Korean J. Math. **25** (2017), No. 2, pp. 163–179
https://doi.org/10.11568/kjm.2017.25.2.163

ON LEFT DERIVATIONS OF BCH-ALGEBRAS

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Abstract. In this paper, we introduce the notion of left derivations of BCH algebras and investigate some properties of left derivations in a BCH-algebra. Moreover, we introduce the notions of fixed set and kernel set of derivations in a BCH-algebra and obtained some interesting properties in medial BCH-algebras. Also, we discuss the relations between ideals in a medial BCH-algebras.

1. Introduction

In 1966, Imai and Iseki introduced two classes of abstract algebras, BCK-algebra and BCI-algebras [6]. It is known that the class of BCI-algebras is a generalization of the class of BCK-algebras In 1983, Hu and Li [3] introduced the notion of a BCH-algebra, which is a generalization of the notions of BCK-algebras and BCI-algebras. They have studied a few properties of these algebras. In this paper, we introduce the notion of left derivations of BCH algebras and investigate some properties of left derivations in a BCH-algebra. Moreover, we introduce the notions of fixed set and kernel set of derivations in a BCH-algebra and obtained some interesting properties in medial BCH-algebras. Also, we discuss the relations between ideals in a medial BCH-algebras.

Received November 4, 2016. Revised April 24, 2017. Accepted April 26, 2017.
2010 Mathematics Subject Classification: 03G25, 06B10, 06D99, 06B35, 06B99.
Key words and phrases: BCH-algebra, derivation, left derivation, isotone, fixed set, medial.
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2. Preliminary

By a *BCH-algebra,* we mean an algebra $(X, *, 0)$ with a single binary operation "$*$" that satisfies the following identities, for any $x, y, z \in X$,

(BCH1) $x * x = 0$,
(BCH2) $x \leq y$ and $y \leq x$ imply $x = y$, where $x \leq y$ if and only if $x * y = 0$.
(BCH3) $(x * y) * z = (x * z) * y$.

In a BCH-algebra X, the following identities are true, for all $x, y \in X$,

(BCH4) $(x * (x * y)) * y = 0$,
(BCH5) $x * 0 = 0$ implies $x = 0$,
(BCH6) $0 * (x * y) = (0 * x) * (0 * y)$,
(BCH7) $x * 0 = x$,
(BCH8) $(x * y) * x = 0 * y$,
(BCH9) $x * y = 0$ implies $0 * x = 0 * y$,
(BCH10) $x * (x * y) \leq y$.

Definition 2.1. Let X be a BCH-algebra. we define a partial order $\leq$ by putting $x \leq y$ if and only if $x * y = 0$, for every $x, y \in X$.

In a BCH-algebra X,, the following identity hold:

(BCH11) $x \leq y$ implies $x * z \leq y * z$ but $x \leq y$ implies $z * y \leq z * x$ does not hold.

Definition 2.2. Let I be a nonempty subset of a BCH-algebra X. Then I is called an *ideal* of X if it satisfies:

(1) $0 \in I$,
(2) $x * y \in I$ and $y \in I$ imply $x \in I$.

Definition 2.3. A BCH-algebra is said to be *medial* if it satisfies

$$(x * y) * (z * w) = (x * z) * (y * w)$$

for all $x, y, z, w \in X$.

In a medial BCH-algebra X, the following identity hold:

(BCH12) $x * (x * y) = y$, for all $x, y \in X$.

Definition 2.4. Let X be a BCH-algebra. Then the set $X_+ = \{x \in X | 0 * x = 0\}$ is called a *BCH-part* of X.

Definition 2.5. A BCH-algebra X is said to be *commutative* if for all $x, y \in X$, $y * (y * x) = x * (x * y)$, i.e., $x \wedge y = y \wedge x$. For a BCH-algebra X, we denote $x \wedge y = y * (y * x)$, for all $x, y \in X$.

