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$([r,s],[t,u])$-INTERVAL-VALUED INTUITIONISTIC FUZZY ALPHA GENERALIZED CONTINUOUS MAPPINGS

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Abstract. In this paper, we introduce the concepts of $([r,s],[t,u])$-interval-valued intuitionistic fuzzy alpha generalized closed and open sets in the interval-valued intuitionistic smooth topological space and $([r,s],[t,u])$-interval-valued intuitionistic fuzzy alpha generalized continuous mappings and then investigate some of their properties.

1. Introduction

After Zadeh [16] introduced the concept of fuzzy sets, there have been various generalizations of the concept of fuzzy sets. Chang [4] introduced the concept of fuzzy topology on a set X by axiomatizing a collection T of fuzzy subsets of X and Coker [6] introduced the concept of intuitionistic fuzzy topology on a set by axiomatizing a collection T of intuitionistic fuzzy subsets of X. Chattopadhyay, Hazra and Samanta [5,7] introduced the concept of gradation of openness of fuzzy subsets. Zadeh [17] introduced the concept of interval-valued fuzzy sets and Atanassov [1] introduced the concept of intuitionistic fuzzy sets. Atanassov and Gargov [2] introduced the concept of interval-valued intuitionistic fuzzy sets

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which is a generalization of both interval-valued fuzzy sets and intuitionistic fuzzy sets. Mondal and Samanta [9,15] introduced the concept of intuitionistic gradation of openness and defined an intuitionistic fuzzy topological space. Jeon, Jun and Park [8] introduced the concepts of intuitionistic fuzzy alpha closed sets and intuitionistic fuzzy alpha continuous mappings. Sakthivel [14] introduced the concepts of intuitionistic fuzzy alpha generalized closed sets and intuitionistic fuzzy alpha generalized continuous mappings.

In this paper, we introduce the concepts of $([r,s],[t,u])$-interval-valued intuitionistic fuzzy alpha generalized closed and open sets in the interval-valued intuitionistic smooth topological space and $([r,s],[t,u])$-interval-valued intuitionistic fuzzy alpha generalized continuous mappings and then investigate some of their properties.

2. Preliminaries

Throughout this paper, let X be a nonempty set, $I=[0,1]$, $I_0=(0,1]$ and $I_1=[0,1)$. The family of all fuzzy sets of X will be denoted by I^X. By 0_X and 1_X we denote the characteristic functions of ϕ and X, respectively. For any $A\in I^X$, A^c denotes the complement of A, i.e., $A^c=1_X-A$.

Definition 2.1. [3,5,13]. A *gradation of openness* (for short, GO) on X, which is also called a *smooth topology* on X, is a mapping $\tau: I^X\to I$ satisfying the following conditions:

(GO1) $\tau(0_X)=\tau(1_X)=1$,

(GO2) $\tau(A\cap B)\geq\tau(A)\wedge\tau(B)$ for each $A,B\in I^X$,

(GO3) $\tau(\cup_{i\in\Gamma}A_i)\geq\wedge_{i\in\Gamma}\tau(A_i)$ for each subfamily $\{A_i: i\in\Gamma\}\subseteq I^X$.

The pair (X,τ) is called a *smooth topological space* (for short, STS).

Definition 2.2. [9]. An *intuitionistic gradation of openness* (for short, IGO) on X, which is also called an *intuitionistic smooth topology* on X, is an ordered pair (τ,τ^*) of mappings from I^X to I satisfying the following conditions:

(IGO1) $\tau(A)+\tau^*(A)\leq 1$ for each $A\in I^X$,

(IGO2) $\tau(0_X)=\tau(1_X)=1$ and $\tau^*(0_X)=\tau^*(1_X)=0$,

(IGO3) $\tau(A\cap B)\geq\tau(A)\wedge\tau(B)$ and $\tau^*(A\cap B)\leq\tau^*(A)\vee\tau^*(B)$ for each $A,B\in I^X$, (IGO4) $\tau(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau(A_i)$ and $\tau^*(\cup_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \tau^*(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subseteq I^X$.

The triple (X, τ, τ^*) is called an *intuitionistic smooth topological space* (for short, ISTS). τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Let $D(I)$ be the set of all closed subintervals of the unit interval I. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$ and $M = [M^L, M^U]$, where M^L and M^U are respectively the lower and the upper end points. Especially, we denote $\mathbf{r} = [r, r]$ for each $r \in I$. The complement of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$. Note that $M = N$ iff $M^L = N^L$ and $M^U = N^U$ and that $M \leq N$ iff $M^L \leq N^L$ and $M^U \leq N^U$.

Definition 2.3. [17]. A mapping $A = [A^L, A^U] : X \to D(I)$ is called an *interval-valued fuzzy set* (for short, IVFS) on X, where $A(x) = [A^L(x), A^U(x)]$ for each $x \in X$. $A^L(x)$ and $A^U(x)$ are called the *lower* and *upper end points* of $A(x)$, respectively.

Definition 2.4. [10]. Let A and B be IVFSs on X.

(i) $A = B$ iff $A^L(x) = B^L(x)$ and $A^U(x) = B^U(x)$ for all $x \in X$.

(ii) $A \subseteq B$ iff $A^L(x) \leq B^L(x)$ and $A^U(x) \leq B^U(x)$ for all $x \in X$.

(iii) The *complement* A^c of A is defined by $A^c(x) = [1 - A^U(x), 1 - A^L(x)]$ for all $x \in X$.

(iv) For a family of IVFSs $\{A_i : i \in \Gamma\}$, the union $\cup_{i\in\Gamma} A_i$ and the intersection $\cap_{i\in\Gamma} A_i$ are respectively defined by

$$\cup_{i\in\Gamma} A_i(x) = [\vee_{i\in\Gamma} A_i^L(x), \vee_{i\in\Gamma} A_i^U(x)],$$
$$\cap_{i\in\Gamma} A_i(x) = [\wedge_{i\in\Gamma} A_i^L(x), \wedge_{i\in\Gamma} A_i^U(x)]$$

for all $x \in X$.

Definition 2.5. [2]. A mapping $A = (\mu_A, \nu_A) : X \to D(I) \times D(I)$ is called an *interval-valued intuitionistic fuzzy set* (for short, IVIFS) on X, where $\mu_A : X \to D(I)$ and $\nu_A : X \to D(I)$ are interval-valued fuzzy sets on X with the condition $\sup_{x\in X} \mu_A^U(x) + \sup_{x\in X} \nu_A^U(x) \leq 1$. The intervals $\mu_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ and $\nu_A(x) = [\nu_A^L(x), \nu_A^U(x)]$ denote the degree of belongingness and the degree of nonbelongingness of the element x to the set A, respectively.

Definition 2.6. [11]. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IVIFSs on X.

