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Point and interval estimation for a simple step-stress model with Type-I censored data from geometric distribution

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Abstract

The estimation problem of expected time to failure of units is studied in a discrete set up. A simple step-stress accelerated life testing is considered with a Type-I censored sample from geometric distribution that is a commonly used distribution to model the lifetime of a device in discrete case. Maximum likelihood estimators as well as the associated distributions are derived. Exact, approximate and bootstrap approaches construct confidence intervals that are compared via a simulation study. Optimal confidence intervals are suggested in view of the expected width and coverage probability criteria. An illustrative example is also presented to explain the results of the paper. Finally, some conclusions are stated.

Keywords: accelerated life testing, expected time to failure, Fisher information, maximum likelihood estimator, bootstrap sample

1. Introduction

There are situations in reliability and survival analysis for which the experiment may not terminate at an adequate time under normal conditions. In such situations, accelerated life testing experiments have been offered to obtain adequate life data. See, for example, Bagdonavicius and Nikulin (2002) and Nelson (1990). A special class of accelerated life testing is the step-stress testing for which the stress levels of the experiment change at some pre-specified times. Balakrishnan and Xie (2007a) obtained similar results when Type-I hybrid censored sample from exponential distribution is considered. Analogous work has been done under Type-II hybrid censoring by Balakrishnan and Xie (2007b). Balakrishnan *et al.* (2009) derived exact inference for a simple step-stress model from the exponential distribution when there is a time constraint on the duration of the experiment. See also, Balakrishnan and Han (2008) and Han and Balakrishnan (2010) for simple step-stress models under Type-II and Type-I censoring schemes, respectively.

In some situations, the life testing experiment must be investigated in a discrete setup. For example, suppose for example that the lifetimes of the units in an experiment depend on the number of times the units are switched on and off, the number of pages a printer prints, or the number of rotations of a machine. Let w be the number of aforementioned shocks the units receive until they fail, so, w is considered as the associated failure time. Censored samples in discrete setup have been studied by some authors. For example, Rezaei and Arghami (2002) investigated Type-I and Type-II

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right censoring in a discrete life model. Davarzani and Parsian (2011) considered discrete middle censored samples and used both classical and Bayesian approaches to do inference about the parameter of interest. Balakrishnan *et al.* (2011) studied characterizations of geometric distribution based on some properties of progressively Type-II right censored order statistics.

Suppose n identical units are simultaneously placed on a test under an initial stress level s_0 and the stress levels are increased to $s_1, \ldots, s_m$ at pre-fixed integer numbers $w_1 < \cdots < w_m$. If the units are subjected to shocks repeatedly w_{m+1} times, where w_{m+1} is a pre-fixed integer number, Type-I censoring scheme has been performed. Under this scheme, failure times greater than w_{m+1} are not observed. In this paper, a simple step-stress scheme with only two stress levels s_0 and s_1 under Type-I censoring is considered. Assuming the failure times in each level of stress are geometrically distributed, Arefi and Razmkhah (2013) obtained the optimal time to change the stress level. Under the same assumptions, the present paper uses a classical approach to investigate the point and interval estimates of the expected times to failure, which are useful for reliability engineers. The performance of the proposed estimators in this paper may be improved when the optimal change times are used. Analogous work has been done by Arefi *et al.* (2011) in a Bayesian approach. Wang *et al.* (2012) investigated the parameter inference in step-stress accelerated life tests under a Type-II censoring model with geometric distribution.

The rest of this paper is as follows. The model considered in the paper is discussed in detail in Section 2. The maximum likelihood estimators (MLEs) of the parameters of interest are also obtained and the conditions for their existence are stated. The conditional distributions of the MLEs are investigated in Section 3. In Section 4, several methods are used to construct confidence intervals (CIs) for the unknown parameters. In Section 5, a simulation study is done to compare various CIs. In Section 7, some conclusions are stated and suggestions are made for applying suitable CIs.

2. Model description and MLE

Suppose n identical units are simultaneously placed on a test under an initial stress level s_0 and the stress level increases to s_1 at the pre-fixed integer number w_1. Moreover, assume that the failure times of the units in stress levels s_0 and s_1 are geometrically distributed with success probabilities p_1 and p_2, respectively. That is, the cumulative distribution function (cdf) is given by

$$F_k(x; p_k) = 1 - q_k^x, \quad x = 1, 2, \ldots,$$

where $q_k = 1 - p_k$ $(k = 1, 2)$. Using the memoryless property of geometric distribution, it can be shown that the probability mass function (pmf) of the failure times in the simple step-stress model is as follows

$$g(x) = \begin{cases} g_1(x) = p_1 q_1^{x-1}, & x = 1, 2, \ldots, w_1, \\ g_2(x) = p_2 q_1^{w_1} q_2^{x-(w_1+1)}, & x = w_1 + 1, w_1 + 2, \ldots. \end{cases} \tag{2.1}$$

Using (2.1), the corresponding cdf is

$$G(x) = \begin{cases} G_1(x) = 1 - q_1^x, & x = 1, 2, \ldots, w_1, \\ G_2(x) = 1 - q_1^{w_1} q_2^{x-w_1}, \ldots & x = w_1 + 1, w_1 + 2, \ldots, \end{cases} \tag{2.2}$$

