which is based on the result of Bell in [Bel81] about the transformation formulas for the Bergman projection and the Bergman kernel under proper holomorphic maps. In §4 we study on uniqueness of Toeplitz operators. The first author proved in [Chu09] that the correspondence $\varphi \to HT_\varphi^\Omega$ is one-to-one for general domain Ω. We also prove that Toeplitz operators on Bergman spaces associated to general domains have uniqueness property.

A. Brown and P. R. Halmos[BH64] classified commuting Toeplitz operators on the Hardy space in the case of the unit disc. And S. Axler and Ž. Čučković[AČ91] also solved the commuting problem of two Toeplitz operators on the Bergman space with bounded harmonic symbols for the unit disc case. On the other hand, the first author in [Chua], [Chub], and [Chuc] obtained many results of algebraic properties of the Laurent and the Toeplitz operators on the Hardy spaces associated to general bounded domains in the plane by constructing an orthonormal basis for L^2-space. In §5 and §6, we study on commuting problems for Toeplitz operators on both Hardy spaces and Bergman spaces for general domains. S. Axler, Ž. Čučković and N. V. Rao[AČR00] proved a partial result for the commuting problem of Toeplitz operators with analytic symbols on Bergman spaces for general domains using an approximation theorem. We generalize the corresponding result for Toeplitz operators on Hardy spaces associated to general domains by using the orthogonal decomposition of L^2-functions.

2. Preliminaries and Notes

Suppose that Ω is a finitely connected bounded domain in the plane with C^∞ smooth boundary. The Cauchy integral formula says that for any homomorphic function f in a neighborhood of $\overline{\Omega}$ and for any point a in Ω, the value of f at a is represented by the boundary values of f via

$$f(a) = \frac{1}{2\pi i}\int_{b\Omega} \frac{f(z)}{z-a}dz. \tag{2.1}$$

If we introduce the classical L^2 inner product $< \, , \, >$ defined by

$$< u, v >= \int_{b\Omega} u\,\overline{v}\; ds,$$

where ds is the differential element of arc length on the boundary $b\Omega$, the integral formula (2.1) is equivalent to the identity

$$f(a) =< f, C_a >,$$

where $C_a(z) = \overline{\dfrac{1}{2\pi i}\dfrac{T(z)}{z-a}}$ is the Cauchy kernel and T is the unit tangent vector function on $b\Omega$ pointing in the direction of the standard orientation of $b\Omega$. This motivates to studying on the Hardy space of the boundary of Ω as follows.

Let $L^2(b\Omega)$ be the Hilbert space completion of $C^\infty(b\Omega)$ with respect to the inner product $< \, , \, >$ and let $H^2(b\Omega)$ denote the classical Hardy space associated to Ω which is the space of holomorphic functions on Ω with L^2-boundary values in $b\Omega$. Since $H^2(b\Omega)$ can be regarded as the completion of the restrictions of holomorphic functions in $C^\infty(\overline{\Omega})$ to $b\Omega$ in $L^2(b\Omega)$, it follows from the inequality $|f(a)| \leq \|f\|_{L^2(b\Omega)}\|C_a\|_{L^2(b\Omega)}$ that the evaluation function at $a \in \Omega$ is a continuous linear functional on $H^2(b\Omega)$. Thus, given $a \in \Omega$, we can apply the Riesz Representation Theorem to the linear functional on $H^2(b\Omega)$ to get a unique function $S_a \in H^2(b\Omega)$ such that for all $f \in H^2(b\Omega)$,

$$f(a) =< f, S_a >= \int_{b\Omega} f\,\overline{S_a}\; ds.$$

On the other hand, since $H^2(b\Omega)$ is a closed subspace of $L^2(b\Omega)$, there exists the orthogonal projection of $L^2(b\Omega)$ onto $H^2(b\Omega)$ called the Szegő projection which is denoted by

$$P_\Omega : L^2(b\Omega) \to H^2(b\Omega).$$

Since for all $f \in H^2(b\Omega)$,

$$< f, S_a >= f(a) =< f, C_a >=< f, P_\Omega(C_a) >$$

and $P_\Omega(C_a) \in H^2(b\Omega)$, the uniqueness property for the function S_a implies that

$$P_\Omega(C_a) = S_a$$

and we call S_a the Szegő kernel for the the Szegő projection P_Ω and S_a is denoted by $S_a(z) = S(z, a)$ when it is considered as a function of two varibales z and a.

It is well known (see [Bel90b], [Bel91a]) that any $u \in L^2(b\Omega)$ has an orthogonal decomposition as a direct sum of the Hardy space $H^2(b\Omega)$ and the orthogonal complement $H^2(b\Omega)^\perp$ of the Hardy space via

$$u = P_\Omega(u) + \overline{T}\ \overline{P_\Omega(\overline{uT}\,)}. \tag{2.2}$$

There is also a special kernel function which is the kernel for the orthogonal projection $P_\Omega^\perp$ of the Szegő projection P_Ω in some sense. The Garabedian kernel function $L(z, a)$ is defined by

(2.3)
$$L(z, a) = \frac{1}{2\pi(z-a)} + P_\Omega\left(\overline{iC_aT}\right)(z) = \frac{1}{2\pi(z-a)} + < \overline{iC_aT}, S_z > .$$