(i) $A \subseteq B$ iff $\mu_A^L(x) \leq \mu_B^L(x)$, $\mu_A^U(x) \leq \mu_B^U(x)$ and $\nu_A^L(x) \geq \nu_B^L(x)$, $\nu_A^U(x) \geq \nu_B^U(x)$ for all $x \in X$.

(ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(iii) The *complement* A^c of A is defined by $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$ for all $x \in X$.

(iv) For a family of IVIFSs $\{A_i : i \in \Gamma\}$, the union $\cup_{i\in\Gamma} A_i$ and the intersection $\cap_{i\in\Gamma} A_i$ are respectively defined by

$$\mu_{\cup_{i\in\Gamma} A_i}(x) = \cup_{i\in\Gamma}\mu_{A_i}(x), \nu_{\cup_{i\in\Gamma} A_i}(x) = \cap_{i\in\Gamma}\nu_{A_i}(x),$$
$$\mu_{\cap_{i\in\Gamma} A_i}(x) = \cap_{i\in\Gamma}\mu_{A_i}(x), \nu_{\cap_{i\in\Gamma} A_i}(x) = \cup_{i\in\Gamma}\nu_{A_i}(x)$$

for all $x \in X$.

3. $([r,s],[t,u])$-interval-valued intuitionistic fuzzy alpha closed and open sets

Definition 3.1. [12]. An *interval-valued intuitionistic gradation of openness* (for short, IVIGO) on X, which is also called an *interval-valued intuitionistic smooth topology* on X, is an ordered pair (τ, τ^*) of mappings $\tau = [\tau^L, \tau^U] : I^X \to D(I)$ and $\tau^* = [\tau^{*L}, \tau^{*U}] : I^X \to D(I)$ satisfying the following conditions:

(IVIGO1) $\tau^L(A) \leq \tau^U(A)$, $\tau^{*L}(A) \leq \tau^{*U}(A)$ and $\tau^U(A) + \tau^{*U}(A) \leq 1$ for each $A \in I^X$,

(IVIGO2) $\tau(0_X) = \tau(1_X) = \mathbf{1}$ and $\tau^*(0_X) = \tau^*(1_X) = \mathbf{0}$,

(IVIGO3) $\tau^L(A \cap B) \geq \tau^L(A) \wedge \tau^L(B)$, $\tau^U(A \cap B) \geq \tau^U(A) \wedge \tau^U(B)$ and $\tau^{*L}(A \cap B) \leq \tau^{*L}(A) \vee \tau^{*L}(B)$, $\tau^{*U}(A \cap B) \leq \tau^{*U}(A) \vee \tau^{*U}(B)$ for each $A, B \in I^X$,

(IVIGO4) $\tau^L(\cup_{i\in\Gamma}\ A_i) \geq \wedge_{i\in\Gamma}\ \tau^L(A_i)$, $\tau^U(\cup_{i\in\Gamma}\ A_i) \geq \wedge_{i\in\Gamma}\ \tau^U(A_i)$ and $\tau^{*L}(\cup_{i\in\Gamma}\ A_i) \leq \vee_{i\in\Gamma}\ \tau^{*L}(A_i)$, $\tau^{*U}(\cup_{i\in\Gamma}\ A_i) \leq \vee_{i\in\Gamma}\ \tau^{*U}(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subseteq I^X$.

The triple (X, τ, τ^*) is called an *interval-valued intuitionistic smooth topological space* (for short, IVISTS). τ and τ^* may be interpreted as interval-valued gradation of openness and interval-valued gradation of nonopenness, respectively.

Definition 3.2. [12]. Let (X, τ, τ^*) be an IVISTS, $A \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s + u \leq 1$.

(i) A is called an $([r,s],[t,u])$*-interval-valued intuitionistic fuzzy open set* (for short, $([r,s],[t,u])$-IVIFOS) if $\tau(A) \geq [r,s]$ and $\tau^*(A) \leq [t,u]$.

(ii) A is called an $([r, s], [t, u])$-*interval-valued intuitionistic fuzzy closed set* (for short, $([r, s], [t, u])$-IVIFCS) if $\tau(A^c) \geq [r, s]$ and $\tau^*(A^c) \leq [t, u]$.

DEFINITION 3.3. [12]. Let (X, τ, τ^*) be an IVISTS, $A \in I^X$ and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$. The $([r, s], [t, u])$-*interval-valued intuitionistic fuzzy closure* and $([r, s], [t, u])$-*interval-valued intuitionistic fuzzy interior* of A are defined by

$cl_{[r,s],[t,u]}(A) = \cap\{K \in I^X : A \subseteq K,\ K$ is an $([r, s], [t, u])$-IVIFCS$\}$,

$int_{[r,s],[t,u]}(A) = \cup\{G \in I^X : G \subseteq A,\ G$ is an $([r, s], [t, u])$-IVIFOS$\}$.

DEFINITION 3.4. Let (X, τ, τ^*) be an IVISTS, $A \in I^X$ and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$.

(i) A is called an $([r, s], [t, u])$-*interval-valued intuitionistic fuzzy α-closed set* (for short, $([r, s], [t, u])$-IVIFαCS) if $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(A))) \subseteq A$.

(ii) A is called an $([r, s], [t, u])$-*interval-valued intuitionistic fuzzy α-open set* (for short, $([r, s], [t, u])$-IVIFαOS) if A^c is an $([r, s], [t, u])$-IVIFαCS, or equivalently, $A \subseteq int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(A)))$.

Note that if A is an $([r, s], [t, u])$-IVIFCS then A is an $([r, s], [t, u])$-IVIFαCS and that if A is an $([r, s], [t, u])$-IVIFOS then A is an $([r, s], [t, u])$-IVIFαOS.

EXAMPLE 3.5. Every $([r, s], [t, u])$-IVIFαCS need not be an $([r, s], [t, u])$-IVIFCS and every $([r, s], [t, u])$-IVIFαOS need not be an $([r, s], [t, u])$-IVIFOS

Let $X = \{a, b\}$. Define $F_1, F_2, F_3, F_4 \in I^X$ as follows:

$F_1 = \{(a, 0.4), (b, 0.4)\}$, $F_2 = \{(a, 0.5), (b, 0.6)\}$, $F_3 = \{(a, 0.5), (b, 0.4)\}$, $F_4 = \{(a, 0.6), (b, 0.6)\}$.

Define $\tau, \tau^* : I^X \to D(I)$ as follows:

$$\tau(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{0_X, 1_X\}, \\ [0.7, 0.8] & \text{if } A = F_2, \\ [0.4, 0.5] & \text{if } A = F_1, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$\tau^*(A) = \begin{cases} \mathbf{0} & \text{if } A \in \{0_X, 1_X\}, \\ [0.1, 0.2] & \text{if } A = F_2, \\ [0.3, 0.4] & \text{if } A = F_1, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Let $[r,s]=[0.5,0.6]$ and $[t,u]=[0.2,0.3]$. Then F_1 is an $([r,s],[t,u])$-IVIFαCS, but F_1 is not an $([r,s],[t,u])$-IVIFCS. Also F_4 is an $([r,s],[t,u])$-IVIFαOS, but F_4 is not an $([r,s],[t,u])$-IVIFOS.