(see also, Arefi *et al.*, 2011). Now, assume that the step-stress testing terminates when the units are subject to shocks repeatedly w_2 times, where w_2 is a pre-fixed integer number. Moreover, let Z_i be the number of failures occur at time i ($i = 1, 2, \ldots, w_2$). Rezaei and Arghami (2002) showed that the joint distribution of the $Z_1, \ldots, Z_{w_2}$ is multinomial with parameters $n, \pi_1, \ldots, \pi_{w_2+1}$, where

$$\pi_i = \begin{cases} p_1 q_1^{i-1}, & i = 1, 2, \ldots, w_1, \\ p_2 q_1^{w_1} q_2^{i-(w_1+1)}, & i = w_1 + 1, \ldots, w_2, \\ 1 - \sum_{i=1}^{w_2} \pi_i = q_1^{w_1} q_2^{w_2 - w_1}, & i = w_2 + 1. \end{cases} \tag{2.3}$$

That is,

$$\begin{aligned} P(Z_1 = z_1, \ldots, Z_{w_2} = z_{w_2}) &= \frac{n!}{z_1! \ldots z_{w_2}! \left(n - \sum_{i=1}^{w_2} z_i\right)!} \pi_1^{z_1} \ldots \pi_{w_2}^{z_{w_2}} \pi_{w_2+1}^{n - \sum_{i=1}^{w_2} z_i} \\ &= \frac{n!}{z_1! \ldots z_{w_2}! \left(n - \sum_{i=1}^{w_2} z_i\right)!} p_1^{\sum_{i=1}^{w_1} z_i} q_1^{\sum_{i=1}^{w_1}(i-1)z_i + w_1\left(n - \sum_{i=1}^{w_1} z_i\right)} \\ &\quad \times p_2^{\sum_{i=w_1+1}^{w_2} z_i} q_2^{\sum_{i=w_1+1}^{w_2}(i-w_1-1)z_i + (w_2-w_1)\left(n - \sum_{i=1}^{w_2} z_i\right)}. \end{aligned} \tag{2.4}$$

This paper infers on the expected time to failure of the products with geometric distributions in both levels of stress, i.e., $\theta_1 = 1/p_1$ and $\theta_2 = 1/p_2$. Using (2.4), the MLEs of θ_1 and θ_2 are readily obtained as

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{w_1} iZ_i + w_1(n - R_1)}{R_1}, \tag{2.5}$$

and

$$\hat{\theta}_2 = \frac{\sum_{i=w_1+1}^{w_2} iZ_i - w_1 R_2 + (w_2 - w_1)(n - R_1 - R_2)}{R_2}, \tag{2.6}$$

respectively, where $R_1 = \sum_{i=1}^{w_1} Z_i$ and $R_2 = \sum_{i=w_1+1}^{w_2} Z_i$ represent the number of failures occur at stress levels s_0 and s_1, respectively. From (2.5) and (2.6), it is observed that the estimates of θ_1 and θ_2 are finite, when $R_1 \neq 0$ and $R_2 \neq 0$. Therefore, we consider the event

$$A = \{1 \leq R_1 \leq n - 1, 1 \leq R_2 \leq n - R_1\}, \tag{2.7}$$

which ensures existence of $\hat{\theta}_1$ and $\hat{\theta}_2$. Note that the estimators proposed by Arefi and Razmkhah (2013) were constructed based on order statistics that needed overly heavy computations. But, the MLEs in (2.5) and (2.6) accelerate the computations. In the next section, the conditional cdfs of $\hat{\theta}_1$ and $\hat{\theta}_1$ are determined when event A occurs.

Remark 1. In this paper, we have not assumed any relationships between the two stress levels. In fact, we assume that $\theta_1 > \theta_2$ or equivalently $p_1 < p_2$. Under this circumstance, it is emphasized that the expected lifetime at the higher stress level s_1 is less than the initial stress level s_0. In this way, the effect of accelerating stress on distribution of the lifetime is determined. In some situations, we may know that some particular relationships hold between the stress levels or parameters; for instance, $\theta_2 = \lambda\theta_1$ with known λ ($0 < \lambda < 1$). In such situations, it is adequate to estimate only θ_1. Moreover, one can also use the likelihood ratio test to test the hypothesis $H_0 : \theta_2 = \lambda\theta_1$ for a specified λ. See, Balakrishnan and Han (2008), Balakrishnan and Xie (2007a, 2007b), Balakrishnan *et al.* (2009).