It is easy to see from (2.3) that for fixed $a \in \Omega$, $L(z, a)$ is a meromorphic function on Ω with a single simple pole at $z = a$ having residue $\frac{1}{2\pi}$ which extends C^∞ smoothly up to the boundary of Ω. It is also known(see [Bel90b]) that $L(z, a)$ never vanishes for all $(z, a) \in \overline{\Omega} \times \Omega$ with $z \neq a$. An important property about the Szegő kernel and the Garabedian kernel to which we often refer in this paper is

$$L(z, a) = i\ \overline{S(z, a)}\ \overline{T(z)}, \quad (z, a) \in \mathrm{b}\Omega \times \Omega. \tag{2.4}$$

It is very interesting to see that when Ω is simply connected, given $a \in \Omega$, the quotient map

$$f_a(z) = \frac{S(z, a)}{L(z, a)}$$

is the Riemann mapping function associated to the pair (Ω, a) which is a biholomorphic mapping of Ω onto the unit disc with $f_a(a) = 0$ and $f_a'(a) > 0$, having the extremal property as follows: the function f_a maximizes $h'(a)$ among all holomorphic functions h mapping Ω into the unit disc making $h'(a)$ real valued (see [Gar49]).

It is natural to ask whether we can do the same thing for the case of a finitely connected domain. There is a kind of generalization of the Riemann mapping function to a finitely connected domain which is called the Ahlfors map. For a finitely connected n-connected domain Ω in the plane with C^∞ smooth boundary and given $a \in \Omega$, the Ahlfors map f_a (we use the same notation as the Riemann mapping function for convenience) associated to the pair (Ω, a) is the unique solution to the extremal problem: among all holomorphic functions h mapping Ω into the unit disc, find the one making $h'(a)$ real-valued and as large as possible. It is well known (see [Gar49], [Bel91b], [Bel99]) that the function f_a is an n-to one proper holomorphic covering map of Ω onto the unit disc and is equal to the quotient

$$f_a(z) = \frac{S(z,a)}{L(z,a)} \tag{2.5}$$

of the Szeő kernel and Garabedian kernel functions.

Like the Hardy space, there is the space of holomorphic functions on Ω which are square integrable on Ω with respect to area measure dA which is called the Bergman space and is denoted by $H^2(\Omega)$. And since $H^2(\Omega)$ is a closed subspace of the Hilbert space $L^2(\Omega)$ with the inner product $\langle u, v\rangle_\Omega = \iint_\Omega u\bar{v}\, dA$, there exists the orthogonal projection B_Ω of $L^2(\Omega)$ onto $H^2(\Omega)$ called the Bergman projection. Furthermore given $w \in \Omega$, since evaluation at w is a continuous linear functional on the Hilbert space $H^2(\Omega)$, by the Riesz Representation theorem, there is the unique function $K(\cdot, w)$ which is called the Bergman kernel function such that for all $u \in L^2(\Omega)$

$$(B_\Omega(u))(w) = \langle B_\Omega(u), K(\cdot, w)\rangle_\Omega = \langle u, K(\cdot, w)\rangle_\Omega = \iint_\Omega K(w,z)u(z)\, dA.$$

Now in order to extend the previous results on Toeplitz operators for the unit disc to general domains, we need the following transformation formulas for the Bergman projections and the Szegő projections under biholomorphic (and proper holomorphic) mappings between domains proved by Bell[Bel81] (see also [Bel92] for Szegő projections).

Proposition 2.1. *Suppose that $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between C^∞ smoothly bounded domains Ω_1 and Ω_2 in the plane. Let P_{Ω_i} be the Szegő projections of $L^2(b\Omega_i)$ onto $H^2(b\Omega_i)$, $i = 1, 2$. Then for all $\varphi \in L^2(b\Omega_2)$, the Szegő projections transform via*

$$P_{\Omega_1}\left(\sqrt{f'}(\varphi \circ f)\right) = \sqrt{f'}\left((P_{\Omega_2}\varphi) \circ f\right), \tag{2.6}$$

where $\sqrt{f'}$ is one of the square roots of f' which is well defined.

Proposition 2.2. *Suppose that $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded domains in the plane. Let B_{Ω_i} be the Bergman projections of $L^2(\Omega_i)$ onto $H^2(\Omega_i)$, $i = 1, 2$. Then for all $\varphi \in L^2(\Omega_2)$, the Bergman projections transform via*

$$B_{\Omega_1}\left(f'(\varphi \circ f)\right) = f'\left((B_{\Omega_2}\varphi) \circ f\right). \tag{2.7}$$

Proposition 2.3. *Suppose that $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded domains in the plane. Let m be the multiplicity of f and let $F_1, F_2, \cdots, F_m$ be the local inverses to f. Let $K_{\Omega_i}(z, w)$ be the Bergman kernel functions associated to Ω_j, $i = 1, 2$. Then for all $z \in \Omega_1, w \in \Omega_2$, the Bergman kernel functions transform via*

$$f'(z)K_{\Omega_2}(f(z), w) = \sum_{k=1}^{m} K_{\Omega_1}(z, F_k(w))\overline{F_k'(w)}. \tag{2.8}$$

3. Transformation rules for Toeplitz operators

In this section, we find transformation formulas for Toeplitz operators on Bergman spaces under bihomorphic (and proper holomorphic) mappings and on Hardy spaces under biholomorphic mappings.