Remark 3.6. Let (X,τ,τ^*) be an IVISTS, $A\in I^X$ and $[r,s]\in D(I_0)$, $[t,u]\in D(I_1)$ with $s+u\le 1$. Then

(i) Any intersection of $([r,s],[t,u])$-IVIFαCSs is an $([r,s],[t,u])$-IVIFαCS.

(ii) Any union of $([r,s],[t,u])$-IVIFαOSs is an $([r,s],[t,u])$-IVIFαOS.

Definition 3.7. Let (X,τ,τ^*) be an IVISTS, $A\in I^X$ and $[r,s]\in D(I_0)$, $[t,u]\in D(I_1)$ with $s+u\le 1$. The $([r,s],[t,u])$-*interval-valued intuitionistic fuzzy α-closure* and $([r,s],[t,u])$-*interval-valued intuitionistic fuzzy α-interior* of A are defined by

$\alpha cl_{[r,s],[t,u]}(A)=\cap\{K\in I^X : A\subseteq K,\ K$ is an $([r,s],[t,u])$-IVIFαCS$\}$,

$\alpha int_{[r,s],[t,u]}(A)=\cup\{G\in I^X : G\subseteq A,\ G$ is an $([r,s],[t,u])$-IVIFαOS$\}$.

Note that $int_{[r,s],[t,u]}(A)\subseteq \alpha int_{[r,s],[t,u]}(A)\subseteq A\subseteq \alpha cl_{[r,s],[t,u]}(A)\subseteq cl_{[r,s],[t,u]}(A)$.

Theorem 3.8. *Let (X,τ,τ^*) be an IVISTS, $A,B\in I^X$ and $[r,s]\in D(I_0)$, $[t,u]\in D(I_1)$ with $s+u\le 1$. Then*

(i) $\alpha cl_{[r,s],[t,u]}(0_X)=0_X$.

(ii) $A\subseteq \alpha cl_{[r,s],[t,u]}(A)$.

(iii) $\alpha cl_{[r,s],[t,u]}(A)\subseteq \alpha cl_{[r,s],[t,u]}(B)$ *if* $A\subseteq B$.

(iv) $\alpha cl_{[r,s],[t,u]}(A\cup B)\supseteq \alpha cl_{[r,s],[t,u]}(A)\cup \alpha cl_{[r,s],[t,u]}(B)$,
$\alpha cl_{[r,s],[t,u]}(A\cap B)\subseteq \alpha cl_{[r,s],[t,u]}(A)\cap \alpha cl_{[r,s],[t,u]}(B)$.

(v) $A=\alpha cl_{[r,s],[t,u]}(A)$ *if and only if A is an $([r,s],[t,u])$-IVIFαCS.*

(vi) $\alpha cl_{[r,s],[t,u]}(\alpha cl_{[r,s],[t,u]}(A))=\alpha cl_{[r,s],[t,u]}(A)$.

(vii) $\alpha cl_{[r,s],[t,u]}(A^c)=(\alpha int_{[r,s],[t,u]}(A))^c$.

Proof. (i), (ii) and (iii) follow directly from Definition 3.7.

(iv) It follows directly from (iii).

(v) It follows directly from Definition 3.7 and Remark 3.6.

(vi) By Definition 3.7 and Remark 3.6, $\alpha cl_{[r,s],[t,u]}(A)$ is an $([r,s],[t,u])$-IVIFαCS. By (v), $\alpha cl_{[r,s],[t,u]}(\alpha cl_{[r,s],[t,u]}(A))=\alpha cl_{[r,s],[t,u]}(A)$.

(vii) By Definition 3.7, we have

$$
\begin{aligned}
&\alpha cl_{[r,s],[t,u]}(A^c)\\
&= \cap\{K \in I^X : A^c \subseteq K,\ K \text{ is an } ([r,s],[t,u])\text{-IVIF}\alpha\text{CS}\}\\
&= \cap\{G^c \in I^X : A^c \subseteq G^c,\ G^c \text{ is an } ([r,s],[t,u])\text{-IVIF}\alpha\text{CS}\}\\
&= (\cup\{G \in I^X : G \subseteq A,\ G \text{ is an } ([r,s],[t,u])\text{-IVIF}\alpha\text{OS}\})^c\\
&= (\alpha int_{[r,s],[t,u]}(A))^c.
\end{aligned}
$$

□

THEOREM 3.9. *Let (X,τ,τ^*) be an IVISTS, $A, B \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$. Then*

(i) $\alpha int_{[r,s],[t,u]}(1_X) = 1_X$.

(ii) $\alpha int_{[r,s],[t,u]}(A) \subseteq A$.

(iii) $\alpha int_{[r,s],[t,u]}(A) \subseteq \alpha int_{[r,s],[t,u]}(B)$ *if* $A \subseteq B$.

(iv) $\alpha int_{[r,s],[t,u]}(A \cup B) \supseteq \alpha int_{[r,s],[t,u]}(A) \cup \alpha int_{[r,s],[t,u]}(B)$,
$\alpha int_{[r,s],[t,u]}(A \cap B) \subseteq \alpha int_{[r,s],[t,u]}(A) \cap \alpha int_{[r,s],[t,u]}(B)$.

(v) $A = \alpha int_{[r,s],[t,u]}(A)$ *if and only if A is an $([r,s],[t,u])$-IVIFαOS.*

(vi) $\alpha int_{[r,s],[t,u]}(\alpha int_{[r,s],[t,u]}(A)) = \alpha int_{[r,s],[t,u]}(A)$.

(vii) $\alpha int_{[r,s],[t,u]}(A^c) = (\alpha cl_{[r,s],[t,u]}(A))^c$.

Proof. The proof is similar to Theorem 3.8. □

4. $([r,s],[t,u])$-interval-valued intuitionistic fuzzy alpha continuous mappings

DEFINITION 4.1. Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. f is called an $([r,s],[t,u])$-*interval-valued intuitionistic fuzzy α-continuous mapping* (for short, $([r,s],[t,u])$-IVIFα-continuous mapping) if $f^{-1}(B)$ is an $([r,s],[t,u])$-IVIFαCS of X for each $([r,s],[t,u])$-IVIFCS B of Y.

Note that $f : X \to Y$ is an $([r,s],[t,u])$-IVIFα-continuous mapping if and only if $f^{-1}(B)$ is an $([r,s],[t,u])$-IVIFαOS of X for each $([r,s],[t,u])$-IVIFOS B of Y.

THEOREM 4.2. *Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. Then the following statements are equivalent.*

(i) *f is an $([r,s],[t,u])$-IVIFα-continuous mapping.*

(ii) $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(B)))) \subseteq f^{-1}(cl_{[r,s],[t,u]}(B))$ *for each* $B \in I^Y$.