3. Conditional distributions of MLEs

In this section, the conditional cdfs of the MLEs $\hat{\theta}_1$ and $\hat{\theta}_2$ are determined and used to find the exact CIs in the next sections. First, note that from (2.4), the following results deduce:

- The random variables R_1 and R_2 have trinomial distribution with pmf

$$P(R_1 = r_1, R_2 = r_2) = c_{r_1,r_2}\beta_1^{r_1}\beta_2^{r_2}\beta_3^{n-r_1-r_2}, \tag{3.1}$$

where $0 \le r_1 \le n, 0 \le r_2 \le n - r_1$ and

$$\begin{aligned} c_{r_1,r_2} &= \frac{n!}{r_1!r_2!(n-r_1-r_2)!}, \\ \beta_1 &= \sum_{i=1}^{w_1} \pi_i = 1 - q_1^{w_1}, \\ \beta_2 &= \sum_{i=w_1+1}^{w_2} \pi_i = q_1^{w_1}\left(1 - q_2^{w_2-w_1}\right), \\ \beta_3 &= 1 - \beta_1 - \beta_2 = q_1^{w_1}q_2^{w_2-w_1}. \end{aligned}$$

Note that the above notations are used throughout the paper. Using (2.7) and (3.1), we have

$$\begin{aligned} P(A) &= \sum_{r_1=1}^{n-1}\sum_{r_2=1}^{n-r_1} P(R_1 = r_1, R_2 = r_2) \\ &= 1 - (1-\beta_1)^n - (1-\beta_2)^n + \beta_3^n. \end{aligned}$$

Therefore, for $1 \le r_1 \le n-1$ and $1 \le r_2 \le n - r_1$, we get

$$P(R_1 = r_1, R_2 = r_2|A) = \frac{c_{r_1,r_2}\,\beta_1^{r_1}\beta_2^{r_2}\beta_3^{n-r_1-r_2}}{1 - (1-\beta_1)^n - (1-\beta_2)^n + \beta_3^n}. \tag{3.2}$$

Using (3.2), we get

$$P(R_1 = r_1|A) = \binom{n}{r_1}\frac{\beta_1^{r_1}\left((1-\beta_1)^{n-r_1} - \beta_3^{n-r_1}\right)}{1 - (1-\beta_1)^n - (1-\beta_2)^n + \beta_3^n} \tag{3.3}$$

and

$$P(R_2 = r_2|R_1 = r_1, A) = \binom{n-r_1}{r_2}\left(\frac{\beta_2}{1-\beta_1}\right)^{r_2}\left(1-\frac{\beta_2}{1-\beta_1}\right)^{n-r_1-r_2}\times\frac{1-\beta_1^n - (1-\beta_1)^n}{1 - (1-\beta_1)^n - (1-\beta_2)^n + \beta_3^n}. \tag{3.4}$$

Moreover, by performing some algebraic calculations, it is deduced that given A, the conditional expected value of R_1 is as follows

$$E_{\theta_1,\theta_2}(R_1|A) = \frac{n\beta_1\left(1 - (1-\beta_2)^{n-1}\right)}{1 - (1-\beta_1)^n - (1-\beta_2)^n + \beta_3^n}, \tag{3.5}$$

where E_{θ_1,θ_2} denotes the fact that the expectation depends on θ_1 and θ_2. Similarly, we get

$$E_{\theta_1,\theta_2}(R_2|A) = \frac{n\beta_2\left(1-(1-\beta_1)^{n-1}\right)}{1-(1-\beta_1)^n-(1-\beta_2)^n+\beta_3^n}. \tag{3.6}$$

- The joint conditional distribution of $Z_1, Z_2, \ldots, Z_{w_1}$, given $R_1 = r_1$, is multinomial with parameters $r_1, \pi_1/\sum_{i=1}^{w_1}\pi_i, \ldots, \pi_{w_1}/\sum_{i=1}^{w_1}\pi_i$, where π_i is as defined in (2.3). So, it is easy to find the corresponding pmf is as follows

$$P\left(Z_1 = z_1, \ldots, Z_{w_1} = z_{w_1}|R_1 = r_1\right) = \frac{r_1!}{\prod_{i=1}^{w_1} z_i!}\frac{p_1^{r_1} q_1^{\sum_{i=1}^{w_1} iz_i - r_1}}{\left(1-q_1^{w_1}\right)^{r_1}}. \tag{3.7}$$

- The joint conditional distribution of $Z_{w_1+1}, \ldots, Z_{w_2}$, given $R_2 = r_2$, is multinomial with parameters $r_2, \pi_{w_1+1}/\sum_{i=w_1+1}^{w_2}\pi_i, \ldots, \pi_{w_2}/\sum_{i=w_1+1}^{w_2}\pi_i$. It can be shown that

$$P(Z_{w_1+1} = z_{w_1+1}, \ldots, Z_{w_2} = z_{w_2}|R_2 = r_2) = \frac{r_2!}{\prod_{i=w_1+1}^{w_2} z_i!}\frac{p_2^{r_2} q_2^{\sum_{i=w_1+1}^{w_2} iz_i-(w_1+1)r_2}}{\left(1-q_2^{w_2-w_1}\right)^{r_2}}. \tag{3.8}$$

Now, using the above results, we present conditional cdfs of $\hat\theta_1$ and $\hat\theta_2$ in the following theorems.