Theorem 3.1. *Suppose that Ω is a C^∞ smoothly bounded simply connected domain and let $a \in \Omega$ be fixed. Let f_a be the Riemann mapping function associated to the pair (Ω, a). Let HT_φ^Ω and HT_ψ^U denote the Toeplitz operators on Hardy spaces associated to the pair of Ω and the symbol $\varphi \in L^\infty(b\Omega)$ and to the pair of the unit disc U and the symbol $\psi \in L^\infty(bU)$, respectively. Then for all $\varphi \in L^\infty(b\Omega)$ and $h \in H^2(b\Omega)$, the Toeplitz operators transform under the Riemann mapping function via*

$$\sqrt{F_a'}\left(HT_\varphi^\Omega(h) \circ F_a\right) = HT_{\varphi \circ F_a}^U\left(\sqrt{F_a'}(h \circ F_a)\right), \tag{3.1}$$

where F_a is the inverse to f_a.

Proof. Let $\varphi \in L^\infty(b\Omega)$ and $h \in H^2(b\Omega)$. Notice that $\sqrt{F_a'}(h \circ F_a)$ is in $H^2(bU)$ by change of variables. The proof is straightforward from Proposition 2.1 as follows.

$$\begin{aligned}
&\sqrt{F_a'}\left(HT_\varphi^\Omega(h) \circ F_a\right) = \sqrt{F_a'}\left(P_\Omega(\varphi h) \circ F_a\right) \\
&= P_U\left(\sqrt{F_a'}\left((\varphi h) \circ F_a\right)\right) = P_U\left(\sqrt{F_a'}(\varphi \circ F_a)(h \circ F_a)\right) \\
&= HT_{\varphi \circ F_a}^U\left(\sqrt{F_a'}(h \circ F_a)\right).
\end{aligned}$$

□

Similar to the Hardy space case, we have the transformation formula for Toeplitz operators on Bergman spaces of simply connected domains.

Theorem 3.2. *Suppose that Ω is a bounded simply connected domain and let $a \in \Omega$ be fixed. Let f_a the Riemann mapping function associated to the pair (Ω, a). Let BT^{Ω}_{φ} and BT^{U}_{ψ} denote the Toeplitz operators on Bergman spaces associated to the pair of Ω and the symbol $\varphi \in L^{\infty}(\Omega)$ and to the pair of the unit disc U and $\psi \in L^{\infty}(U)$, respectively. Then for all $\varphi \in L^{\infty}(\Omega)$ and $h \in H^2(\Omega)$, the Toeplitz operators transform under the Riemann mapping function via*

$$F'_a\left(BT^{\Omega}_{\varphi}(h) \circ F_a\right) = BT^{U}_{\varphi \circ F_a}\left(F'_a(h \circ F_a)\right), \tag{3.2}$$

where F_a is the inverse to f_a.

Proof. This is also straightforward by Proposition 2.2. Let $\varphi \in L^{\infty}(\Omega)$ and $h \in H^2(\Omega)$. Applying the biholomorphic mapping $F_a : U \to \Omega$ to Proposition 2.2, we obtain

$$\begin{aligned}
&F'_a\left(BT^{\Omega}_{\varphi}(h) \circ F_a\right) = F'_a\left(B_{\Omega}(\varphi h) \circ F_a\right) \\
&= B_U\left(F'_a\left((\varphi h) \circ F_a\right)\right) = B_U\left(F'_a(\varphi \circ F_a)(h \circ F_a)\right) \\
&= BT^{U}_{\varphi \circ F_a}\left(F'_a(h \circ F_a)\right).
\end{aligned}$$

□

Theorem 3.3. *Suppose that Ω is a finitely n-connected domain and let $a \in \Omega$ be fixed. Let f_a the Ahlfors mapping function of Ω onto the unit disc U associated to the pair (Ω, a). Let BT^{Ω}_{φ} and BT^{U}_{ψ} denote the Toeplitz operators on Bergman spaces associated to the pair of Ω and the symbol $\varphi \in L^{\infty}(\Omega)$ and to the pair of U and the symbol $\psi \in L^{\infty}(U)$, respectively. Then for all $\Phi \in L^{\infty}(U)$ and $H \in H^2(U)$, the Toeplitz operators transform under the Ahlfors map via*

$$\sum_{k=1}^{n} F'_k\left[\left(BT^{\Omega}_{\Phi \circ f_a}[f'_a(H \circ f_a)]\right)\right) \circ F_k\right] = n\ BT^{U}_{\Phi}(H), \tag{3.3}$$

where $F_1, F_2, \cdots, F_n$ are the local inverses to f_a.

Proof. Let $\Phi \in L^{\infty}(U)$ and $H \in H^2(U)$. Since the Ahlfors map f_a is a proper holomorphic mapping from Ω onto the unit disc U, we can apply Proposition 2.3 with f_a as follows. Fix $w \in U$. Then using the change of variables, we have

$$
\begin{aligned}
&\sum_{k=1}^{n} F_k' \left[\left(BT^{\Omega}_{\Phi\circ f_a}[f_a'(H\circ f_a)] \right) \circ F_k \right](w) \\
&= \sum_{k=1}^{n} F_k'(w) \left[B_\Omega \left((\Phi\circ f_a) f_a'(H\circ f_a) \right) \right] (F_k(w)) \\
&= \sum_{k=1}^{n} F_k'(w) \left\langle (\Phi\circ f_a) f_a'(H\circ f_a), K_\Omega(\cdot, F_k(w)) \right\rangle \\
&= \iint_\Omega \Phi(f_a(z)) f_a'(z) H(f_a(z)) \sum_{k=1}^{n} F_k'(w) K_\Omega(F_k(w), z) dA_{\Omega_z} \\
&= \iint_\Omega \Phi(f_a(z)) H(f_a(z)) f_a'(z) \overline{f_a'(z)} K_U(w, f_a(z)) dA_{\Omega_z} \\
&= n \iint_U \Phi(\zeta) H(\zeta) K_U(w, \zeta) dA_{U_\zeta} \\
&= n < \Phi H, K_U(\cdot, w) >= n(B_U(\Phi H))(w) \\
&= n(BT^U_\Phi(H))(w).
\end{aligned}
$$

□

4. Uniqueness of Toeplitz operators

In the case of the unit disc, it is well known that the correspondence $\varphi \to BT^U_\varphi$ and $\varphi \to HT^U_\varphi$ are one to one. The first author in [Chu09] also proved the following uniqueness property for Toeplitz operators on Hardy spaces of general domains.