Definition 2.6. Let X be a BCH-algebra. A map $d : X \to X$ is a *left-right derivation* (briefly, (l, r)-*derivation*) of X if it satisfies the identity

$$d(x * y) = (d(x) * y) \wedge (x * d(y))$$

for all $x, y \in X$. If d satisfies the identity

$$d(x * y) = (x * d(y)) \wedge (d(x) * y)$$

for all $x, y \in X$, then d is a *right-left derivation* (briefly, (r, l)-*derivation*) of X. Moreover, if d is both an (l, r) and (r, l)-derivation of X, then d is a *derivation* of X.

3. Left derivations of BCH-algebras

In what follows, let X denote a BCH-algebra unless otherwise specified.

Definition 3.1. Let X be a BCH-algebra. By a *left derivation* of X, we mean a self-map d satisfying

$$d(x * y) = (x * d(y)) \wedge (y * d(x))$$

for all $x, y \in X$.

Example 3.2. Let $X = \{0, 1, 2\}$ be a BCH-algebra with Cayley table as follows:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define a self-map $d : X \to X$ by

$$d(x) = \begin{cases} 2 & \text{if } x = 0, 1 \\ 0 & \text{if } x = 2 \end{cases}$$

Then it is easy to check that d is a left derivation of a BCH-algebra X.

Example 3.3. Let $X = \{0, 1, 2, 3\}$ be a BCH-algebra with Cayley table as follows:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define a self-map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 3 \\ 1 & \text{if } x = 2 \\ 2 & \text{if } x = 1 \\ 3 & \text{if } x = 0 \end{cases}$$

Then it is easy to check that d is a left derivation of a BCH-algebra X.

Definition 3.4. A self-map d of a BCH-algebra X is said to be *regular* if $d(0) = 0$.

Proposition 3.5. *Let d be a left derivation of X. If $0 * x = 0$, for every $x \in X$, then d is regular.*

Proof. Let $0 * x = 0$, for all $x \in X$. Then we have

$$\begin{aligned} d(0) &= d(0 * x) = (0 * d(x)) \wedge (x * d(0)) \\ &= 0 \wedge (x * d(0)) = (x * d(0)) * ((x * d(0)) * 0) \\ &= (x * d(0)) * ((x * d(0)) = 0 \end{aligned}$$

Hence d is regular. □

Proposition 3.6. *Let d be a left derivation of X. If there exists $a \in X$ such that $a * d(x) = 0$, for all $x \in X$, then d is regular.*

Proof. Let $a * d(x) = 0$ for all $x \in X$. Then

$$\begin{aligned} 0 &= a * d(a * x) = a * ((a * d(x)) \wedge (x * d(a))) \\ &= a * (0 \wedge (x * d(a))) \\ &= a * ((x * d(a)) * ((x * d(a)) * 0)) \\ &= a * 0 \\ &= a. \end{aligned}$$

Hence we have

$$\begin{aligned} d(0) &= d(a) \\ &= d(a * 0) \\ &= (a * d(0)) \wedge (0 * d(a)) \\ &= 0 \wedge (0 * d(a)) \\ &= 0. \end{aligned}$$

Hence d is regular. □

Proposition 3.7. *Let d be a regular left derivation of X. Then, for all $x \in X$,*
(1) $x \leq d(x)$,
(2) $x \leq d(d(x))$.

Proof. (1) Let d be a regular left derivation of X. Then we have

$$\begin{aligned} 0 = d(x * x) &= (x * d(x)) \wedge (x * d(x)) \\ &= (x * d(x)) * ((x * d(x)) * (x * d(x))) = (x * d(x)) * 0 \\ &= x * d(x), \end{aligned}$$

which implies $x \leq d(x)$, for all $x \in X$.
(2) From (1), $x \leq d(x) \leq d(d(x))$, for all $x \in X$. □

Theorem 3.8. *Let d be a left derivation of a medial BCH-algebra X. Then $d(x * y) = x * d(y)$, for all $x, y \in X$.*

Proof. Let $x, y \in X$. Then we have

$$d(x*y) = (x*d(y)) \wedge (y*d(x)) = (y*d(x))*((y*d(x))*(x*d(y))) = x*d(y).$$

□

Proposition 3.9. *Let X be a BCH-algebra. Then*

$$d_n(d_{n-1}(...(d_2(d_1(x)))...)) \leq x$$

for $n \in \mathbb{N}$, where $d_1, d_2, ..., d_n$ are regular left derivations of X.