Theorem 1. *The conditional cdf of $\hat\theta_1$, given A, is*

$$F_{\hat\theta_1|A}(x;\theta_1,\theta_2) = \sum_{r_1=1}^{n-1}\sum_{r_2=1}^{n-r_1}\sum_{\Im_1\left(r_1x-w_1(n-r_1)\right)}\frac{r_1!c_{r_1,r_2}}{\prod_{i=1}^{w_1} z_i!}\frac{\beta_2^{r_2}\beta_3^{n-r_1-r_2}p_1^{r_1}q_1^{\sum_{i=1}^{w_1} iz_i-r_1}}{1-(1-\beta_1)^n-(1-\beta_2)^n+\beta_3^n}, \tag{3.9}$$

where

$$\Im_1(x) = \left\{(z_1, \ldots, z_{w_1}) : \sum_{i=1}^{w_1} z_i = r_1, \sum_{i=1}^{w_1} iz_i \le x\right\}.$$

Proof: Notice that the conditional cdf of $\hat\theta_1$, given A, can be written as follows

$$F_{\hat\theta_1|A}(x) = P\left(\hat\theta_1 \le x|A\right) \tag{3.10}$$

$$= \sum_{r_1=1}^{n-1}\sum_{r_2=1}^{n-r_1} P(\hat\theta_1 \le x \mid R_1 = r_1, R_2 = r_2)P(R_1 = r_1, R_2 = r_2|A). \tag{3.11}$$

Furthermore, from (2.5) and (3.7), we find

$$P\left(\hat\theta_1 \le x|R_1 = r_1, R_2 = r_2\right) = P\left(\sum_{i=1}^{w_1} iZ_i \le r_1x - w_1(n-r_1)|R_1 = r_1\right)$$

$$= \sum_{\Im_1(r_1x-w_1(n-r_1))}\frac{r_1!}{\prod_{i=1}^{w_1} z_i!}\frac{p_1^{r_1}q_1^{\sum_{i=1}^{w_1} iz_i-r_1}}{\left(1-q_1^{w_1}\right)^{r_1}}, \tag{3.12}$$

where the summation index $\Im_1(x)$ extends over all integers $z_1, \ldots, z_{w_1}$ for which $\sum_{i=1}^{w_1} z_i = r_1$ and $\sum_{i=1}^{w_1} iz_i \leq x$. Substituting (3.2) and (3.12) in (3.11), the result follows. □

Theorem 2. *The conditional cdf of $\hat{\theta}_2$, given A, is*

$$F_{\hat{\theta}_2|A}(x;\theta_1,\theta_2) = \sum_{r_1=1}^{n-1}\sum_{r_2=1}^{n-r_1}\sum_{\Im_2(b(x))} \frac{r_2!c_{r_1,r_2}}{\prod_{i=w_1+1}^{w_2} z_i!} \frac{\beta_1^{r_1}\beta_3^{n-r_1-r_2}}{1-(1-\beta_1)^n-(1-\beta_2)^n+\beta_3^n} \times \left(q_1^{w_1}p_2\right)^{r_2} q_2^{\sum_{i=w_1+1}^{w_2} iz_i - r_2(w_1+1)}, \tag{3.13}$$

where $b(x) = r_2x - (w_2 - w_1)(n - r_1 - r_2) + w_1r_2$ and

$$\Im_2(x) = \left\{(z_{w_1+1}, \ldots, z_{w_2}) : \sum_{i=w_1+1}^{w_2} z_i = r_2, \sum_{i=w_1+1}^{w_2} iz_i \leq x\right\}.$$

Proof: Using (2.6) and (3.8), analogous to proof of Theorem 1, the result deduces. □

4. Confidence intervals

In this section, we use different methods to construct CIs for the unknown parameters θ_1 and θ_2.

4.1. Exact CIs

Suppose that the tail probability of $\hat{\theta}_k$, i.e., $\mathrm{P}_{\theta_k}(\hat{\theta}_k > \xi)$, is monotone increasing function of θ_k $(k = 1, 2)$. If $\hat{\theta}_k^{\text{obs}}$ denotes the observed value of $\hat{\theta}_k$, then (θ_k^L, θ_k^U) is an exact $100(1-\alpha)\%$ CI for θ_k, where θ_k^L and θ_k^U can be obtained by solving the equations

$$\mathrm{P}_{\theta_k^L}\left(\hat{\theta}_k > \hat{\theta}_k^{\text{obs}}\right) = \frac{\alpha}{2} \quad \text{and} \quad \mathrm{P}_{\theta_k^U}\left(\hat{\theta}_k > \hat{\theta}_k^{\text{obs}}\right) = 1 - \frac{\alpha}{2}.$$

This approach has been applied to construct exact CI in different contexts by several authors. See, for example, Balakrishnan and Han (2008), Balakrishnan and Xie (2007a, 2007b), Balakrishnan *et al.* (2009), Chen and Bhattacharyya (1988), Childs *et al.* (2003), Gupta and Kundu (1998), and Kundu and Basu (2000). Further, the required monotonicity in this approach follows those of Balakrishnan and Iliopoulos (2010). According to the above scenario, the interval (θ_1^L, θ_1^U) is the exact $100(1-\alpha)\%$ CI for θ_1, if the following two non-linear equations hold

$$\begin{cases} 1 - \dfrac{\alpha}{2} = F_{\hat{\theta}_1|A}\left(\hat{\theta}_1^{\text{obs}}; \theta_1^L, \hat{\theta}_2^{\text{obs}}\right), \\ \dfrac{\alpha}{2} = F_{\hat{\theta}_1|A}\left(\hat{\theta}_1^{\text{obs}}; \theta_1^U, \hat{\theta}_2^{\text{obs}}\right), \end{cases} \tag{4.1}$$

