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$\mathcal{A}$-Frequent Hypercyclicity in an Algebra of Operators

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Abstract

We study a notion of $\mathcal{A}$-frequent hypercyclicity of linear maps between the Banach algebras consisting of operators on a separable infinite dimensional Banach space. We prove a sufficient condition for a linear map to satisfy the $\mathcal{A}$-frequent hypercyclicity in the strong operator topology.

Keywords: Operator Algebras, Strong Operator Topology, Frequently Hypercyclicity

1. Preliminaries

Definition 1.1. A subfamily $\mathcal{A}$ of subsets of $\mathbb{Z}_+$ is called a hypercyclicity set if it satisfies

1. $\varnothing \notin \mathcal{A}$,
2. for any $A \in, \mathcal{A}$ if $A \subset B$, then $B \in \mathcal{A}$,
3. there is a disjoint sequence (A_k) in $\mathcal{A}$ such that for any $m \in A_k$, any $n \in A_j$, $m \neq n$,
$|m-n| \geq \max\{k, j\}$.

If $\mathcal{A}$ satisfies 1 and 2, it is called a non-trivial hereditarily upward family.

Definition 1.2. Let $\mathcal{A}$ be a non-trivial hereditarily upward family and let $T \in \mathcal{L}(X)$. The operator T is said to be $\mathcal{A}$-frequently hypercyclic if there is a vector $x \in X$ such that for any non-empty open set U of X, the return set

$$N(x, U) = \{n \in \mathbb{N} \mid T^n x \in U\}$$

is in $\mathcal{A}$. Such a vector x is called an $\mathcal{A}$-frequently hypercyclic vector for T.

Example 1.3. 1. If $\mathcal{A}$ is the family of infinite subsets of $\mathbb{Z}_+$, the $\mathcal{A}$-frequently hypercyclic operators are hypercyclic operators.

2. An operator T is frequently hypercyclic, if $\mathcal{A}$ is the family of positive lower density sets.
3. An operator T is q-frequently hypercyclic if $\mathcal{A}$ is the family of positive lower q-density sets[1]. Similarly, if $\mathcal{A}$ is the family of positive lower (m_k) -density sets, it defines an (m_k)-hypercyclic operators[2].

Proposition 1.4.[3] Let X be an infinite-dimensional separable Banach space. For a non-trivial hereditarily upward family $\mathcal{A}$, an operator T is $\mathcal{A}$-frequently hypercyclic, then $\mathcal{A}$ is a hypercyclicity set.

Theorem 1.5.[3] Let X be a separable Banach space, $\mathcal{A}$ be a hypercyclicity set and $T \in \mathcal{L}(X)$. Suppose that there exist a dense subset Y_0 of X and $S: Y_0 \to Y_0$, and disjoint sets $A_k \in \mathcal{A}$, $k \geq 1$, such that for each $y \in Y_0$,

1. $TS = 1$, the identity on Y_0,
2. there exists $k_0 \geq 1$ such that $\sum_{i \in A_k} S^n y$ converges unconditionally in X and uniformly in $k \geq k_0$,
3. for any $k_0 \geq 1$, there exists $k \geq k_0$ such that $\sum_{i \in A_k} T^n S^i y$ unconditionally convergent, uniformly in $n \in U_{l \geq 1} A_l$,
4. there exists $l_0 \geq 1$ such that $\sum_{i \in A_k} T^n S^i y$ unconditionally convergent, uniformly in $n \in U_{l \geq l_0} A_l$.

Then T is $\mathcal{A}$-frequently hypercyclic.

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2. SOT-$\mathcal{A}$-Frequently Hypercyclic Operators

Let X be a separable infinite dimensional Banach space and let $\mathcal{L}(X)$ be the algebra of bounded linear operators on X. In general, the operator algebra $\mathcal{L}(X)$ is not separable under the operator norm topology.

Theorem 2.1.[4] If X_0 is a dense subset of a separable infinite dimensional Banach space X, then there is a countable dense subset $\mathcal{L}_0(X_0)$ of $\mathcal{L}(X)$ in the strong operator topology consisting of finite rank operators whose range is contained in the span of X_0. If X is a separable infinite dimensional Banach spce, then the algebra $\mathcal{L}(X)$ is separable in the strong operator topology.