Proof. For $n = 1$,

$$d_1(x) = d_1(x * 0) = (x * d_1(0)) \wedge (0 * d_1(x)) = x \wedge (0 * d_1(x)) \leq x,$$

by (BCH10). Hence we have $d_1(x) \leq x$. □

Let $n \in \mathbb{N}$ and $d_n(d_{n-1}(...(d_2(d_1(x)))...)) \leq x$. For simplicity, let

$$D_n = d_n(d_{n-1}(...(d_2(d_1(x)))...)).$$

Then

$$\begin{aligned} d_{n+1}(D_n) &= d_{n+1}(D_n * 0) = (D_n * d_{n+1}(0)) \wedge (0 * d_{n+1}(D_n)) \\ &= (0 * d_{n+1}(D_n)) * ((0 * d_{n+1}(D_n)) * D_n) \leq D_n. \end{aligned}$$

Hence $d_{n+1}(D_n) \leq D_n$, that is, $D_{n+1} \leq D_n \leq x$ by assumption, which implies $D_{n+1} \leq x$.

Theorem 3.10. *Let X be a medial BCH-algebra and let d be a left derivation of X. Then d is regular if and only if $d(x) = x$, for all $x \in X$.*

Proof. Let d be regular. Then we have $d(0) = 0$. Hence,

$$\begin{aligned} d(x) &= d(x * 0) \\ &= (x * d(0)) \wedge (0 * d(x)) \\ &= (x * 0) \wedge (0 * d(x)) \\ &= (0 * d(x)) * ((0 * d(x)) * x) \\ &= x. \end{aligned}$$

Hence $d(x) = x$ for $x \in X$. Conversely, assume that $d(x) = x$ for all $x \in X$. Then it is clear that $d(0) = 0$. This implies that d is regular. □

Proposition 3.11. *Let d be a left derivation of X. Then we have*
(1) $x * d(x) = y * d(y)$,
(2) $d(x * y) \leq x * d(y)$,
(3) $d(d(x) * x) = 0$, *for every $x, y \in X$.*

Proof. (1) Since $x * x = 0$ for all $x \in X$, we have $d(0) = d(x * x) = (x * d(x)) \wedge (x * d(x)) = x * d(x)$. Similarly, $d(0) = y * d(y))$, which implies $x * d(x) = y * d(y)$.

(2) Let d be a left derivation of a BCH-algebra of X and $x, y \in X$. Then

$$\begin{aligned} d(x * y) &= (x * d(y)) \wedge (y * d(x)) \\ &= (y * d(x)) * ((y * d(x)) * (x * (d(y)))) \\ &\leq x * d(y). \end{aligned}$$

(3) Let d be a left derivation of a BCH-algebra of X and $x \in X$. Then

$$d(d(x) * x) = (d(x) * d(x)) \wedge (x * d(d(x))) = 0 \wedge (x * d(d(x))) = 0.$$

□

PROPOSITION 3.12. *Let d be a regular left derivation of X. Then $d : X \to X$ is an identity map if it satisfies $d(x) * y = x * d(y)$, for all $x, y \in X$.*

Proof. Since d is regular, we have $d(0) = 0$. Let $x * d(y) = d(x) * y$ for all $x, y \in X$. Then $d(x) = d(x) * 0 = x * d(0) = x * 0 = x$. Thus d is an identity map. □

PROPOSITION 3.13. *Let X be a medial BCH-algebra X and let d be a regular left derivation of X. Define $d^2(x) = d(d(x))$, for all $x \in X$. Then $d^2 = d$.*

Proof. Let X be a medial BCH-algebra X and let d be a regular left derivation of X. Then we have for all $x \in X$,

$$\begin{aligned} d^2(x) &= d(d(x)) = d(d(x * 0)) \\ &= (d(x) * d(0)) \wedge (d(0) * d(d(x))) \\ &= (d(x) * 0) \wedge (0 * d(d(x))) \\ &= d(x) \wedge (0 * d(d(x))) \\ &= (0 * d(d(x))) * [(0 * d(d(x))) * d(x)] = d(x). \end{aligned}$$

□

DEFINITION 3.14. Let X be a BCH-algebra. A self-map d on X is said to be *isotone* if $x \leq y$ implies $d(x) \leq d(y)$, for $x, y \in X$.