Proposition 4.1. *Suppose that Ω is a C^∞ smoothly bounded connected domain and let $\varphi \in C^\infty(b\Omega)$. Suppose the Toeplitz operator HT^Ω_φ is the zero operator on $H^2(b\Omega)$. Then φ vanishes on $b\Omega$.*

Here we prove the uniqueness property of Toeplitz operators on Bergman spaces of general domains by using the transformation formula (3.2) for the simply connected case and by using the Green's operator for the multiply connected case.

Theorem 4.2. *Suppose that Ω is a bounded simply connected domain and let $\varphi \in C^\infty(\overline{\Omega})$. Suppose the Toeplitz operator BT^Ω_φ is the zero operator on $H^2(\Omega)$. Then φ is the zero function on Ω.*

Proof. Suppose that for all $h \in H^2(\Omega)$, $BT^{\Omega}_{\varphi}(h) = 0$. Then it follows from the transformation formula (3.2) that $BT^{U}_{\varphi\circ F_a}(F_a'(h \circ F_a)) = 0$. Thus given $H \in H^2(U)$, we obtain $f_a'(H \circ f_a) \in H^2(\Omega)$ and

$$BT^{U}_{\varphi\circ F_a}(H) = BT^{U}_{\varphi\circ F_a}\left[(F_a'(\left[f_a'(H \circ f_a)\right] \circ F_a)\right] = 0.$$

From uniqueness of the Toeplitz operators on the Bergman space $H^2(U)$ of the unit disc, $\varphi \circ F_a = 0$. Hence $\varphi = 0$ and we are done. □

In order to prove the uniqueness for multiply connected domains, we need the following observations which were proved in [Bel90a] and [Bel92]. Suppose that Ω is a bounded domain with C^∞ smooth boundary. The classical Green's operator G is the operator on $C^\infty(\overline{\Omega})$ which solves the following Dirichlet problem: given $\psi \in C^\infty(\overline{\Omega})$,

$$\Delta(G(\psi)) = \psi$$

under the boundary condition $G(\psi) = 0$ on $b\Omega$. The Bergman projection is related to the Green's operator via Spencer's formula

$$B_\Omega(\psi) = \psi - 4\frac{\partial}{\partial z}G\left(\frac{\partial}{\partial \overline{z}}\psi\right). \tag{4.1}$$

(see [Bel90a]). In particular, the operator G maps $C^\infty(\overline{\Omega})$ into itself. The Bergman kernel function $K_\Omega(z,w)$ is related to the Green function $g(z,w)$ via

$$K_\Omega(z,w) = -\frac{2}{\pi}\frac{\partial^2 g(z,w)}{\partial z \partial \overline{w}}.$$

On the other hand, the supplemental function $\Lambda_\Omega(z,w)$ defined by

$$\Lambda_\Omega(z,w) = -\frac{2}{\pi}\frac{\partial^2 g(z,w)}{\partial z \partial w}.$$

is holomorphic in z and w and is in $C^\infty(\overline{\Omega} \times \overline{\Omega} - \{(z,z) : z \in \overline{\Omega}\})$. For fixed $w_0 \in \Omega$, $\Lambda_\Omega(z,w_0)$ has a double pole at w_0 with principal part $-\dfrac{1}{\pi}\dfrac{1}{(z-w_0)^2}$ and $\Lambda_\Omega(z,w) = \Lambda_\Omega(w,z)$ for $z \neq w$. The two functions $K_\Omega(z,w)$ and $\Lambda_\Omega(z,w)$ satisfy the identity

$$\Lambda_\Omega(w,z)T(z) = -K_\Omega(w,z)\overline{T(z)}, \quad w \in \Omega,\ z \in b\Omega. \tag{4.2}$$

(see [Bel92] and [Ber50]).

Now we are ready to prove the following theorem.

Theorem 4.3. *Suppose that Ω is a C^∞ smoothly bounded domain in the plane and let $\varphi \in C^\infty(\overline{\Omega})$. Suppose the Toeplitz operator BT^{Ω}_{φ}*

on the Bergman space $H^2(\Omega)$ is the zero operator. Then φ vanishes on Ω.