where $F_{\hat{\theta}_1|A}(x;\theta_1,\theta_2)$ is as defined in (3.9). Similarly, (θ_2^L, θ_2^U) is the exact $100(1-\alpha)\%$ CI for θ_2, if

$$\begin{cases} 1 - \dfrac{\alpha}{2} = F_{\hat{\theta}_2|A}\left(\hat{\theta}_2^{\text{obs}}; \hat{\theta}_1^{\text{obs}}, \theta_2^L\right), \\ \dfrac{\alpha}{2} = F_{\hat{\theta}_2|A}\left(\hat{\theta}_2^{\text{obs}}; \hat{\theta}_1^{\text{obs}}, \theta_2^U\right), \end{cases} \tag{4.2}$$

where $F_{\hat{\theta}_2|A}(x;\theta_1,\theta_2)$ is as defined in (3.13).

4.2. Approximate CI

Consider a random sample of size n from the cdf in (2.2) and let $\hat{\theta}_k$ be the MLE of θ_k ($k = 1, 2$), then for large n, an approximate $100(1-\alpha)\%$ CI for θ_k is given by

$$\hat{\theta}_k - z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{\hat{I}_k(\theta_1,\theta_2)}} \le \theta_k \le \hat{\theta}_k + z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{\hat{I}_k(\theta_1,\theta_2)}}, \quad k = 1, 2, \tag{4.3}$$

where $\hat{I}_k(\theta_1,\theta_2)$ is the approximate Fisher information (FI) about θ_k. In this section, we use two approaches to evaluate $\hat{I}_k(\theta_1,\theta_2)$.

Approach I. Observed FI is a reasonable approximation for the expected FI in (4.3) (see Casella and Berger, 1990, p. 326). Therefore, we use the following approximation to construct CI for θ_k:

$$\hat{I}_k(\theta_1,\theta_2) = -\frac{\partial^2}{\partial\theta_k^2}\ell(\theta_1,\theta_2)\bigg|_{\theta_1=\hat{\theta}_1,\theta_2=\hat{\theta}_2},$$

where $\ell(\theta_1,\theta_2)$ is the log-likelihood function of θ_1 and θ_2, which is obtained by taking the logarithm of the pmf in (2.4), when the reparameterization $p_i = 1/\theta_i$ ($i = 1, 2$) is performed. By doing some algebraic calculations it can be shown that

$$\hat{I}_1(\theta_1,\theta_2) = \frac{R_1}{\hat{\theta}_1\left(\hat{\theta}_1 - 1\right)} \qquad \text{and} \qquad \hat{I}_2(\theta_1,\theta_2) = \frac{R_2}{\hat{\theta}_2\left(\hat{\theta}_2 - 1\right)}.$$

The approximate CIs obtained from this approach are denoted by Appr-I CIs in the sequel.

Approach II. Another approximation for the expected FI can be derived by substituting $\hat{\theta}_k$ instead of θ_k ($k = 1, 2$) in the associated quantity, i.e.,

$$\hat{I}_k(\theta_1,\theta_2) = E\left(-\frac{\partial^2}{\partial\theta_k^2}\ell(\theta_1,\theta_2)\right)\bigg|_{\theta_1=\hat{\theta}_1,\theta_2=\hat{\theta}_2}.$$

By performing some algebraic calculations, we have

$$\hat{I}_1(\theta_1,\theta_2) = \frac{2\hat{\theta}_1 - 1}{\hat{\theta}_1^2\left(\hat{\theta}_1 - 1\right)^2}\left\{\left(\hat{\theta}_1 - \frac{w_1\left(\hat{\theta}_1 - 1\right)^{w_1}}{\hat{\theta}_1^{w_1} - \left(\hat{\theta}_1 - 1\right)^{w_1}} - (w_1 + 1)\right)E_{\hat{\theta}_1,\hat{\theta}_2}(R_1|A) + nw_1\right\} - \frac{1}{\hat{\theta}_1^2}E_{\hat{\theta}_1,\hat{\theta}_2}(R_1|A)$$

and

$$\hat{I}_2(\theta_1,\theta_2) = \frac{2\hat{\theta}_2 - 1}{\hat{\theta}_2^2\left(\hat{\theta}_2 - 1\right)^2}\left\{\left(\hat{\theta}_2 - \frac{(w_2 - w_1)\left(\hat{\theta}_2 - 1\right)^{w_2-w_1}}{\hat{\theta}_2^{w_2-w_1} - \left(\hat{\theta}_2 - 1\right)^{w_2-w_1}} - 1\right)E_{\hat{\theta}_1,\hat{\theta}_2}(R_2|A) + (w_2 - w_1)E_{\hat{\theta}_1,\hat{\theta}_2}(n - R_1 - R_2|A)\right\} - \frac{1}{\hat{\theta}_2^2}E_{\hat{\theta}_1,\hat{\theta}_2}(R_2|A),$$

where $E_{\theta_1,\theta_2}(R_1|A)$ and $E_{\theta_1,\theta_2}(R_2|A)$ are as defined in (3.5) and (3.6), respectively.