(iii) $f(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(A)))) \subseteq cl_{[r,s],[t,u]}(f(A))$ *for each* $A \in I^X$.

Proof. (i)$\Rightarrow$(ii). Let $B \in I^Y$. Then $cl_{[r,s],[t,u]}(B)$ is an $([r,s],[t,u])$-IVIFCS of Y. Since f is an $([r,s],[t,u])$-IVIFα-continuous mapping, $f^{-1}(cl_{[r,s],[t,u]}(B))$ is an $([r,s],[t,u])$-IVIFαCS of X. Hence

$$\begin{aligned} f^{-1}(cl_{[r,s],[t,u]}(B)) &\supseteq cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(cl_{[r,s],[t,u]}(B))))) \\ &\supseteq cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(B)))). \end{aligned}$$

(ii)$\Rightarrow$(iii). Let $A \in I^X$. Then $f(A) \in I^Y$. By (ii),

$$\begin{aligned} f^{-1}(cl_{[r,s],[t,u]}(f(A))) &\supseteq cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(f(A))))) \\ &\supseteq cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(A))). \end{aligned}$$

Hence

$$\begin{aligned} cl_{[r,s],[t,u]}(f(A)) &\supseteq f(f^{-1}(cl_{[r,s],[t,u]}(f(A)))) \\ &\supseteq f(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(A)))). \end{aligned}$$

(iii)$\Rightarrow$(i). Let B be an $([r,s],[t,u])$-IVIFCS of Y. Then $cl_{[r,s],[t,u]}(B) = B$ and $f^{-1}(B) \in I^X$. By (iii),

$$\begin{aligned} f(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(B))))) &\subseteq cl_{[r,s],[t,u]}(f(f^{-1}(B))) \\ &\subseteq cl_{[r,s],[t,u]}(B) = B. \end{aligned}$$

Hence

$$\begin{aligned} &cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(B)))) \\ &\quad \subseteq f^{-1}(f(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(B)))))) \\ &\quad \subseteq f^{-1}(B). \end{aligned}$$

Thus $f^{-1}(B)$ is an $([r,s],[t,u])$-IVIFαCS of X. Hence f is an $([r,s],[t,u])$-IVIFα-continuous mapping. □

Theorem 4.3. *Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. Then the following statements are equivalent.*

(i) *f is an $([r,s],[t,u])$-IVIFα-continuous mapping.*

(ii) $f(\alpha cl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f(A))$ *for each* $A \in I^X$.

(iii) $\alpha cl_{[r,s],[t,u]}(f^{-1}(B)) \subseteq f^{-1}(cl_{[r,s],[t,u]}(B))$ *for each* $B \in I^Y$.

(iv) $f^{-1}(int_{[r,s],[t,u]}(B)) \subseteq \alpha int_{[r,s],[t,u]}(f^{-1}(B))$ *for each* $B \in I^Y$.

Proof. (i)⇒(ii). Let $A \in I^X$. Then $cl_{[r,s],[t,u]}(f(A))$ is an $([r,s],[t,u])$-IVIFCS of Y. Since f is an $([r,s],[t,u])$-IVIFα-continuous mapping, $f^{-1}(cl_{[r,s],[t,u]}(f(A)))$ is an $([r,s],[t,u])$-IVIFαCS of X. By Theorem 3.8,

$$\begin{aligned}\alpha cl_{[r,s],[t,u]}(A) &\subseteq \alpha cl_{[r,s],[t,u]}(f^{-1}(f(A)))\\ &\subseteq \alpha cl_{[r,s],[t,u]}(f^{-1}(cl_{[r,s],[t,u]}(f(A))))\\ &= f^{-1}(cl_{[r,s],[t,u]}(f(A))).\end{aligned}$$

Hence

$$\begin{aligned}f(\alpha cl_{[r,s],[t,u]}(A)) &\subseteq f(f^{-1}(cl_{[r,s],[t,u]}(f(A))))\\ &\subseteq cl_{[r,s],[t,u]}(f(A)).\end{aligned}$$

(ii)⇒(iii). Let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. By (ii),

$$f(\alpha cl_{[r,s],[t,u]}(f^{-1}(B))) \subseteq cl_{[r,s],[t,u]}(f(f^{-1}(B))) \subseteq cl_{[r,s],[t,u]}(B).$$

Hence

$$\alpha cl_{[r,s],[t,u]}(f^{-1}(B)) \subseteq f^{-1}(f(\alpha cl_{[r,s],[t,u]}(f^{-1}(B)))) \subseteq f^{-1}(cl_{[r,s],[t,u]}(B)).$$

(iii)⇒(iv). Let $B \in I^Y$. By (iii) and Theorem 3.8,

$$\begin{aligned}(\alpha int_{[r,s],[t,u]}(f^{-1}(B)))^c &= \alpha cl_{[r,s],[t,u]}(f^{-1}(B^c))\\ &\subseteq f^{-1}(cl_{[r,s],[t,u]}(B^c))\\ &= (f^{-1}(int_{[r,s],[t,u]}(B)))^c.\end{aligned}$$

Hence $f^{-1}(int_{[r,s],[t,u]}(B)) \subseteq \alpha int_{[r,s],[t,u]}(f^{-1}(B))$.

(iv)⇒(i). Let B be an $([r,s],[t,u])$-IVIFOS of Y. Then $int_{[r,s],[t,u]}(B) = B$ and $f^{-1}(B) \in I^X$. By (iv),

$$f^{-1}(B) = f^{-1}(int_{[r,s],[t,u]}(B)) \subseteq \alpha int_{[r,s],[t,u]}(f^{-1}(B)) \subseteq f^{-1}(B).$$

Thus $f^{-1}(B) = \alpha int_{[r,s],[t,u]}(f^{-1}(B))$. By Theorem 3.8, $f^{-1}(B)$ is an $([r,s],[t,u])$-IVIFαOS of X. Hence f is an $([r,s],[t,u])$-IVIFα-continuous mapping. □

THEOREM 4.4. *Let* (X,τ,τ^*) *and* (Y,η,η^*) *be two IVISTSs and* $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ *with* $s+u \leq 1$ *and let* $f : X \to Y$ *be a bijective mapping. Then* f *is an* $([r,s],[t,u])$*-IVIF*α*-continuous mapping if and only if* $int_{[r,s],[t,u]}(f(A)) \subseteq f(\alpha int_{[r,s],[t,u]}(A))$ *for each* $A \in I^X$.