Proof. Assume that $\varphi \in C^\infty(\overline{\Omega})$ and $BT_\varphi^\Omega = 0$ on $H^2(\Omega)$. Fix $a \in \Omega$. Applying the identity (4.1) with the function $\psi(z) = \varphi(z)K_\Omega(z,a)$, we get

$$0 = B_\Omega(\varphi K_\Omega(z,a)) = \varphi(z)K_\Omega(z,a) - \frac{\partial}{\partial z} G\left(\frac{\partial}{\partial \overline{z}}\left[\varphi K_\Omega(z,a)\right]\right).$$

It follows that

$$\varphi(z)K_\Omega(z,a) = \frac{\partial}{\partial z} G\left(\frac{\partial}{\partial \overline{z}}(\varphi(z)K_\Omega(z,a))\right). \tag{4.3}$$

On the other hand, by the identities (4.2) and (2.4), we obtain that for $z \in b\Omega$

$$\begin{aligned} K_\Omega(z,a) &= \overline{K_\Omega(a,z)} = -\overline{\Lambda_\Omega(a,z)}\ \overline{T(z)}^2 \\ &= -\overline{\Lambda_\Omega(a,z)}\left(\frac{L(z,a)}{i\overline{S(z,a)}}\right)^2 = \overline{\Lambda_\Omega(a,z)}\left(\frac{L(z,a)}{\overline{S(z,a)}}\right)^2. \end{aligned}$$

Substituting the last identity with $K_\Omega(z,a)$ in (4.3), we have

$$\varphi(z)L(z,a)^2\overline{\left(\frac{\Lambda_\Omega(a,z)}{S(z,a)^2}\right)} = \frac{\partial}{\partial z} G\left(\frac{\partial}{\partial \overline{z}}\left[\varphi(z)K_\Omega(z,a)\right]\right). \tag{4.4}$$

The right hand side of (4.4) is holomorphic in $z \in \Omega$ which extends C^∞ smoothly to the boundary of Ω. However, the function

$$L(z,a)^2\overline{\left(\frac{\Lambda_\Omega(a,z)}{S(z,a)^2}\right)}$$

in the left hand side of (4.4) has singularity at $z = a$ as a double pole holomorphically and as a double pole antiholomorphically. Hence by letting z tend to a, we obtain $\varphi(a) = 0$ and since the point a is arbitrary given, the proof is finished. □

5. Commuting properties of Toeplitz operators on Hardy spaces

In this section, we study on commuting properties of Toeplitz operators on Hardy spaces of general domains and find a generalization of the result of the unit disc case.

Theorem 5.1. *Suppose that Ω is a C^∞ smoothly bounded simply connected domain and let $\varphi, \psi \in L^\infty(b\Omega)$. Let $a \in \Omega$ be fixed and let f_a be the Riemann mapping function associated to the pair (Ω, a). Suppose the Toeplitz operators HT^Ω_φ and HT^Ω_ψ with symbols φ and ψ on the Hardy space $H^2(b\Omega)$ commute. Then the Toeplitz operators $HT^U_{\varphi\circ F_a}$ and $HT^U_{\psi\circ F_a}$ with symbols $\varphi \circ F_a$ and $\psi \circ F_a$ on the Hardy space $H^2(bU)$ of the unit disc U commute where F_a is the inverse of f_a.*

Proof. Assume that for all $h \in H^2(b\Omega)$, $HT^\Omega_\varphi\, HT^\Omega_\psi(h) = HT^\Omega_\psi\, HT^\Omega_\varphi(h)$ and let $H \in H^2(bU)$. It follows from the change of variables that $\sqrt{f_a'}(H \circ f_a) \in H^2(b\Omega)$. Thus by using the transformation formula (3.1), we obtain

$$
\begin{aligned}
&HT^U_{\varphi\circ F_a} HT^U_{\psi\circ F_a}(H) \\
&= HT^U_{\varphi\circ F_a} HT^U_{\psi\circ F_a} \left(\sqrt{F_a'}\left(\left[\sqrt{f_a'}(H\circ f_a)\right]\circ F_a\right)\right) \\
&= HT^U_{\varphi\circ F_a} \left(\sqrt{F_a'}\ \left(HT^\Omega_\psi \left[\sqrt{f_a'}(H\circ f_a)\right]\circ F_a\right)\right) \\
&= \sqrt{F_a'}\ \left[\left(HT^\Omega_\varphi HT^\Omega_\psi \left[\sqrt{f_a'}(H\circ f_a)\right]\right)\circ F_a\right] \\
&= \sqrt{F_a'}\ \left[\left(HT^\Omega_\psi HT^\Omega_\varphi \left[\sqrt{f_a'}(H\circ f_a)\right]\right)\circ F_a\right] \\
&= HT^U_{\psi\circ F_a} \left[\sqrt{F_a'}\ \left(HT^\Omega_\varphi \left[\sqrt{f_a'}(H\circ f_a)\right]\circ F_a\right)\right] \\
&= HT^U_{\psi\circ F_a} HT^U_{\varphi\circ F_a} \left[\sqrt{F_a'}\left(\left[\sqrt{f_a'}(H\circ f_a)\right]\circ F_a\right)\right] \\
&= HT^U_{\psi\circ F_a} HT^U_{\varphi\circ F_a}(H),
\end{aligned}
\tag{5.1}
$$

which proves the commuting property of $HT^U_{\varphi\circ F_a}$ and $HT^U_{\psi\circ F_a}$. □

It is well known(see [BH64]) that if the Toeplitz operators HT^U_φ and HT^U_ϕ on the Hardy space $H^2(bU)$ commute, either both φ and ψ are analytic or both φ and ψ are co-analytic or $a\varphi + b\psi$ is constant for some constants a and b not both 0. Now we can generalize the result to simply connected domains in the plane.