The notions with respect to the strong operator topology will be denoted by using a prefix SOT. Thus, $\mathcal{L}(X)$ is SOT-separable and a dense subset in the strong operator topology is denoted by SOT-dense. Inspired by the definitions given in[4], we introduce a notion of SOT-$\mathcal{A}$-frequent hypercyclicity.

Definition 2.2. Let $\mathcal{A}$ be a non-trivial hereditarily upward family of subsets of $\mathbb{Z}_+$. A bounded linear mapping $\mathrm{T}:\mathcal{L}(X)\to\mathcal{L}(X)$ is said to be SOT-$\mathcal{A}$-frequently hypercyclic if there is an operator $B\in\mathcal{L}(X)$ such that for each non-empty SOT-open subset $\mathcal{U}$ of $\mathcal{L}(X)$, the return set

$$\mathrm{N(B,}\mathcal{U}\mathrm{)} = \{n\in\mathbb{N}|\mathrm{T}^n(B)\in\mathcal{U}\}$$

is in $\mathcal{A}$.

Analogous to the $\mathcal{A}$-frequent hypercyclicity criterion, we define the corresponding criterion.

Definition 2.3. Let X be a separable Banach space and let $\mathcal{A}$ be a hypercyclicity set. A linear map $\mathrm{T}:\mathcal{L}(X)\to\mathcal{L}(X)$ satisfies the SOT-$\mathcal{A}$-frequent hypercyclicity criterion if there exist a countable SOT-dense subset $\mathcal{I}_0$ of $\mathcal{L}(X)$ and a map $\mathrm{S}:\mathcal{I}_0\to\mathcal{I}_0$, and disjoint sets $A_k\in\mathcal{A}$, $k\geq 1$, such that for each $B\in\mathcal{I}_0$,

1. $\mathrm{TS}=I$, the identity on $\mathcal{I}_0$,
2. there exists $k_0\geq 1$ such that $\sum_{n\in A_k}\mathrm{S}^n B$ converges unconditionally, uniformly in $k\geq k_0$,
3. for any $k_0\geq 1$, there exists $k\geq k_0$ such that $\sum_{i\in A_k}\mathrm{T}^n\mathrm{S}^i B$ unconditionally convergent, uniformly in $n\in\cup_{l\geq 1}A_l$,
4. there exists $l_0\geq 1$ such that $\sum_{i\in A_k}\mathrm{T}^n\mathrm{S}^i B$ unconditionally convergent, uniformly in $n\in\cup_{l\geq l_0}A_l$.

Theorem 2.4. Let X be a separable Banach space. If a linear map $\mathrm{T}:\mathcal{L}(X)\to\mathcal{L}(X)$ satisfies the SOT-$\mathcal{A}$-frequent hypercyclicity criterion, then T is SOT-$\mathcal{A}$-frequently hypercyclic.

Theorem 2.5. Let T be an operator on a separable Banach space X. If T satisfies the $\mathcal{A}$-frequent hypercyclicity criterion, then the left multiplication operator $\mathrm{L_T}$ satisfies the SOT-$\mathcal{A}$-frequent hypercyclicity criterion.

Proof. Let X_0 be a dense subset of X and let $(y_l)\subset X_0$ be a dense sequence. Without loss of generality, we may assume that T satisfies the $\mathcal{A}$-frequent hypercyclicity criterion for $X_0=\{y_l\mid l\geq 1\}$. Let $S:X_0\to X_0$ be the map given in the $\mathcal{A}$-frequent hypercyclicity criterion. Let $\overline{X_0}$ be the linear span of X_0 and extending S by linearly to $\overline{X_0}$, we may assume that S is linear. Since X_0 is countable, by Theorem 2.1, there is a countable SOT-dense subset $\mathcal{L}_0(X_0)$ of $\mathcal{L}(X)$ consisting of finite rank operators whose range is contained in the span of X_0. If $B\in\mathcal{L}_0(X_0)$, then since B is finite rank operator, B can be represented as

$$B(x)=\sum_{i=1}^{l}\lambda_i(x)y_i\,,\ x\in X,\ y_i\in X_0$$

where λ_i are bounded linear functional on X. Define $\Lambda_S:\mathcal{L}_0(X_0)\to\mathcal{L}_0(X_0)$ by $\Lambda_S(B)=SB$ for all $B\in\mathcal{L}_0(X_0)$. Since

$$\Lambda_S(B)(x)=SB(x)=S\left(\sum_{i=1}^{l}\lambda_i(x)y_i\right)=\sum_{i=1}^{l}\lambda_i(x)Sy_i,$$

$\Lambda_S(B)$ is a finite rank operator, and thus Λ_S is well-defined.