Hereinafter, we show the approximate CIs obtained from this approach by the Appr-II CIs.

4.3. Bootstrap CIs

Here, we present several bootstrap methods to construct CIs for θ_1 and θ_2. Toward this end, we use the following resampling algorithm for given n, w_1 and w_2:

Step 1. Using the original Type-I censored sample, we obtain $\hat{\theta}_1$ and $\hat{\theta}_2$ from the equations in (2.5) and (2.6), respectively.

Step 2. By substituting $\hat{\theta}_1$ and $\hat{\theta}_2$ in the conditional pmf (3.3), a value for R_1 is generated.

Step 3. For obtained $R_1 = r_1$, by substituting $\hat{\theta}_1$ and $\hat{\theta}_2$ in (3.4), a value for R_2 is generated.

Step 4. Based on the observed values of $R_1 = r_1$ and $R_2 = r_2$, the random samples $(Z_1, \ldots, Z_{w_1})$ and $(Z_{w_1+1}, \ldots, Z_{w_1+w_2})$ are generated from (3.7) and (3.8), respectively.

Step 5. The estimators $\hat{\theta}_1^*$ and $\hat{\theta}_2^*$ are derived form (2.5) and (2.6), respectively.

Step 6. Repeat Steps 2–5, M times and arrange all $\hat{\theta}_1^*$'s and $\hat{\theta}_2^*$'s in ascending order to obtain the bootstrap sample

$$\left\{\hat{\theta}_k^{*(1)}, \hat{\theta}_k^{*(2)}, \ldots, \hat{\theta}_k^{*(M)}\right\}, \quad k = 1, 2. \tag{4.4}$$

Using the above algorithm, we can construct some bootstrap CIs in different methods that are studied in the sequel. For more details one may refer to Efron and Tibshirani (1993).

4.3.1. Percentile interval

Using the bootstrap samples presented in (4.4), a two-sided $100(1-\alpha)\%$ percentile bootstrap CI for θ_k is given by

$$\left(\hat{\theta}_k^{*\left(\frac{\alpha M}{2}\right)},\ \hat{\theta}_k^{*\left(\left(1-\frac{\alpha}{2}\right)M\right)}\right). \tag{4.5}$$

If $\alpha M/2$ or $(1-\alpha/2)M$ in (4.5) are not integer, we use the integer parts of $(\alpha/2)(M+1)$ or $(1-\alpha/2)(M+1)$, respectively. Note that the CI in (4.5) is an equi-tailed CI. The shortest width $100(1-\alpha)\%$ percentile bootstrap CI is

$$\left(\hat{\theta}_k^{*L}, \hat{\theta}_k^{*U}\right), \tag{4.6}$$

where it is obtained by considering all possible $100(1-\alpha)\%$ CIs of the form

$$\left(\hat{\theta}_k^{*(i)},\ \hat{\theta}_k^{*((1-\alpha)M+i)}\right), \quad i = 1, \ldots, \alpha M,\ k = 1, 2,$$

and choosing the interval with minimum width.

In the next sections, the notations P and ShWP will show the percentile and shortest width percentile bootstrap CIs, respectively.

4.3.2. Studentized-t interval

Here, we use the statistic

$$T_k^{*(j)} = \frac{\hat{\theta}_k^{*(j)} - \hat{\theta}_k}{\widehat{\mathrm{se}}\left(\hat{\theta}_k^*\right)}, \quad j = 1, \ldots, M, \;\; k = 1, 2,$$

where $\widehat{\mathrm{se}}(\hat{\theta}_k^*)$ is the estimated standard error of $\hat{\theta}_k^*$ for the bootstrap samples. A two-sided equi-tailed $100(1-\alpha)\%$ studentized-t (St) bootstrap CI for θ_k is

$$\left(\hat{\theta}_k - T_k^{*\left((1-\frac{\alpha}{2})M\right)} \widehat{\mathrm{se}}\left(\hat{\theta}_k\right),\; \hat{\theta}_k - T_k^{*\left(\frac{\alpha M}{2}\right)} \widehat{\mathrm{se}}\left(\hat{\theta}_k\right)\right), \tag{4.7}$$

where $\widehat{\mathrm{se}}(\hat{\theta}_k)$ can be obtained from the asymptotic variance of the original Type-I censored sample. That is, $\widehat{\mathrm{se}}(\hat{\theta}_k) \approx \sqrt{1/\hat{I}_k(\theta_1, \theta_2)}$, where the expected FI can be approximated via Approaches I and II, presented in Subsection 4.2. For the corresponding St bootstrap CIs, we shall use the notations St-I and St-II, respectively, in the next sections. Moreover, the shortest width $100(1-\alpha)\%$ St bootstrap CI is

$$\left(\hat{\theta}_k - T_k^{*U} \widehat{\mathrm{se}}\left(\hat{\theta}_k\right),\; \hat{\theta}_k - T_k^{*L} \widehat{\mathrm{se}}\left(\hat{\theta}_k\right)\right), \tag{4.8}$$

where (T_k^{*L}, T_k^{*U}) has the minimum width among all possible $100(1-\alpha)\%$ CIs of the form

$$\left(T_k^{*(i)},\; T_k^{*((1-\alpha)M+i)}\right), \quad i = 1, \ldots, \alpha M, \;\; k = 1, 2.$$

If Approach I or II is used to approximate $\widehat{\mathrm{se}}(\hat{\theta}_k)$ in (4.8), the corresponding shortest width bootstrap CIs are denoted by ShWSt-I or ShWSt-II, respectively.