Corollary 5.2. *Suppose that Ω is a bounded simply connected domain and let $\varphi, \psi \in L^\infty(b\Omega)$. Suppose the Toeplitz operators HT^Ω_φ and HT^Ω_ψ on the Hardy space $H^2(b\Omega)$ commute. Then either both φ and ψ are analytic or both φ and ψ are co-analytic or $a\varphi + b\psi$ is constant for some constants a and b not both 0.*

Proof. Suppose that HT^{Ω}_{φ} and HT^{Ω}_{ϕ} on the Hardy space $H^2(b\Omega)$ commute. Then from Theorem 5.1 it follows that $HT^{U}_{\varphi\circ F_a}$ and $HT^{U}_{\psi\circ F_a}$ commute. By the result for the case of the unit circle, either both $\varphi\circ F_a$ and $\psi\circ F_a$ are analytic or both $\varphi\circ F_a$ and $\psi\circ F_a$ are co-analytic or $a(\varphi\circ F_a)+b(\psi\circ F_a)$ is constant for some constants a and b not both 0. But $\varphi=(\varphi\circ F_a)\circ f_a, \psi=(\psi\circ F_a)\circ f_a$ and $\overline{\varphi}=(\overline{\varphi}\circ F_a)\circ f_a, \overline{\psi}=(\overline{\psi}\circ F_a)\circ f_a$, which implies that the statements of Corollary 5.2 hold. □

For multiply connected domains, we have the following necessary condition.

Theorem 5.3. *Suppose that Ω is a C^∞ smoothly bounded connected domain and let P_Ω be the Szegő projection of Ω. Let $\varphi,\psi\in L^\infty(b\Omega)$. Suppose the Toeplitz operators HT^{Ω}_{φ} and HT^{Ω}_{ψ} on the Hardy space $H^2(b\Omega)$ commute. Then for all $h\in H^2(b\Omega)$,*

(5.2)
$$\begin{aligned}
&P_\Omega\left(P_\Omega(\varphi)P_\Omega\left(\overline{T\,P_\Omega(\overline{\psi T})\ h}\right)\right)-P_\Omega\left(P_\Omega(\psi)P_\Omega\left(\overline{T\,P_\Omega(\overline{\varphi T})\ h}\right)\right)\\
&+P_\Omega\left(\overline{T\,P_\Omega(\overline{\varphi T})}P_\Omega(\psi)h\right)-P_\Omega\left(\overline{T\,P_\Omega(\overline{\psi T})}P_\Omega(\varphi)h\right)\\
&+P_\Omega\left(\overline{T\,P_\Omega(\overline{\varphi T})}\ P_\Omega\left(\overline{T\,P_\Omega(\overline{\psi T})h}\right)\right)-P_\Omega\left(\overline{T\,P_\Omega(\overline{\psi T})}\ P_\Omega\left(\overline{T\,P_\Omega(\overline{\varphi T})h}\right)\right)\\
&=0.
\end{aligned}$$

Proof. This is straightforward from the identity (2.2) by letting

$$\varphi=P_\Omega(\varphi)+\overline{T}\overline{P_\Omega(\overline{\varphi T})}$$

and

$$\psi=P_\Omega(\psi)+\overline{T}\overline{P_\Omega(\overline{\psi T})}$$

and by factoring out. □

Corollary 5.4. *Suppose that Ω is a C^∞ smoothly bounded connected domain and let P_Ω be the Szegő projection of Ω. Let $\varphi,\psi\in L^\infty(b\Omega)$. Suppose the Toeplitz operators HT^{Ω}_{φ} and HT^{Ω}_{ψ} on the Hardy space $H^2(b\Omega)$ commute. Then we have the identity*

$$P_\Omega\left(\overline{T\,P_\Omega(\overline{\varphi T})}P_\Omega(\psi)\right)=P_\Omega\left(\overline{T\,P_\Omega(\overline{\psi T})}P_\Omega(\varphi)\right). \tag{5.3}$$

Proof. We substitute $f=1$ into the left hand side of the identity (5.2). Then it follows that all terms except the third and fourth ones vanish because the forms $\overline{T}\,\overline{H}$ where H is in $H^2(b\Omega)$ are orthogonal to $H^2(b\Omega)$ and hence we are done. □

S. Axler, Ž. Čučković and N. V. Rao[AČR00] proved that for a bounded domain Ω, if φ is a nonconstant bounded *analytic* function and if ψ is in $L^\infty(\Omega)$ such that Toeplitz operators BT^Ω_φ and BT^Ω_ψ on Bergman spaces commute, then ψ is *analytic*. In the following theorem we generalize the corresponding statement for Toeplitz operators on Hardy spaces.

Theorem 5.5. *Suppose that Ω is a C^∞ smoothly bounded connected domain and suppose that φ is a nonconstant holomorphic function in $H^2(b\Omega)$ that has no zeroes in $\overline{\Omega}$. If ψ is a function in $C^\infty(\overline{\Omega})$ such that two Toeplitz operators HT^Ω_φ and HT^Ω_ψ on the Hardy space $H^2(b\Omega)$ commute, then the function ψ is a holomorphic function in $H^2(b\Omega)$.*

Proof. Notice that any function of the form $\overline{TH}$ where H is in $H^2(b\Omega)$ is orthogonal to $H^2(b\Omega)$. It thus follows from (5.2) that for any $h \in H^2(b\Omega)$,

$$P_\Omega\left[\overline{TP_\Omega(\overline{\psi T})}\varphi h\right] = P_\Omega\left(\varphi P_\Omega\left[\overline{TP_\Omega(\overline{\psi T})}h\right]\right).$$