Proof. Let f be an $([r,s],[t,u])$-IVIFα-continuous mapping and let $A \in I^X$. Then $int_{[r,s],[t,u]}(f(A))$ is an $([r,s],[t,u])$-IVIFOS of Y. Since f is an $([r,s],[t,u])$-IVIFα-continuous mapping, $f^{-1}(int_{[r,s],[t,u]}(f(A)))$ is an $([r,s],\ [t,u])$-IVIFαOS of X. By Theorem 3.8 and injectivity of f,

$$\begin{aligned} f^{-1}(int_{[r,s],[t,u]}(f(A))) &= \alpha int_{[r,s],[t,u]}(f^{-1}(int_{[r,s],[t,u]}(f(A)))) \\ &\subseteq \alpha int_{[r,s],[t,u]}(f^{-1}(f(A))) \\ &= \alpha int_{[r,s],[t,u]}(A). \end{aligned}$$

By surjectivity of f,

$$int_{[r,s],[t,u]}(f(A)) = f(f^{-1}(int_{[r,s],[t,u]}(f(A)))) \subseteq f(\alpha int_{[r,s],[t,u]}(A)).$$

Conversely, let B be an $([r,s],[t,u])$-IVIFOS of Y. Then $int_{[r,s],[t,u]}(B) = B$ and $f^{-1}(B) \in I^X$. By hypothesis and surjectivity of f,

$$\begin{aligned} B = int_{[r,s],[t,u]}(B) &= int_{[r,s],[t,u]}(f(f^{-1}(B))) \\ &\subseteq f(\alpha int_{[r,s],[t,u]}(f^{-1}(B))) \\ &\subseteq f(f^{-1}(B)) = B. \end{aligned}$$

Thus $B = f(\alpha int_{[r,s],[t,u]}(f^{-1}(B)))$. By injectivity of f,

$$f^{-1}(B) = f^{-1}(f(\alpha int_{[r,s],[t,u]}(f^{-1}(B)))) = \alpha int_{[r,s],[t,u]}(f^{-1}(B)).$$

By Theorem 3.8, $f^{-1}(B)$ is an $([r,s],\ [t,u])$-IVIFαOS of X. Hence f is an $([r,s],[t,u])$-IVIFα-continuous mapping. □

From Theorem 4.3 and Theorem 4.4, we can obtain the following corollary.

Corollary 4.5. *Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \le 1$ and let $f : X \to Y$ be a bijective mapping. Then the following statements are equivalent.*

(i) *f is an $([r,s],[t,u])$-IVIFα-continuous mapping.*

(ii) *$f(\alpha cl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$.*

(iii) *$\alpha cl_{[r,s],[t,u]}(f^{-1}(B)) \subseteq f^{-1}(cl_{[r,s],[t,u]}(B))$ for each $B \in I^Y$.*

(iv) *$f^{-1}(int_{[r,s],[t,u]}(B)) \subseteq \alpha int_{[r,s],[t,u]}(f^{-1}(B))$ for each $B \in I^Y$.*

(v) *$int_{[r,s],[t,u]}(f(A)) \subseteq f(\alpha int_{[r,s],[t,u]}(A))$ for each $A \in I^X$.*

5. $([r,s],[t,u])$-interval-valued intuitionistic fuzzy alpha generalized closed and open sets

DEFINITION 5.1. Let (X,τ,τ^*) be an IVISTS, $A \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$.

(i) A is called an $([r,s],[t,u])$*-interval-valued intuitionistic fuzzy* α*-generalized closed set* (for short, $([r,s],[t,u])$-IVIFαGCS) if $\alpha cl_{([r,s],[t,u])}(A) \subseteq U$ whenever $A \subseteq U$ and U is an $([r,s],[t,u])$-IVIFOS.

(ii) A is called an $([r,s],[t,u])$*-interval-valued intuitionistic fuzzy* α*-generalized open set* (for short, $([r,s],[t,u])$-IVIFαGOS) if A^c is an $([r,s],[t,u])$-IVIFαGCS, or equivalently, $U \subseteq \alpha int_{[r,s],[t,u]}(A)$ whenever $U \subseteq A$ and U is an $([r,s],[t,u])$-IVIFCS.

Note that if A is an $([r,s],[t,u])$-IVIFαCS then A is an $([r,s],[t,u])$-IVIFαGCS and that if A is an $([r,s],[t,u])$-IVIFαOS then A is an $([r,s],[t,u])$-IVIFαGOS.

EXAMPLE 5.2. Every $([r,s],[t,u])$-IVIFαGCS need not be an $([r,s],[t,u])$-IVIFαCS and every $([r,s],[t,u])$-IVIFαGOS need not be an $([r,s],[t,u])$-IVIFαOS.

Let $X = \{a,b\}$. Define $F_1, F_2, F_3, F_4 \in I^X$ as follows:

$F_1 = \{(a,0.4),(b,0.4)\}$, $F_2 = \{(a,0.5),(b,0.6)\}$, $F_3 = \{(a,0.6),(b,0.6)\}$, $F_4 = \{(a,0.5),(b,0.4)\}$.

Define $\tau, \tau^* : I^X \to D(I)$ as follows:

$$\tau(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{0_X, 1_X\}, \\ [0.7,0.8] & \text{if } A = F_1, \\ [0.4,0.5] & \text{if } A = F_2, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$\tau^*(A) = \begin{cases} \mathbf{0} & \text{if } A \in \{0_X, 1_X\}, \\ [0.1,0.2] & \text{if } A = F_1, \\ [0.3,0.4] & \text{if } A = F_2, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Let $[r,s] = [0.5,0.6]$ and $[t,u] = [0.2,0.3]$. Then F_2 is an $([r,s],[t,u])$-IVIFαGCS, but F_2 is not an $([r,s],[t,u])$-IVIFαCS. Also F_4 is an $([r,s],[t,u])$-IVIFαGOS, but F_4 is not an $([r,s],[t,u])$-IVIFαOS.

EXAMPLE 5.3. The intersection of two $([r,s],[t,u])$-IVIFαGCSs need not be an $([r,s],[t,u])$-IVIFαGCS and the union of two $([r,s],[t,u])$-IVIFαGOSs need not be an $([r,s],[t,u])$-IVIFαGOS.