PROPOSITION 3.15. *Let X be a medial commutative BCH-algebra and let d be a regular left derivation of X. Then the following properties are equivalent:*

(1) *d is an isotone derivation of X,*

(2) *$x \leq y$ implies $d(x * y) = d(x) * y$.*

Proof. (1) $\Rightarrow$ (2). Let $x, y \in X$ such that $x \leq y$. Then we have $d(x * y) = d(x * y) = d(0) = 0$. Since d is isotone, we get $d(x) * d(y) = 0$. Thus

$$\begin{aligned} d(x * y) = 0 &= d(x) * d(y) \\ &= d(x) * (d(y) \wedge y) \\ &= d(x) * (y * (y * d(y))) \\ &= d(x) * (d(y) * (d(y) * y)) \\ &= d(x) * y. \end{aligned}$$

(2) $\Rightarrow$ (1). Let $x, y \in X$ such that $x \leq y$. Thus

$$\begin{aligned} d(x) * d(y) &= d(x) * (d(y) \wedge y) \\ &= d(x) * (y * (y * d(y))) \\ &= d(x) * (d(y) * (d(y) * y)) \\ &= d(x) * y = d(x * y) \\ &= d(0) = 0. \end{aligned}$$

Hence $d(x) \leq d(y)$, which implies that d is an isotone derivation of X. □

Proposition 3.16. *Let X be a medial commutative BCH-algebra and let d be a regular left derivation of X. Then the following properties are equivalent:*

(1) $d(x * y) = d(x) * y$,

(2) $d(x * y) = d(x) * d(y)$, *for all* $x, y \in X$.

Proof. (1) $\Rightarrow$ (2). Let X be a medial commutative BCH-algebra and let d be a regular left derivation of X. Then we have

$$\begin{aligned} d(x * y) &= d(x) * y = d(x) * (y \wedge d(y)) = d(x) * ((dy) * (d(y) * y)) \\ &= d(x) * (y * (y * d(y)) = d(x) * d(y). \end{aligned}$$

Hence $d(x * y) = d(x) * d(y)$.

(2) $\Rightarrow$ (1). Let X be a medial BCH-algebra and let d be a regular left derivation of X. Then we have

$$\begin{aligned} d(x * y) &= d(x) * d(y) \\ &= d(x) * (d(y) \wedge y) \\ &= d(x) * (y \wedge d(y)) \\ &= d(x) * y, \end{aligned}$$

which implies $d(x * y) = d(x) * y$. □

From the Proposition 3.15 and 3.16, we have the following theorem.

Theorem 3.17. *Let X be a medial commutative BCH-algebra and let d be a regular left derivation of X. Then the following properties are equivalent:*

(1) *d is an isotone derivation of X,*

(2) *$x \leq y$ implies* $d(x * y) = d(x) * y$,

(3) $d(x * y) = d(x) * d(y)$, *for all* $x, y \in X$.

THEOREM 3.18. *Let X be a medial commutative BCH-algebra and let d be a regular left derivation of X. Then the following properties are equivalent:*

(1) *d is an isotone derivation of X,*

(2) $d(x * y) = d(x) * d(y)$,

(3) $d(x \wedge y) = d(x) \wedge d(y)$, *for all $x, y \in X$.*

Proof. (1) $\Rightarrow$ (2). It follows from Theorem 3.17.

(2) $\Rightarrow$ (3). Let $d(x * y) = d(x) * d(y)$, for all $x, y \in X$. Then we have

$$\begin{aligned} d(x \wedge y) &= d(y * (y * x)) \\ &= d(y) * d(y * x) \\ &= d(y) * (d(y) * d(x)) \\ &= d(x) \wedge d(y). \end{aligned}$$

(3) $\Rightarrow$ (1). Let $d(x \wedge y) = d(x) \wedge d(y)$ and $x \leq y$. Then $x * y = 0$, for all $x, y \in X$. Hence we have

$$\begin{aligned} d(x) &= d(x * 0) \\ &= d(x * (x * y)) \\ &= d(y \wedge x) \\ &= d(x) * (d(x) * d(y)) \\ &\leq d(y). \end{aligned}$$

Hence $d(x) \leq d(y)$, which implies that d is an isotone derivation of X. □

DEFINITION 3.19. An ideal I of X is said to be *d-invariant* if $d(I) \subset I$.