In particular, letting $h = \dfrac{1}{\varphi}$ which is holomorphic by assumption implies that $\overline{TP_\Omega(\overline{\psi T})} - \varphi P_\Omega\left[\overline{TP_\Omega(\overline{\psi T})}\,\dfrac{1}{\varphi}\right]$ is a function in $H^2(b\Omega)^\perp$. Note that the first term $\overline{TP_\Omega(\overline{\psi T})}$ is a function in $H^2(b\Omega)^\perp$. Thus the second term $\varphi P_\Omega\left[\overline{TP_\Omega(\overline{\psi T})}\,\dfrac{1}{\varphi}\right]$ is a function in both $H^2(b\Omega)$ and $H^2(b\Omega)^\perp$, which implies from the non-vanishing property of φ that $P_\Omega\left[\overline{TP_\Omega(\overline{\psi T})}\,\dfrac{1}{\varphi}\right] = 0$.

It is known (see [Bel90b]) that any $v \in L^2(b\Omega)$ which is orthogonal to $H^2(b\Omega)$ is of the form $v = \overline{T}\,\overline{H}$ for some $H \in H^2(b\Omega)$. In fact, the function H is given by $H = \mathcal{C}(\overline{v}\,\overline{T})$ where $\mathcal{C}$ is the Cauchy transform defined by

$$(\mathcal{C}u)(z) = \frac{1}{2\pi i}\int_{b\Omega}\frac{u(\zeta)}{\zeta - z}d\zeta$$

for $u \in C^\infty(b\Omega)$. Thus there exists a function $H \in H^2(b\Omega)$ such that

$$\overline{TP_\Omega(\overline{\psi T})}\,\frac{1}{\varphi} = \overline{TH}.$$

If the holomorphic function $P_\Omega(\overline{\psi T})$ has a non-zero point z_0 in Ω, then the holomorphic function $\dfrac{1}{\varphi}$ is equal to the anti-holomorphic function

$\overline{\left[\dfrac{H}{P_\Omega(\overline{\psi T})}\right]}$ in a neighborhood of z_0 which implies that $\dfrac{1}{\varphi}$ is constant which contradicts the assumption of φ.

Therefore the holomorphic function $P_\Omega(\overline{\psi T})$ is the zero function and hence the symbol ψ is a holomorphic function in $H^2(b\Omega)$.

□

6. Commuting properties of Toeplitz operators on Bergman spaces

In this final section, we study on commuting properties for Toeplitz operators on Bergman spaces according to the same procedure of the previous section.

Theorem 6.1. *Suppose that Ω is a bounded simply connected domain with C^∞ smooth boundary and let $\varphi, \psi \in L^\infty(\Omega)$. Let $a \in \Omega$ be fixed and let f_a be the Riemann mapping function associated to the pair (Ω, a). Suppose the Toeplitz operators BT^Ω_φ and BT^Ω_ψ on the Bergman space $H^2(\Omega)$ commute. Then the Toeplitz operators $BT^U_{\varphi\circ F_a}$ and $BT^U_{\psi\circ F_a}$ on the Bergman space $H^2(U)$ commute where F_a is the inverse of f_a.*

Proof. Suppose that for all $h \in H^2(\Omega)$, $BT^\Omega_\varphi\, BT^\Omega_\psi(h) = BT^\Omega_\psi\, BT^\Omega_\varphi(h)$. Let $H \in H^2(U)$. It follows from the change of variables that $f_a'(H\circ f_a) \in H^2(\Omega)$. Thus by using the transformation formula (3.2), we obtain

$$
\begin{aligned}
& BT^U_{\varphi\circ F_a} BT^U_{\psi\circ F_a}(H) \\
&= BT^U_{\varphi\circ F_a} BT^U_{\psi\circ F_a}\left[F_a'\left(\left[f_a'(H\circ f_a)\right]\circ F_a\right)\right] \\
&= BT^U_{\varphi\circ F_a}\left[F_a'\ \left(BT^\Omega_\psi\left[f_a'(H\circ f_a)\right]\circ F_a\right)\right] \\
&= F_a'\ \left[\left(BT^\Omega_\varphi BT^\Omega_\psi\left[f_a'(H\circ f_a)\right]\right)\circ F_a\right] \\
&= F_a'\ \left[\left(BT^\Omega_\psi BT^\Omega_\varphi\left[f_a'(H\circ f_a)\right]\right)\circ F_a\right] \\
&= BT^U_{\psi\circ F_a}\left[F_a'\ \left(BT^\Omega_\varphi\left[f_a'(H\circ f_a)\right]\circ F_a\right)\right] \\
&= BT^U_{\psi\circ F_a} BT^U_{\varphi\circ F_a}\left[F_a'\left(\left[f_a'(H\circ f_a)\right]\circ F_a\right)\right] \\
&= BT^U_{\psi\circ F_a} BT^U_{\varphi\circ F_a}(H),
\end{aligned}
\tag{6.1}
$$

which proves the commuting property of $BT^U_{\varphi\circ F_a}$ and $BT^U_{\psi\circ F_a}$. □

Therefore like S. Axler and Ž. Čučković[AČ91] solved a commuting problem of two Toeplitz operators on Bergman spaces with bounded harmonic symbols for the unit disc case, we obtain an immediate result for simply connected domains as follows.

Corollary 6.2. *Suppose that Ω is a bounded simply connected domain. Suppose the Toeplitz operators BT_{φ}^{Ω} and BT_{ψ}^{Ω} on the Bergman space $H^2(\Omega)$ with bounded harmonic symbols φ, ψ commute. Then either both φ and ψ are analytic or both φ and ψ are co-analytic or $a\varphi+b\psi$ is constant for some constants a and b not both* 0.