Let $X = \{a,b,c\}$. Define $G_1, G_2, G_3 \in I^X$ as follows:

$$G_1 = \{(a,1),(b,0),(c,0)\},\ G_2 = \{(a,1),(b,1),(c,0)\},$$
$$G_3 = \{(a,1),(b,0),(c,1)\}.$$

Define $\tau, \tau^* : I^X \to D(I)$ as follows:

$$\tau(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{0_X, 1_X\}, \\ [0.7,0.8] & \text{if } A = G_1, \\ [0.5,0.6] & \text{if } A = G_2, \\ [0.3,0.4] & \text{if } A = G_3, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$\tau^*(A) = \begin{cases} \mathbf{0} & \text{if } A \in \{0_X, 1_X\}, \\ [0.1,0.2] & \text{if } A = G_1, \\ [0.3,0.4] & \text{if } A = G_2, \\ [0.5,0.6] & \text{if } A = G_3, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Let $[r,s] = [0.6,0.7]$ and $[t,u] = [0.2,0.3]$. Then the only $([r,s],[t,u])$-IVIFOSs are 0_X, 1_X and G_1 and the only $([r,s],[t,u])$-IVIFCSs are 0_X, 1_X and G_1^c. Also $G_1 \subseteq G_2$ and $G_1 \subseteq G_3$. Let $G_2 \subseteq U$ and let U be an $([r,s],[t,u])$-IVIFOS. Then $U = 1_X$ and so $\alpha cl_{[r,s],[t,u]}(G_2) \subseteq 1_X = U$. Hence G_2 is an $([r,s],[t,u])$-IVIFαGCS. Similarly, G_3 is also an $([r,s],[t,u])$-IVIFαGCS. Now $G_2 \cap G_3 = G_1$. $cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(G_1))) = cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(1_X)) = cl_{[r,s],[t,u]}\ (1_X) = 1_X \nsubseteq G_1$. Hence G_1 is not an $([r,s],[t,u])$-IVIFαCS. Let $G_1 \subseteq U$ and let U be an $([r,s],[t,u])$-IVIFOS. Then $U = G_1$ or $U = 1_X$. In the case $U = G_1$, by Theorem 3.8(v) $G_1 \neq \alpha cl_{[r,s],[t,u]}(G_1)$ since G_1 is not an $([r,s],[t,u])$-IVIFαCS. Thus $\alpha cl_{[r,s],[t,u]}(G_1) \supsetneq G_1$. Hence $\alpha cl_{[r,s],[t,u]}(G_1) \nsubseteq G_1 = U$. Thus G_1 is not an $([r,s],[t,u])$-IVIFαGCS.

By taking the complementation in the above example, the union of two $([r,s],[t,u])$-IVIFαGOSs need not be an $([r,s],[t,u])$-IVIFαGOS.

DEFINITION 5.4. Let (X,τ,τ^*) be an IVISTS, $A \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s + u \leq 1$. The $([r,s],\ [t,u])$-*interval-valued* *intuitionistic fuzzy α-generalized closure* and $([r,s],[t,u])$*-interval-valued intuitionistic fuzzy α-generalized interior* of A are defined by

$\alpha gcl_{[r,s],[t,u]}(A) = \cap\{K \in I^X : A \subseteq K,\ K$ is an $([r,s],[t,u])$-IVIFαGCS$\}$,

$\alpha gint_{[r,s],[t,u]}(A) = \cup\{G \in I^X : G \subseteq A,\ G$ is an $([r,s],[t,u])$-IVIFαGOS$\}$.

Note that $\alpha int_{[r,s],[t,u]}(A) \subseteq \alpha gint_{[r,s],[t,u]}(A) \subseteq A \subseteq \alpha gcl_{[r,s],[t,u]}(A) \subseteq \alpha cl_{[r,s],[t,u]}\ (A)$.

THEOREM 5.5. *Let (X,τ,τ^*) be an IVISTS, $A, B \in I^X$ and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$. Then*

(i) $\alpha gcl_{[r,s],[t,u]}(0_X) = 0_X$.

(ii) $A \subseteq \alpha gcl_{[r,s],[t,u]}(A)$.

(iii) $\alpha gcl_{[r,s],[t,u]}(A) \subseteq \alpha gcl_{[r,s],[t,u]}(B)$ *if* $A \subseteq B$.

(iv) $\alpha gcl_{[r,s],[t,u]}(A \cup B) \supseteq \alpha gcl_{[r,s],[t,u]}(A) \cup \alpha gcl_{[r,s],[t,u]}(B)$,
$\alpha gcl_{[r,s],[t,u]}(A \cap B) \subseteq \alpha gcl_{[r,s],[t,u]}(A) \cap \alpha gcl_{[r,s],[t,u]}(B)$.

(v) $A = \alpha gcl_{[r,s],[t,u]}(A)$ *if A is an $([r,s],[t,u])$-IVIFαGCS.*

(vi) $\alpha gcl_{[r,s],[t,u]}(\alpha gcl_{[r,s],[t,u]}(A)) = \alpha gcl_{[r,s],[t,u]}(A)$.

(vii) $\alpha gcl_{[r,s],[t,u]}(A^c) = (\alpha gint_{[r,s],[t,u]}(A))^c$.

Proof. (i), (ii) and (iii) follow directly from Definition 5.4.

(iv) It follows directly from (iii).

(v) It follows directly from Definition 5.4.

(vi) By (ii) and (iii), $\alpha gcl_{[r,s],[t,u]}(A) \subseteq \alpha gcl_{[r,s],[t,u]}(\alpha gcl_{[r,s],[t,u]}(A))$. Suppose that $\alpha gcl_{[r,s],[t,u]}(\alpha gcl_{[r,s],[t,u]}(A)) \nsubseteq \alpha gcl_{[r,s],[t,u]}(A)$. Then there exists $x \in X$ such that $(\alpha gcl_{[r,s],[t,u]}(A))(x) < (\alpha gcl_{[r,s],[t,u]}(\alpha gcl_{[r,s],[t,u]}(A)))(x)$. Choose $a \in (0,1)$ with $(\alpha gcl_{[r,s],[t,u]}(A))(x) < a < (\alpha gcl_{[r,s],[t,u]}(\alpha gcl_{[r,s],[t,u]}(A)))(x)$. Since $(\alpha gcl_{[r,s],[t,u]}(A))(x) < a$, by Definition 5.4 there exists an $([r,s],[t,u])$-IVIFαGCS K such that $A \subseteq K$ and $K(x) < a$. Since K is an $([r,s],[t,u])$-IVIFαGCS with $A \subseteq K$, $\alpha gcl_{[r,s],[t,u]}(A) \subseteq K$ and also $\alpha gcl_{[r,s],[t,u]}\ (\alpha gcl_{[r,s],[t,u]}(A)) \subseteq K$. Hence $(\alpha gcl_{[r,s],[t,u]}(\alpha gcl_{[r,s],[t,u]}(A)))(x) \leq K(x) < a$. This is a contradiction. Hence $\alpha gcl_{[r,s],[t,u]}(\alpha gcl_{[r,s],[t,u]}(A)) \subseteq \alpha gcl_{[r,s],[t,u]}(A)$. Therefore $\alpha gcl_{[r,s],[t,u]}(\alpha gcl_{[r,s],[t,u]}(A)) = \alpha gcl_{[r,s],[t,u]}(A)$.

