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3계 마코프 도착과정의 계수과정과 적률근사*

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Counting Process of MAP(3)s and Moment Fittings

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■ Abstract ■

Moments of stationary intervals and those of the counting process can be used for moment fittings of the point processes. As for the Markovian arrival processes, the moments of stationary intervals are given as a polynomial function of parameters whereas the moments of the counting process involve exponential terms. Therefore, moment fittings are more complicated with the counting process than with stationary intervals. However, in queueing network analysis, cross-correlation between point processes can be modeled more conveniently with counting processes than with stationary intervals. A Laplace-Stieltjies transform of the stationary intervals of MAP(3)s is recently proposed in minimal number of parameters. We extend the results and present the Laplace transform of the counting process of MAP(3)s. We also show how moments of the counting process such as index of dispersions for counts, IDC, and limiting IDC can be used for moment fittings. Examples of exact MAP(3) moment fittings are also presented on the basis of moments of stationary intervals and those of the counting process.

Keywords : Markovian Arrival Process, Counting Process, Laplace Transform, Moment Fitting, Queueing Network

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1. Introduction

As a generalization of the Poisson process, the Markovian arrival process of order n, MAP(n), is a mixture of Poisson processes of which the arrival rate is dependent on n phases. Transitions take place from one of n phases to another as a continuous time Markov chain with or without an event of arrival. A MAP(n) is described by $2n^2-n$ parameters in two transition rate matrices. As a special case of MAP(n)s, the Markov modulated Poisson process of order n, MMPP(n), is a mixture of exactly n Poisson processes and described by n^2 parameters. Both MAP(n)s and MMPP(n)s can be used for modeling non-renewal processes and can be used for queueing network analysis such as in decomposition approximation; see Heindl [7] and Ferng and Chang [6]. One of the main tasks in queueing network decomposition analysis is the approximation of point processes by moment fittings which can be done with stationary intervals and/or the counting process: see Kuehn [13], Shanthikumar and Buzacott [16], and Whitt [18]. In fact, counting processes are easier to deal with when cross-correlation between point processes should be taken into account; see Kim [9] and [10]. However, the Laplace transform (LT) of the counting process is much more complicated than the Laplace-Stieltjies transform (LST) of the stationary intervals of MAP (n)s. In fact, exact moment fitting procedures have been available only for stationary intervals of MAP(2)s; see Bodrog et al. [2]. Recently, Kim [11] proposed six different ways of exact moment fittings based on both stationary intervals and the counting process of MAP(2)s. Kim [12] also proposed an exact MAP(3) moment fitting based on stationary intervals as an application of minimal LST representation of MAP(n)s. We extend the results in [12] and present the LT of the counting process of MAP(3)s. We also show how moments of the counting process such as index of dispersions for counts, IDC, and limiting IDC can be used for moment fittings. More researches on Markov processes and applications can be found in Chae [5], Jang and Bai [8], and Yoon [19].

The rest of the paper is organized as follows. In Section 2, we present preliminary results in the literature as well as definitions and notations for MAP(3)s. In Section 3, we show that the LT of the counting process of MAP(3)s can be written in minimal number of parameters. By differentiation and inverse Laplace transformation, we obtain moments of the counting process. In Section 4, we propose exact moment fitting procedures based on moments of both stationary intervals and the counting process. Numerical examples of exact moment fittings are also presented for MAP(3)s in Section 5 followed by a conclusion.

2. Preliminaries

In this section, we introduce definitions and notations for MAP(3)s. Since minimal representations are crucial for exact moment fittings we also briefly review two minimal representations for MAP(3)s given in [12].

2.1 Notations and definitions

In general, a MAP(3) is represented by two rate matrices, $(\mathbf{D}_0, \mathbf{D}_1)$, given in terms of 15 transition rate parameters σ_{ij}'s without arrivals and λ_{ij}'s with arrivals.

$$\mathbf{D}_0=\begin{bmatrix} -\sigma_1-\lambda_1 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & -\sigma_2-\lambda_2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & -\sigma_3-\lambda_3 \end{bmatrix},$$

$$\mathbf{D}_1=\begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix},$$

where $\sigma_i=\Sigma_{j\neq i}\sigma_{ij}$ and $\lambda_i=\lambda_{i1}+\lambda_{i2}+\lambda_{i3}$. Then, $\mathbf{Q}\equiv\mathbf{D}_0+\mathbf{D}_1$ is the infinitesimal generator of the continuous time Markov chain governing the transitions among 3 phases. Let $\boldsymbol{\pi}$ be the steady-state probability vector for $\mathbf{Q}$, i.e. $\boldsymbol{\pi}\mathbf{Q}=\mathbf{0}$ and $\boldsymbol{\pi e}=1$ where $\boldsymbol{e}$ is a column vector of ones. Also let $\boldsymbol{p}$ be the stationary probability vector for the embedded Markov chain $\mathbf{P}\equiv-\mathbf{D}_0^{-1}\mathbf{D}_1$, i.e. $\boldsymbol{p}\mathbf{P}=\boldsymbol{p}$ and $\boldsymbol{pe}=1$. If we let λ_A be the arrival rate of a MAP(3), then we have $\lambda_A\equiv\boldsymbol{\pi}\mathbf{D}_1\boldsymbol{e}$.