PROPOSITION 3.20. *Let d be a left derivation of a medial BCH-algebra X. Then d is regular if and only if every ideal of X is d-invariant.*

Proof. Let d is regular. Then by Theorem 3.10, $d(x) = x$ for all $x \in X$. Let $y \in d(A)$, where A is an ideal of X. Then we have $y = d(x)$ for some $x \in A$. Thus

$$\begin{aligned} y * x &= d(x) * x \\ &= x * x \\ &= 0 \in A. \end{aligned}$$

This implies that $y \in A$ and $d(A) \subset A$. Conversely, let every ideal of X be d-invariant. Then $d(\{0\}) \subset \{0\}$, and so $d(0) = 0$, which implies that d is regular. □

THEOREM 3.21. *In a medial BCH-algebra X, a self-map d is a left derivation of X if and only if it is a (r, l)-derivation of X.*

Proof. Let d be a left derivation of a medal BCH-algebra X. First, we show that d is a (r, l)-derivation of X. Then

$$\begin{aligned} d(x * y) &= x * d(y) \\ &= (d(x) * y) * [(d(x) * y) * (x * d(y))] \\ &= (x * d(y)) \wedge (d(x) * y) \end{aligned}$$

for all $x, y \in X$. Conversely, let d be a (r, l)-derivation of X. Then

$$\begin{aligned} d(x * y) &= (x * d(y)) \wedge (d(x) * y) \\ &= (d(x) * y) * [(d(x) * y) * (x * d(y))] \\ &= x * d(y) = (y * d(x)) * [(y * d(x)) * (x * d(y))] \\ &= (x * d(y)) \wedge (y * d(y)). \end{aligned}$$

Hence, d is a left derivation of X. □

Let d be a left derivation of X. Define a set $Fix_d(x)$ by

$$Fix_d(X) = \{x \in X \mid d(x) = x\}.$$

PROPOSITION 3.22 *Let d be a left derivation of X. Then $Fix_d(X)$ is a subalgebra of X.*

Proof. Let $x, y \in Fix_d(X)$. Then $d(x) = x$ and $d(y) = y$, and so we have

$$\begin{aligned} d(x * y) &= (x * d(y)) \wedge (y * d(x))) = (x * y) \wedge (y * x) \\ &= (y * x) * ((y * x) * (x * y)) \\ &= x * y, \end{aligned}$$

which implies $x * y \in Fix_d(X)$. □

PROPOSITION 3.23. *Let d be a left derivation of a medial BCH-algebra X. If $x, y \in Fix_d(X)$, then $x \wedge y \in Fix_d(X)$.*

Proof. Let $x, y \in Fix_dX)$, Then $d(x) = x$ and $d(y) = y$, and so we have

$$\begin{aligned} d(x \wedge y) &= d(y * (y * x)) = (y * d(y * x)) \wedge ((y * x) * d(y)) \\ &= y * d(y * x) = (y * [(y * d(x)) \wedge (x * d(y))]) \\ &= y * (y * d(x)) = y * (y * x) \\ &= x \wedge y, \end{aligned}$$

which implies $x \wedge y \in Fix_d(X)$. □

PROPOSITION 3.24. *Let d be a left derivation of X. If $x \in Fix_d(X)$, then we have $(d \circ d)(x) = x$.*

Proof. Let $x \in Fix_d(X)$. Then we have

$$(d \circ d)(x) = d(d(x)) = d(x) = x.$$

This completes the proof. □

PROPOSITION 3.25. *Let d be a left derivation of X. If there exists $x, y \in X$ such that $x \leq y$ and $y \in Fix_d(X)$, then d is regular.*