(vii) By Definition 5.4, we have

$$\begin{aligned}
&\alpha gcl_{[r,s],[t,u]}(A^c) \\
&= \cap\{K \in I^X : A^c \subseteq K,\ K \text{ is an } ([r,s],[t,u])\text{-IVIF}\alpha\text{GCS}\} \\
&= \cap\{G^c \in I^X : A^c \subseteq G^c,\ G^c \text{ is an } ([r,s],[t,u])\text{-IVIF}\alpha\text{GCS}\} \\
&= (\cup\{G \in I^X : G \subseteq A,\ G \text{ is an } ([r,s],[t,u])\text{-IVIF}\alpha\text{GOS}\})^c \\
&= (\alpha gint_{[r,s],[t,u]}(A))^c.
\end{aligned}$$

□

THEOREM 5.6. *Let* (X, τ, τ^*) *be an IVISTS,* $A, B \in I^X$ *and* $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ *with* $s + u \leq 1$. *Then*

(i) $\alpha gint_{[r,s],[t,u]}(1_X) = 1_X$.
(ii) $\alpha gint_{[r,s],[t,u]}(A) \subseteq A$.
(iii) $\alpha gint_{[r,s],[t,u]}(A) \subseteq \alpha gint_{[r,s],[t,u]}(B)$ *if* $A \subseteq B$.
(iv) $\alpha gint_{[r,s],[t,u]}(A \cup B) \supseteq \alpha gint_{[r,s],[t,u]}(A) \cup \alpha gint_{[r,s],[t,u]}(B)$,
$\alpha gint_{[r,s],[t,u]}(A \cap B) \subseteq \alpha gint_{[r,s],[t,u]}(A) \cap \alpha gint_{[r,s],[t,u]}(B)$.
(v) $A = \alpha gint_{[r,s],[t,u]}(A)$ *if* A *is an* $([r, s], [t, u])$*-IVIF*α*GOS.*
(vi) $\alpha gint_{[r,s],[t,u]}(\alpha gint_{[r,s],[t,u]}(A)) = \alpha gint_{[r,s],[t,u]}(A)$.
(vii) $\alpha gint_{[r,s],[t,u]}(A^c) = (\alpha gcl_{[r,s],[t,u]}(A))^c$.

Proof. The proof is similar to Theorem 5.5. □

6. $([r, s], [t, u])$-interval-valued intuitionistic fuzzy alpha generalized continuous mappings

DEFINITION 6.1. Let (X, τ, τ^*) and (Y, η, η^*) be two IVISTSs and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$ and let $f : X \to Y$ be a mapping. f is called an $([r, s], [t, u])$*-interval-valued intuitionistic fuzzy* α*-generalized continuous mapping* (for short, $([r, s], [t, u])$-IVIFαG continuous mapping) if $f^{-1}(B)$ is an $([r, s], [t, u])$-IVIFαGCS of X for each $([r, s], [t, u])$-IVIFCS B of Y.

Note that $f : X \to Y$ is an $([r, s], [t, u])$-IVIFαG continuous mapping if and only if $f^{-1}(B)$ is an $([r, s], [t, u])$-IVIFαGOS of X for each $([r, s], [t, u])$-IVIFOS B of Y and that if $f : X \to Y$ is an $([r, s], [t, u])$-IVIFα-continuous mapping then $f : X \to Y$ is an $([r, s], [t, u])$-IVIFαG continuous mapping.

EXAMPLE 6.2. Every $([r, s], [t, u])$-IVIFαG continuous mapping need not be an $([r, s], [t, u])$-IVIFα-continuous mapping.

Let $X = \{a, b\}$ and $Y = \{c, d\}$. Define $F_1, F_2, F_3 \in I^X$ and $G_1, G_2 \in I^Y$ as follows:

$$F_1 = \{(a, 0.4), (b, 0.4)\},\ F_2 = \{(a, 0.5), (b, 0.6)\},\ F_3 = \{(a, 0.6), (b, 0.6)\},$$
$$G_1 = \{(c, 0.5), (d, 0.4)\},\ G_2 = \{(c, 0.5), (d, 0.6)\}.$$

Define $\tau, \tau^* : I^X \to D(I),\ \eta, \eta^* : I^Y \to D(I)$ as follows:

$$\tau(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{0_X, 1_X\}, \\ [0.7, 0.8] & \text{if } A = F_1, \\ [0.4, 0.5] & \text{if } A = F_2, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$\tau^*(A) = \begin{cases} \mathbf{0} & \text{if } A \in \{0_X, 1_X\}, \\ [0.1, 0.2] & \text{if } A = F_1, \\ [0.3, 0.4] & \text{if } A = F_2, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

$$\eta(B) = \begin{cases} \mathbf{1} & \text{if } B \in \{0_Y, 1_Y\}, \\ [0.8, 0.9] & \text{if } B = G_1, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$\eta^*(B) = \begin{cases} \mathbf{0} & \text{if } B \in \{0_Y, 1_Y\}, \\ [0.1, 0.2] & \text{if } B = G_1, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Define the mapping $f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)$ by $f(a) = c, f(b) = d$ and let $[r, s] = [0.5, 0.6]$ and $[t, u] = [0.2, 0.3]$. Then f is an $([r, s], [t, u])$-IVIFαG continuous mapping, but f is not an $([r, s], [t, u])$-IVIFα-continuous mapping.

Definition 6.3. An IVISTS (X, τ, τ^*) is called an *interval-valued intuitionistic fuzzy alpha* $T^*_{1/2}$ *space* (for short, IVIFαT$^*_{1/2}$ space) if for each $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$, every $([r, s], [t, u])$-IVIFαGCS in X is an $([r, s], [t, u])$-IVIFCS in X.

Theorem 6.4. *Let (X, τ, τ^*) be an IVIF$\alpha T^*_{1/2}$ space and (Y, η, η^*) an IVISTS and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$ and let $f : X \to Y$ be a mapping. Then the following statements are equivalent.*

(i) *f is an $([r, s], [t, u])$-IVIFαG continuous mapping.*

(ii) *$f^{-1}(int_{[r,s],[t,u]}(B)) \subseteq int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(B))))$ for each $B \in I^Y$.*

Proof. (i)$\Rightarrow$(ii). Let $B \in I^Y$. Then $int_{[r,s],[t,u]}(B)$ is an $([r, s], [t, u])$-IVIFOS of Y. Since f is an $([r, s], [t, u])$-IVIFαG continuous mapping, $f^{-1}(int_{[r,s],[t,u]}(B))$ is an $([r, s], [t, u])$-IVIFαGOS of X. Since X is an

IVIF$\alpha T^*_{1/2}$ space, $f^{-1}(int_{[r,s],[t,u]}(B))$ is an $([r,s],[t,u])$-IVIFOS of X. Hence

$$\begin{aligned} f^{-1}(int_{[r,s],[t,u]}(B)) &= int_{[r,s],[t,u]}(f^{-1}(int_{[r,s],[t,u]}(B))) \\ &\subseteq int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(int_{[r,s],[t,u]}(B)))) \\ &= int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(int_{[r,s],[t,u]}(B))))) \\ &\subseteq int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(B)))). \end{aligned}$$

(ii)⇒(i). Let B be an $([r,s],[t,u])$-IVIFCS of Y. Then B^c is an $([r,s],[t,u])$-IVIFOS of Y and so $int_{[r,s],[t,u]}(B^c) = B^c$. By hypothesis,

$$f^{-1}(B^c) = f^{-1}(int_{[r,s],[t,u]}(B^c)) \subseteq int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(f^{-1}(B^c)))).$$

Thus $f^{-1}(B^c)$ is an $([r,s],[t,u])$-IVIFαOS of X. Since every $([r,s],[t,u])$-IVIFαOS is an $([r,s],[t,u])$-IVIFαGOS, $f^{-1}(B^c)$ is an $([r,s],[t,u])$-IVIFαGOS of X. Hence $f^{-1}(B)$ is an $([r,s],[t,u])$-IVIFαGCS of X. Therefore f is an $([r,s],[t,u])$-IVIFαG continuous mapping. □

By taking the complement of the set $B \in I^Y$ in Theorem 6.4, we obtain the following corollary.