By condition 1 of the Theorem 1.5, we have, for all $B \in \mathcal{L}_0(X_0)$,

$$\Lambda_S \mathrm{L}_T(B) = \Lambda_S(TB) = STB.$$

Thus $\Lambda_S \mathrm{L}_T = I$ on $\mathcal{L}_0(X_0)$. For $B \in \mathcal{L}_0(X_0)$, it is enough to consider B is of the form $B(x) = \lambda(x)y$ with $y \in X_0$ and λ is a linear functional on X. Then

$$\begin{aligned}\sum_{n \in A_k} \Lambda_S^n B(x) &= \sum_{n \in A_k} S^n B(x) \\ &= \sum_{n \in A_k} S^n \lambda(x) y = \lambda(x) \sum_{n \in A_k} S^n y\end{aligned}$$

Since $\sum_{n \in A_k} S^n y$ converges unconditionally, uniformly in k, the condition 2 of the SOT-$\mathcal{A}$-frequent hypercyclicity criterion. Since

$$\begin{aligned}\sum_{i \in A_k} \mathrm{L}_T^n \Lambda_S^i B(x) &= \sum_{i \in A_k} \mathrm{L}_T^n \Lambda_S^i \lambda(x) y \\ &= \lambda(x) \sum_{i \in A_k} T^n S^i y,\end{aligned}$$

condition 3 and 4 of SOT-$\mathcal{A}$-frequent hypercyclicity criterion follows from the conditions 3 and 4 of the Theorem 1.5. Thus the left multiplication operator L_T satisfies the SOT-$\mathcal{A}$-frequent hypercyclicity criterion.

Theorem 2.6. Let T be an operator on a separable Banach space X. If $B \in \mathcal{L}(X)$ is an SOT-$\mathcal{A}$-frequently hypercyclic vector for the left multiplication operator L_T and $x \in X$ is any non-zero vector, then Bx is a $\mathcal{A}$-frequently hypercyclic vector for T.

Proof. Recall that for any $V_0 \in \mathcal{L}(X)$ and any $\epsilon > 0$, an SOT-neighborhood of V_0 is of the form

$$\mathcal{U} = \{ V \in \mathcal{L}(X) \mid \| Vx - V_0 x \| < \epsilon \}.$$

Since $B \in \mathcal{L}(X)$ is an SOT-$\mathcal{A}$-frequently hypercyclic vector for L_T, for any $V_0 \in \mathcal{L}(X)$ and $\epsilon > 0$, if $\| \mathrm{L}_T^n Bx - V_0 x \| < \epsilon$, then $n \in \mathcal{A}$. Let y be any vector in X. Then there is an operator V_0 on X such that $V_0 x = y$. By definition of the left multiplication operator, if $\| \mathrm{L}_T^n Bx - V_0 x \| < \epsilon$, then

$$\| T^n Bx - V_0 x \| = \| T^n Bx - y \| < \epsilon.$$

This proves that the vector Bx is an $\mathcal{A}$-frequently hypercyclic vector for T.

An $\mathcal{A}$-frequently hypercyclic subspace for an operator T on X is an infinite-dimensional closed subspace of X all of whose non-zero vectors are $\mathcal{A}$-frequently hypercyclic vectors. Inspired by literature[5], we have the following results.

Theorem 2.7. Let X be a separable infinite-dimensional Banach space, $T \in \mathcal{L}(X)$ and $\mathcal{A}$ a hypercyclicity set. Suppose that

(1) T satisfies the $\mathcal{A}$-frequently hypercyclicity criterion

(2) there exists an infinite-dimensional closed subspace $M_0 \subset X$ and a set $A \in \mathcal{A}$ such that $T^n x \to 0$ as $n \to \infty$, for all $x \in M_0$ and $n \in A$.