Proof. Let x, y be such that $x \leq y$ and $d(y) = y$. Then

$$\begin{aligned} d(0) &= d(x * y) \\ &= (x * d(y)) \wedge (y * d(x)) \\ &= (x * y) \wedge (y * d(x)) \\ &= 0 \wedge (y * d(x)) \\ &= (y * d(x)) * (y * d(x)) \\ &= 0. \end{aligned}$$

Hence d is regular. □

PROPOSITION 3.26. *Let d be a left derivation of a medial commutative BCH-algebra X. If $x \leq y$ and $y \in Fix_d(X)$, then $x \in Fix_d(X)$.*

Proof. Let x, y be such that $x \leq y$ and $d(y) = y$. Then

$$\begin{aligned} d(x) &= d(x \wedge y) \\ &= d((y * (y * x)) \\ &= d(x * (x * y)) \\ &= (x * d(x * y)) \wedge ((x * y) * d(x)) \\ &= (x * ((x * d(y)) \wedge (y * d(x)))) \wedge ((x * y) * d(x)) \\ &= (x * ((x * y))) \wedge (y * d(x)))) \wedge (0 * d(x)) \\ &= (x * (0 \wedge (y * d(x))) \wedge (0 * d(x)) \\ &= x \wedge (0 * d(x)) \\ &= (0 * d(x)) * ((0 * d(x)) * x) \\ &= x. \end{aligned}$$

Hence $x \in Fix_d(X)$. □

THEOREM 3.27. *Let X be a medial BCH-algebra and let d be a left derivation of X. If $Fix_d(X) \neq \phi$, then d is regular.*

Proof. Let $y \in Fix_d(X)$. Then we get $d(y) = y$ and

$$\begin{aligned} d(0) &= d(0 \wedge y) \\ &= d(y * (y * 0)) \\ &= (y * d(y * 0)) \wedge ((y * 0) * d(y)) \\ &= (y * d(y)) \wedge (y * d(y)) \\ &= (y * y) \wedge (y * y) \\ &= 0 * 0 = 0. \end{aligned}$$

Hence d is regular. □

In the following, we will consider a left derivation d with $Fix_d(X) \neq \phi$.

THEOREM 3.28. *Let d be a left derivation of a medial BCH-algebra X. If X is commutative, then $Fix_d(X)$ is an ideal of X.*

Proof. Let X be a medial BCH-algebra and let d be a left derivation of X. Then from Theorem 3.27, d is regular, and so $0 \in Fix_d(X)$. Let $x * y \in Fix_d(X)$ and $y \in Fix_d(X)$. Then we get $d(x * y) = x * y$ and

$d(y) = y$. Thus we have

$$\begin{aligned} d(x) &= d(x \wedge y) = d(y * (y * x)) \\ &= d(x * (x * y)) \\ &= (x * d(x * y)) \wedge ((x * y) * d(x)) \\ &= (x * (x * y)) \wedge ((x * y) * d(x)) \\ &= (y * (y * x)) \wedge ((x * y) * d(x)) \\ &= x \wedge ((x * y) * d(x)) \\ &= ((x * y) * d(x)) * [((x * y) * d(x)) * x] \\ &= x \end{aligned}$$

□

THEOREM 3.29. *Let X be a medial BCH-algebra and let d_1 and d_2 be two isotone regular left derivations on X. Then $d_1 = d_2$ if and only if $Fix_{d_1}(X) = Fix_{d_2}(X)$.*

Proof. It is clear that $d_1 = d_2$ implies $Fix_{d_1}(X) = Fix_{d_2}(X)$. Conversely, let $Fix_{d_1}(X) = Fix_{d_2}(X)$ and $x \in X$. By Proposition 3.13, $d_1(x) \in Fix_{d_1}(X) = Fix_{d_2}(X)$ and so

$$d_2(d_1(x)) = d_1(x).$$

Similarly, we can have $d_1(d_2(x)) = d_2(x)$. Since d_1 and d_2 are isotone, we have $d_2(d_1(x)) \leq d_2(x) = d_1(d_2(x))$ and so $d_2(d_1(x)) \leq d_1(d_2(x))$. Symmetrically, we can also have $d_1(d_2(x)) \leq d_2(d_1(x))$, which implies that

$$d_1(d_2(x)) = d_2(d_1(x)).$$

It follows that $d_1(x) = d_2(d_1(x)) = d_1(d_2(x)) = d_2(x)$, i.e., $d_1 = d_2$. □

Let d be a left derivation of X. Define a $Kerd$ by

$$Kerd = \{x \mid d(x) = 0\}$$

for all $x \in X$.