5. Simulation study

To compare the proposed confidence intervals in terms of coverage probability (CP) and expected width (EW), a simulation study has been conducted using the following algorithm:

1. For given values of n, w_1, w_2, p_1 and p_2, the value of R_1 is generated by use of the (3.3).
2. Based on the observed value of $R_1 = r_1$, the value of $(Z_1, \ldots, Z_{w_1})$ is simulated from (3.7); then the MLE of θ_1 is derived using (2.5).
3. Given $R_1 = r_1$, an observation is generated from the conditional pmf (3.4) to obtain the observed value of R_2.
4. Given $R_2 = r_2$, the value of $(Z_{w_1+1}, \ldots, Z_{w_2})$ is extracted from (3.8); then the value of $\hat{\theta}_2$ is obtained using (2.6).
5. For given α, the exact CIs for θ_1 and θ_2 are derived by solving the equations in (4.1) and (4.2), respectively. The Appr-I and Appr-II CIs for θ_k $(k = 1, 2)$ are obtained using (4.3).
6. To derive the various bootstrap CIs, the bootstrap samples are generated using Steps 2–6 of the algorithm presented in Section 4.3. The percentiles and ShWP bootstrap CIs are then derived using (4.5) and (4.6), respectively. The St-I and St-II bootstrap CIs are obtained using (4.7). ShWSt-I and ShWSt-II bootstrap CIs are also constructed using (4.8).

7. The Steps 1–6 are repeated B times to attain the CPs and EWs of various CIs.

Table 1 represents the results for $n = 10, 15, 20$, $p_1 = 0.1, p_2 = 0.2$, $w_1 = 5, w_2 = 10$, $\alpha = 0.05, 0.1$ and $B = M = 1000$. From Table 1, for given n, it is deduced that:

- Based on minimum EW criterion, ShWSt-I bootstrap CIs may be suggested for both of θ_1 and θ_2.
- In view of closeness the CP to the nominal level, the exact CIs are offered for both of θ_1 and θ_2, but, the EWs of these CIs are too large.
- Considering both criteria EW and CP, it is reasonable to choose the CI with minimum EW among those with CPs near to the nominal level. Therefore, one may suggest the percentile bootstrap CIs or exact CIs to estimate both of θ_1 and θ_2. Note that the EW of exact CIs is excessively larger than the percentile bootstrap CIs for $n = 10$, whereas there is no significant difference for larger samples.
- Comparing between performance the Approaches I and II in approximating the FI, it is observed that using Approach I leads to shorter EW than Approach II for approximate CIs as well as studentized bootstrap CIs for both of θ_1 and θ_2.
- When n increases, the EWs of all CIs decrease.

6. Illustrative example

To illustrate the proposed procedure in this paper, we consider a numerical example. Assuming $w_1 = 5$ and $w_2 = 10$, a random ample of size 20 has been generated from the cdf in (2.2) with $p_1 = 0.1$ and $p_2 = 0.2$. Equivalently, we get $\theta_1 = 10$ and $\theta_2 = 5$. Table 2 presents the generated data.

Using the data in Table 2, we would obtain that $R_1 = 8$ and $R_2 = 9$ are the number of units failed at stress level s_0 and s_1, respectively. Using (2.5) and (2.6), the MLEs have been evaluated as $\hat{\theta}_1 = 10.25$ and $\hat{\theta}_2 = 4$, respectively.

We only derive exact CIs, percentile and ShWSt-I bootstrap CIs, because the results of Section 5 show that these CIs are more important than the others. For given α, the exact CIs for θ_1 and θ_2 are derived by solving the equations in (4.1) and (4.2), respectively. We use $M = 1000$ bootstrap samples to construct the bootstrap CIs. The percentile and ShWSt-I bootstrap CIs are obtained using (4.5) and (4.8), respectively. Table 3 tabulates the results for significant levels $\alpha = 0.05, 0.10$. It, deduces that:

1. All derived CIs contain the exact values of parameters.
2. As mentioned in Section 5, the ShWSt-I bootstrap CIs for both of θ_1 and θ_2 have shorter width than other CIs.
3. The exact CIs for both of θ_1 and θ_2 have larger width than other CIs; however, the results in Section 5 indicate that they have more accurate CPs.