Then T has an $\mathcal{A}$-frequently hypercyclic subspace.

Proof. Suppose that T satisfies the $\mathcal{A}$-frequently hypercyclicity criterion. Then by Theorem 2.4 and Theorem 2.5, the left multiplication operator L_T is an SOT-$\mathcal{A}$-frequently hypercyclic and let $B \in \mathcal{L}(X)$ be an SOT-$\mathcal{A}$-frequently hyperctclic vector for L_T. For any non-zero scalar α, $\alpha \mathrm{L}_T$ is also SOT-$\mathcal{A}$-frequently hypercyclic. Thus, without loss of generality, we may assume that $\| B \| = \frac{1}{2}$. Since

$$\| 1 + B \| \geq 1 - \| B \| \geq \frac{1}{2},$$

the subspace $M = (1+B)(M_0)$ is an infinite dimensional closed subspace of X. For any $y \in M$, $y = x + Bx$, for some $x \in M_0$, and $T^n y = T^n x + T^n Bx$. By Theorem 2.6, for any non-zero vector $x \in M_0$, Bx is an $\mathcal{A}$-frequently hypercyclic vector for T. Let x_0 be any vector in X. Then

$$\begin{aligned}\| T^n y - x_0 \| &= \| T^n Bx + T^n x - x_0 \| \\ &\leq \| T^n Bx - x_0 \| + \| T^n x \| .\end{aligned}$$

Since Bx is an $\mathcal{A}$-frequently hypercyclic for T, for any $\epsilon > 0$, $\| T^n Bx - x_0 \| < \frac{\epsilon}{2}$ for some $n \in A_k$ and by (2), for such n, $T^n x \to 0$ for any $x \in M_0$. Thus, for sufficiently large $n \in A_k$. $\| T^n y - x_0 \| < \epsilon$, for some $n \in A_k$. In other words, $y \in M$ is an $\mathcal{A}$-frequently hypercyclic vector for T.

Proposition 2.8.[3] Let A be a hypercyclicity set and B_W be a weighted shift on $l^p(\mathbb{Z}_+)(1 \leq p < \infty)$. Then B_W is $\mathcal{A}$-frequently hypercyclic if and only if B_W holds the $\mathcal{A}$-frequent hypercyclicity criterion.

Theorem 2.9. Let B_W be a weighted shift on $X = l^p(\mathbb{Z}_+)$ and $\mathcal{A}$ a hypercyclicity set. For each $k \geq 0$, let

$$A(k) = \{ n \geq 0 | \| B_W^n e_k \| \leq C \}, \; C > 0.$$

Suppose that B_W is $\mathcal{A}$-frequently hypercyclic and there is a strictly increasing sequence $(k_j)_{j \geq 1}$ of nonnegative integers such that $\cap_{j \geq 1} A(k_j) \in \mathcal{A}$, then B_W has an A-frequently hypercyclic subspace.

Proof. By Proposition 2.8, B_W is $\mathcal{A}$-frequently hypercyclic if and only if B_W holds the $\mathcal{A}$-frequent hypercyclicity criterion. Let $M_0 = \overline{span}\{ e_{k_j} | j \geq 1 \}$ and let $x \in M_0$. Then for $n \in \cap_{j \geq 1} A(k_j)$, we have

$$\begin{aligned} \| B_W^n x \| &= \| B_W^n \sum_{j=1}^{\infty} x_{k_j} e_{k_j} \| \\ &\leq \sum_{j=1}^{\infty} |x_{k_j}| \| B_W^n e_{k_j} \| = \sum_{k_j \geq n} |x_{k_j}| \| B_W^n e_{k_j} \| \\ &\leq \sum_{k_j \geq 1} C |x_{k_j}| \to 0 \quad as \quad n \to \infty. \end{aligned}$$

Since $A = \cap_{j \geq 1} A(k_j) \in \mathcal{A}$, B_W has the $\mathcal{A}$-frequently hypercyclic subspace.

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