THEOREM 3.30. *Let d be a regular left derivation of a medial BCH-algebra X. Then $Kerd$ is an ideal of X.*

Proof. Clearly, $0 \in Kerd$. Let $x * y \in Kerd$ and $y \in Kerd$. Then we have $0 = d(x * y) = x * d(y) = x * 0 = x$, and so $d(x) = d(0) = 0$. This implies $x \in Kerd$. Hence $Kerd$ is an ideal of X. □

DEFINITION 3.31. Let X be a BCH-algebra and let I be a proper ideal of X. Then I is said to be *prime* if $a \wedge b \in I$ implies $a \in I$, or $b \in I$, for any $a, b \in X$.

THEOREM 3.32. *Let X be a medial BCH-algebra. Then for any left derivation d, $Kerd$ is a prime ideal of X.*

Proof. Let X be a medial BCH-algebra and let d be a left derivation of X. Then $Kerd$ is an ideal of X by Theorem 3.30. Let $x \wedge y \in Kerd$. Then $d(x \wedge y) = 0$. Hence $d(x) = d(x \wedge y) = 0$, which implies $x \in Kerd$. This show that $Kerd$ is a prime ideal of X. □

THEOREM 3.33. *Let d be a regular left derivation of a medial BCH-algebra X. Then the following are equivalent:*

(1) *$Ker(d) = \{0\}$,*
(2) *d is one-to-one,*
(3) *d is the identity derivation.*

Proof. ((1)$\Longrightarrow$(2)) Suppose that $Ker(d) = \{0\}$ and $d(x) = d(y)$, for any $x, y \in X$. Since $x \leq d(x)$, it follows from (BCH11) that $d(x * y) = x * d(y) \leq d(x) * d(y) = 0$, so that $d(x * y) = 0$. Hence $x * y \in Ker(d)$, and so $x * y = 0$, and $x \leq y$. Similarly, we have $y \leq x$. This implies $x = y$. Therefore d is one-to-one.

((2)$\Longrightarrow$(3)) Suppose that d is one-to-one. For every $x \in X$, we have

$$d(d(x) * x) = d(x) * d(x) = 0 = d(0)$$

and so $d(x) * x = 0$, i.e., $d(x) \leq x$. Since $x \leq d(x)$ for every $x \in X$, $d(x) = x$. Hence d is an identity derivation.

((3)$\Longrightarrow$(1)) It is obvious. □

DEFINITION 3.34. Let X be a BCH-algebra and let d_1, d_2 be two self-maps of X. Define $d_1 \circ d_2 : X \to X$ by

$$(d_1 \circ d_2)(x) = d_1(d_2(x))$$

for all $x \in X$.

Let $Der_r(X)$ be the set of all isotone regular derivations of a BCH-algebra X. Then $Der_r(X)$ is a poset with the partial order defined by

$$d_1 \leq d_2 \text{ if and only if } d_1(x) \leq d_2(x) \text{ for all } x \in X,$$

for any $d_1, d_2 \in Der_r(X)$ and identity map id_X is the least element in $Der_r(X)$.

Theorem 3.35. *Let X be a medial BCH-algebra. Then $Der_r(X)$ is a semilattice with $d_1 \vee d_2 = d_1 \circ d_2$ for every $d_1, d_2 \in Der_r(X)$.*

Proof. Let $d_1, d_2 \in Der_r(X)$. Then $x \leq d_2(x)$ implies $d_1(x) \leq d_1(d_2(x))$, for all $x \in X$. Also, $d_2(x) \leq d_1(d_2(x))$, for all $x \in X$. That is, $d_1 \leq d_1 \circ d_2$ and $d_2 \leq d_1 \circ d_2$, hence $d_1 \circ d_2$ is a upper bound of d_1 and d_2.