For the case of multiply connected domains, the commuting properties of Toeplitz operators on Bergman spaces are related to the classical Green's operator as follows.

Theorem 6.3. *Suppose that Ω is a C^∞ smoothly bounded connected domain and let G be the classical Green's operator. Let $\varphi, \psi \in C^\infty(\overline{\Omega})$. Suppose the Toeplitz operators BT_{φ}^{Ω} and BT_{ψ}^{Ω} on the Bergman space $H^2(\Omega)$ commute. Then for all $h \in H^2(\Omega)$,*

(6.2)

$$\begin{aligned}
&B_\Omega\left(B_\Omega(\varphi)B_\Omega\left(h\ \frac{\partial}{\partial z}G\left(\frac{\partial\psi}{\partial\overline{z}}\right)\right)\right) - B_\Omega\left(B_\Omega(\psi)B_\Omega\left(h\ \frac{\partial}{\partial z}G\left(\frac{\partial\varphi}{\partial\overline{z}}\right)\right)\right)\\
&+\ B_\Omega\left(h\ B_\Omega(\psi)\frac{\partial}{\partial z}G\left(\frac{\partial\varphi}{\partial\overline{z}}\right)\right) - B_\Omega\left(h\ B_\Omega(\varphi)\frac{\partial}{\partial z}G\left(\frac{\partial\psi}{\partial\overline{z}}\right)\right)\\
&+4\cdot B_\Omega\left(\frac{\partial}{\partial z}G\left(\frac{\partial\varphi}{\partial\overline{z}}\right)B_\Omega\left(h\ \frac{\partial}{\partial z}G\left(\frac{\partial\psi}{\partial\overline{z}}\right)\right)\right)\\
&-4\cdot B_\Omega\left(\frac{\partial}{\partial z}G\left(\frac{\partial\psi}{\partial\overline{z}}\right)B_\Omega\left(h\ \frac{\partial}{\partial z}G\left(\frac{\partial\varphi}{\partial\overline{z}}\right)\right)\right)\\
&=0.
\end{aligned}$$

Proof. This is done from Spencer's formula (4.1) by letting

$$\varphi = B_\Omega(\varphi) + 4\frac{\partial}{\partial z}G\left(\frac{\partial}{\partial\overline{z}}\varphi\right)$$

and

$$\psi = B_\Omega(\psi) + 4\frac{\partial}{\partial z}G\left(\frac{\partial}{\partial\overline{z}}\psi\right).$$

and by factoring out. In fact, one side of the identity $BT_\varphi BT_\psi(h) = BT_\psi BT_\varphi(h)$ is read off as follows.

$$\begin{aligned}
&BT_\varphi BT_\psi(h) = B_\Omega\left\{\varphi B_\Omega(\psi h)\right\}\\
&= B_\Omega\left[\left(B_\Omega(\varphi)+4\frac{\partial}{\partial z}G\left(\frac{\partial}{\partial \bar z}\varphi\right)\right)B_\Omega\left\{\left(B_\Omega(\psi)+4h\frac{\partial}{\partial z}G\left(\frac{\partial}{\partial \bar z}\psi\right)\right)\right\}\right]\\
&= B_\Omega\left[B_\Omega(\varphi)B_\Omega\left\{B_\Omega(\psi)h\right\}\right]+4B_\Omega\left[B_\Omega(\varphi)B_\Omega\left\{h\frac{\partial}{\partial z}G\left(\frac{\partial}{\partial \bar z}\psi\right)\right\}\right]\\
&+4B_\Omega\left[\frac{\partial}{\partial z}G\left(\frac{\partial}{\partial \bar z}\varphi\right)\left\{B_\Omega(\psi)h\right\}\right]+16B_\Omega\left[\frac{\partial}{\partial z}G\left(\frac{\partial}{\partial \bar z}\varphi\right)B_\Omega\left\{h\frac{\partial}{\partial z}G\left(\frac{\partial}{\partial \bar z}\psi\right)\right\}\right].
\end{aligned}$$

Notice that the first term of the above identity is equal to $B_\Omega\left\{B_\Omega(\varphi)B_\Omega(\psi)h\right\}$ because the function $B_\Omega(\psi)h$ is holomorphic. □

Corollary 6.4. *Suppose that Ω is a C^∞ smoothly bounded connected domain and let B_Ω be the Bergman projection of Ω and G be the classical Green's operator. Let $\varphi, \psi \in C^\infty(\overline{\Omega})$. Suppose the Toeplitz operators BT_φ^Ω and BT_ψ^Ω on the Bergman space $H^2(\Omega)$ commute. Then*

$$B_\Omega\left(B_\Omega(\psi)\frac{\partial}{\partial z}G\left(\frac{\partial\varphi}{\partial \bar z}\right)\right) = B_\Omega\left(B_\Omega(\varphi)\frac{\partial}{\partial z}G\left(\frac{\partial\psi}{\partial \bar z}\right)\right) \tag{6.3}$$

Proof. Substituting $f = 1$ into the identity (6.2), we obtain only the third and the fourth terms (of possibly nonvanishing) because for any function Φ in $C^\infty(\overline{\Omega})$ that vanishes on the boundary of Ω, $\dfrac{\partial\Phi}{\partial z}$ is orthogonal to $H^2(\Omega)$ by virtue of the Green's identity and hence we are done. □

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