Corollary 6.5. *Let (X,τ,τ^*) be an IVIF$\alpha T^*_{1/2}$ space and (Y,η,η^*) an IVISTS and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \le 1$ and let $f : X \to Y$ be a mapping. Then the following statements are equivalent.*

(i) *f is an $([r,s],[t,u])$-IVIFαG continuous mapping.*

(ii) *$cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(B)))) \subseteq f^{-1}(cl_{[r,s],[t,u]}(B))$ for each $B \in I^Y$.*

Definition 6.6. An IVISTS (X,τ,τ^*) is called an *interval-valued intuitionistic fuzzy alpha $T_{1/2}$ space* (for short, IVIFαT$_{1/2}$ space) if for each $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \le 1$, every $([r,s],[t,u])$-IVIFαGCS in X is an $([r,s],[t,u])$-IVIFαCS in X.

Theorem 6.7. *Let (X,τ,τ^*) be an IVIF$\alpha T_{1/2}$ space and (Y,η,η^*) an IVISTS and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \le 1$ and let $f : X \to Y$ be a mapping. Then the following statements are equivalent.*

(i) *f is an $([r,s],[t,u])$-IVIFαG continuous mapping.*

(ii) *$f(\alpha gcl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$.*

(iii) *$\alpha gcl_{[r,s],[t,u]}(f^{-1}(B)) \subseteq f^{-1}(cl_{[r,s],[t,u]}(B))$ for each $B \in I^Y$.*

(iv) *$f^{-1}(int_{[r,s],[t,u]}(B)) \subseteq \alpha gint_{[r,s],[t,u]}(f^{-1}(B))$ for each $B \in I^Y$.*

Proof. (i)$\Rightarrow$(ii). Let $A \in I^X$. Then $cl_{[r,s],[t,u]}(f(A))$ is an $([r,s],\ [t,u])$-IVIFCS of Y. Since f is an $([r,s],[t,u])$-IVIFαG continuous mapping, $f^{-1}(cl_{[r,s],[t,u]}(f(A)))$ is an $([r,s],\ [t,u])$-IVIFαGCS of X. Since $A \subseteq f^{-1}(cl_{[r,s],[t,u]}\ (f(A)))$, by Definition 5.3 $\alpha gcl_{[r,s],[t,u]}(A) \subseteq f^{-1}(cl_{[r,s],[t,u]}(f(A)))$. Hence $f(\alpha gcl_{[r,s],[t,u]}(A)) \subseteq f(f^{-1}(cl_{[r,s],[t,u]}(f(A)))) \subseteq cl_{[r,s],[t,u]}(f(A))$.

(ii)$\Rightarrow$(iii). Let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. By (ii),

$$\begin{aligned} f(\alpha gcl_{[r,s],[t,u]}(f^{-1}(B))) &\subseteq cl_{[r,s],[t,u]}(f(f^{-1}(B))) \\ &\subseteq cl_{[r,s],[t,u]}(B). \end{aligned}$$

Hence

$$\begin{aligned} \alpha gcl_{[r,s],[t,u]}(f^{-1}(B)) &\subseteq f^{-1}(f(\alpha gcl_{[r,s],[t,u]}(f^{-1}(B)))) \\ &\subseteq f^{-1}(cl_{[r,s],[t,u]}(B)). \end{aligned}$$

(iii)$\Rightarrow$(iv). Let $B \in I^Y$. By (iii), $\alpha gcl_{[r,s],[t,u]}(f^{-1}(B^c)) \subseteq f^{-1}(cl_{[r,s],[t,u]}(B^c))$. Thus $(\alpha gint_{[r,s],[t,u]}(f^{-1}(B)))^c \subseteq (f^{-1}(int_{[r,s],[t,u]}(B)))^c$. Hence $f^{-1}(int_{[r,s],[t,u]}\ (B)) \subseteq \alpha gint_{[r,s],[t,u]}(f^{-1}(B))$.

(iv)$\Rightarrow$(i). Let B be an $([r,s],[t,u])$-IVIFCS of Y. Then $f^{-1}(B) \in I^X$ and B^c is an $([r,s],[t,u])$-IVIFOS of Y and so $int_{[r,s],[t,u]}(B^c) = B^c$. Let $f^{-1}(B) \subseteq U$ and let U be an $([r,s],[t,u])$-IVIFOS of X. By (iv),

$$\begin{aligned} (f^{-1}(B))^c = f^{-1}(B^c) &= f^{-1}(int_{[r,s],[t,u]}(B^c)) \\ &\subseteq \alpha gint_{[r,s],[t,u]}(f^{-1}(B^c)) \\ &= (\alpha gcl_{[r,s],[t,u]}(f^{-1}(B)))^c. \end{aligned}$$

Hence $\alpha gcl_{[r,s],[t,u]}(f^{-1}(B)) \subseteq f^{-1}(B)$ and so $\alpha gcl_{[r,s],[t,u]}(f^{-1}(B)) = f^{-1}(B)$. Since (X,τ,τ^*) is an IVIFαT$_{1/2}$ space, $\alpha gcl_{[r,s],[t,u]}(f^{-1}(B)) = \alpha cl_{[r,s],[t,u]}\ (f^{-1}(B))$. Hence $\alpha cl_{[r,s],[t,u]}(f^{-1}(B)) = f^{-1}(B) \subseteq U$. Thus $f^{-1}(B)$ is an $([r,s],[t,u])$-IVIFαGCS of X. Therefore f is an $([r,s],[t,u])$-IVIFαG continuous mapping. $\square$

Since $\alpha gcl_{[r,s],[t,u]}(A) = \alpha cl_{[r,s],[t,u]}\ (A)$ for each $A \in I^X$ in an IVIFαT$_{1/2}$ space X, we obtain the following corollary from Theorem 6.7.

Corollary 6.8. *Let (X,τ,τ^*) be an IVIFαT$_{1/2}$ space and (Y,η,η^*) an IVISTS and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. Then f is an $([r,s],[t,u])$-IVIFαG continuous mapping if and only if $f(\alpha cl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$.*

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