7. Conclusions

This paper considered a simple step-stress model with Type-I censored samples from geometric distribution. The MLEs of the means of geometric distributions in both levels of stress, θ_1 and θ_2, were determined and the associated distributions derived. Several methods were used to construct CIs for

Table 1: The simulated CPs and EWs for various CIs of θ_1 and θ_2

Parameter	Method	90%			95%		
		$n = 10$	$n = 15$	$n = 20$	$n = 10$	$n = 15$	$n = 20$
θ_1	Bootstrap (P)	25.1614* 0.9057**	23.3402 0.8776	18.4297 0.8667	31.5360 0.9358	29.3901 0.9426	23.9198 0.9283
	Bootstrap (ShWP)	22.0043 0.7794	18.1397 0.8585	14.8762 0.8500	26.4382 0.9038	24.9321 0.9006	20.0466 0.9217
	Bootstrap (St-I)	18.4873 0.6573	12.5858 0.7725	11.0341 0.7667	19.9077 0.6901	14.5434 0.7992	13.4059 0.7967
	Bootstrap (ShWSt-I)	17.0203 0.7023	10.5830 0.8011	9.2594 0.7600	19.0453 0.7243	13.3769 0.8337	12.0078 0.8267
	Bootstrap (St-II)	21.3086 0.6595	12.6408 0.7667	11.0573 0.7683	22.8299 0.6880	14.5997 0.7954	13.4333 0.7933
	Bootstrap (ShWSt-II)	19.6041 0.6959	10.6452 0.8050	9.2820 0.7617	21.8643 0.7264	13.4350 0.8317	12.0320 0.8300
	Appr-I	23.2292 0.8736	16.3139 0.8948	13.8510 0.8717	26.5624 0.9166	19.1938 0.9273	16.4145 0.9067
	Appr-II	25.2416 0.8736	16.4659 0.8948	13.9088 0.8717	28.9052 0.9038	19.3378 0.9254	16.4688 0.9067
	Exact	43.9552 0.8951	23.0489 0.9025	17.1317 0.8883	54.4006 0.9487	30.2843 0.9541	22.0887 0.9433
θ_2	Bootstrap (P)	11.7757 0.8779	9.5233 0.8757	7.5525 0.8917	15.0301 0.9188	12.6284 0.8776	9.9035 0.9450
	Bootstrap (ShWP)	9.8730 0.8415	8.0076 0.8509	6.5239 0.8800	12.8614 0.9102	10.6132 0.8585	8.5210 0.9367
	Bootstrap (St-I)	8.7932 0.7131	6.4302 0.7438	5.3088 0.7800	9.5912 0.7414	7.5541 0.7725	6.5471 0.8233
	Bootstrap (ShWSt-I)	8.3315 0.7537	5.6958 0.7954	4.7262 0.8233	9.5671 0.7799	7.1394 0.8011	5.9811 0.8500
	Bootstrap (St-II)	9.5278 0.7109	6.7310 0.7476	5.4697 0.7817	10.3522 0.7521	7.8479 0.7667	6.7123 0.8217
	Bootstrap (ShWSt-II)	9.1996 0.7644	6.0236 0.7954	4.9110 0.8267	10.3318 0.7763	7.4575 0.8050	6.1503 0.8533
	Appr-I	10.3003 0.8629	7.6594 0.8604	6.2390 0.8817	11.8528 0.8867	9.0149 0.8948	7.3936 0.9150
	Appr-II	11.0150 0.8651	7.9019 0.8585	6.3693 0.8833	12.6716 0.8974	9.3000 0.8948	7.5452 0.9150
	Exact	31.9176 0.9058	12.9558 0.9101	8.2693 0.9167	46.6476 0.9487	17.1537 0.9025	10.6431 0.9583

CP = coverage probability; EW = expected width; CI = confidence interval. * and ** stand for the EWs and CPs of the CIs.

Table 2: Generated sample of size 20 from the cdf in (2.2) with $w_1 = 5$, $w_2 = 10$, $\theta_1 = 10$, and $\theta_2 = 5$

Mean failure time	Failure times								
$\theta_1 = 10$	1	2	2	2	2	3	5	5	
$\theta_2 = 5$	6	6	6	6	7	8	9	9	9

θ_1 and θ_2 and they were compared via a simulation study. The ShWSt-I bootstrap CIs for both of θ_1 and θ_2 have shortest width; however, the CPs are significantly different from nominal levels. These CIs are derived using the asymptotic property of the MLEs; therefore, it is recommended to use them in the case of large sample sizes. Exact CIs can be extensively used for each sample size with satis-

Table 3: CIs for θ_1 and θ_2 based on the data in Table 1 for $w_1 = 5$ and $w_2 = 10$

Parameter	Method	90%	95%
θ_1	Bootstrap (P)	(6.4167, 21.2500)	(6.0000, 23.7500)
	Bootstrap (ShWSt-I)	(5.0874, 13.2701)	(2.7283, 14.1818)
	Exact	(6.2697, 19.0887)	(5.7711, 21.8869)
θ_2	Bootstrap (P)	(2.5000, 6.8333)	(2.3077, 7.8000)
	Bootstrap (ShWSt-I)	(2.0909, 5.4656)	(1.5959, 5.8182)
	Exact	(2.6884, 7.1213)	(2.5053, 8.0948)

factory CPs; however, the required computation becomes difficult for large sample sizes. Hence, our recommendation is to apply ShWSt-I bootstrap CIs in the case of large sample sizes, when minimum EW is the only criterion to select the best CI. Exact CIs may be presented, when only the CP criterion is considered to choose the optimal CI. Percentile bootstrap CIs or exact CIs are suggested to estimate θ_1 and θ_2 in situations in which both of EW and CP criteria are important.

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