Suppose that d is a upper bound of d_1 and d_2. Then $d_1(x) \leq d(x)$ and $d_2(x) \leq d(x)$, for all $x \in X$. These imply $d_1(d_2(x)) \leq d_1(d(x)) \leq d(d(x)) = d(x)$, for all $x \in X$, and $d_1 \circ d_2 \leq d$. Hence $d_1 \circ d_2$ is the least upper bound of d_1 and d_2. □

Proposition 3.36. *Let X be a medial BCH–algebra and let d_1, d_2 be two left derivations of X. Then $d_1 \circ d_2$ is a left derivation of X.*

Proof. Let $x, y \in X$. Then we have

$$\begin{aligned}
(d_1 \circ d_2)(x * y) &= d_1(d_2(x * y)) \\
&= d_1((x * d_2(y)) \wedge (y * d_2(x))) \\
&= d_1((y * d_2(x)) * ((y * d_2(x)) * (x * d_2(y)))) \\
&= d_1(x * d_2(y)) = x * d_1(d_2(y)) \\
&= (y * d_1(d_2(x))) * [(y * (d_1(d_2(x))) * (x * d_1(d_2(y))))] \\
&= (x * (d_1 \circ d_2)(y)) \wedge (y * (d_1 \circ d_2)(x)).
\end{aligned}$$

Hence $d_1 \circ d_2$ is a left derivation of X. □

Definition 3.37. Let X be a BCH-algebra and let d_1, d_2 be two self-maps of X. Define $d_1 \wedge d_2 : X \to X$ by

$$(d_1 \wedge d_2)(x) = d_1(x) \wedge d_2(x)$$

for all $x \in X$.

Proposition 3.38. *Let X be a medial BCH-algebra and let d_1 and d_2 be two left derivations of X. Then $d_1 \wedge d_2$ is a left derivation of X.*

Proof. Let $x, y \in X$. Then we have

$$\begin{aligned}
(d_1 \wedge d_2)(x * y) &= d_1(x * y) \wedge d_2(x * y) \\
&= (x * d_1(y)) \wedge (y * d(x)) \\
&= x * d_1(y) \\
&= (x * (d_1 \wedge d_2)(y)) \wedge (y * (d_1 \wedge d_2)(x)).
\end{aligned}$$

Hence $d_1 \wedge d_2$ is a left derivation of X. □

$Der(X)$ denote the set of all left derivations of X.

Proposition 3.39. *Let* $d_1, d_2, d_3 \in Der(X)$. *Then*

$$d_1 \wedge (d_2 \wedge d_3) = (d_1 \wedge d_2) \wedge d_3.$$

Proof. Let d_1, d_2 and d_3 be left derivations on X. Then

$$\begin{aligned} ((d_1 \wedge d_2) \wedge d_3(x * y) &= (d_1 \wedge d_2)(x * y) \wedge d_3(x * y) \\ &= d_3(x * y) * (d_3(x * y) * (d_1 \wedge d_2)(x * y)) \\ &= (d_1 \wedge d_2)(x * y) \\ &= (x * d_2(y)) * ((x * d_2(y)) * (x * d_1(y)) \\ &= x * d_1(y). \end{aligned}$$

Similarly, we have

$$\begin{aligned} d_1 \wedge (d_2 \wedge d_3)(x * y) &= d_1(x \wedge y) \wedge (d_2 \wedge d_3)(x * y) \\ &= d_1(x * y) \wedge ((d_2(x * y) \wedge d_3(x * y)) \\ &= (x * d_1(y)) \wedge ((x * d_3(y)) * ((x * d_3(y)) * x * d_2(y)))) \\ &= x * d_1(y). \end{aligned}$$

This implies that $(d_1 \wedge (d_2 \wedge d_3))(x * y) = ((d_1 \wedge d_2) \wedge d_3)(x * y)$. Put $y = 0$, we have $(d_1 \wedge (d_2 \wedge d_3))(x) = ((d_1 \wedge d_2) \wedge d_3)(x)$. This implies $d_1 \wedge (d_2 \wedge d_3) = (d_1 \wedge d_2) \wedge d_3$. □

Theorem 3.40. *Let* X *be a* BCH*-algebra. Then* $(Der(X), \wedge)$ *is a semigroup.*

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