2.2 A Minimal Moment Representation of MAP(3)s

Let T be a stationary interval of MAP(3)s and let $r_i\equiv \mathrm{E}(T^i)/i!$ be the reduced marginal moment. Also, let T_1 and T_2 be two consecutive stationary intervals and let $r_{ij}\equiv \mathrm{E}(T_1^i T_2^j)/(i!j!)$ be the reduced joint moment. It is shown in Bodrog et al. [3] that the first $2n-1$ marginal moments and the first $(n-1)^2$ lag-1 joint moments uniquely determine all other moments of a MAP(n). That is, the following set of nine moments is a minimal moment representation of a MAP(3); see Casale et al. [4] and Telek et al. [17]

$$(r_1, r_2, r_3, r_4, r_5) \text{ and } (r_{11}, r_{12}, r_{21}, r_{22}) \tag{1}$$

2.3 A Minimal LST Representation of MAP(3)s

Kim [12] proposed a minimal Laplace-Stieljes transform of stationary intervals of MAP(n)s. The joint LST of two consecutive stationary intervals of a MAP(3) is given as follows

$$\begin{aligned}\tilde{f}(s,t) &\equiv \boldsymbol{p}(s\mathbf{I}-\mathbf{D}_0)^{-1}\mathbf{D}_1(t\mathbf{I}-\mathbf{D}_0)^{-1}\mathbf{D}_1\boldsymbol{e}\\ &= (c_{22}s^2t^2+c_{21}s^2t+c_{12}st^2+c_{11}st\\ &\quad +a_0b_2(s^2+t^2)+a_0b_1(s+t)+a_0^2)/\\ &\quad ((s^3+a_2s^2+a_1s+a_0)(t^3+a_2t^2+a_1t+a_0)).\end{aligned}$$

by which a set of minimal moments in (1) can be written in terms $\boldsymbol{a}\equiv(a_0, a_1, a_2)$, $\boldsymbol{b}\equiv(b_1, b_2)$, and $\mathbf{c}\equiv(c_{11}, c_{12}, c_{21}, c_{22})$. It can be shown that

$$(r_1,r_2,r_3,r_4,r_5)=\left(\frac{a_1-b_1}{a_0},\frac{a_1r_1-a_2+b_2}{a_0},\frac{a_1r_2-a_2r_1+1}{a_0},\frac{a_1r_3-a_2r_2+r_1}{a_0},\frac{a_1r_4-a_2r_3+r_2}{a_0}\right) \tag{2}$$

$$r_{11}=\frac{c_{11}-a_1b_1}{a_0^2}+\frac{a_1r_1}{a_0}, \tag{3}$$

$$r_{12}=\frac{a_1(c_{11}-a_1b_1)}{a_0^3}+\frac{a_2b_1-c_{12}}{a_0^2}+\frac{a_1r_2}{a_0}, \tag{4}$$

$$r_{21}=\frac{a_1(c_{11}-a_1b_1)}{a_0^3}+\frac{a_2b_1-c_{21}}{a_0^2}+\frac{a_1r_2}{a_0}, \tag{5}$$

$$r_{22}=\frac{a_1(2a_1b_2-c_{12}-c_{21})}{a_0^3}+\frac{a_1^2r_{11}-a_2^2+c_{22}}{a_0^2}-\frac{2a_2r_2}{a_0}. \tag{6}$$

2.4 Another Minimal Representation for MAP(3)s

For $\mathbf{Q}\equiv(q_{ij})$, let $q_i=-q_{ii}$ and let $\bar{q}_i$ be the 2×2 principal minor of $\mathbf{Q}$. Then, the following set of 9 parameters $\boldsymbol{\Sigma}\equiv(\Sigma_1,\Sigma_2)$, $\boldsymbol{\Gamma}\equiv(\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ and $\boldsymbol{\Lambda}\equiv(\Lambda_0, \Lambda_1, \Lambda_2)$ is also proposed by Kim [12] as an alternative minimal representation for MAP(3)s.

$$\begin{aligned}\Sigma_1 &= \mathrm{Tr}(-\mathbf{Q}),\\ \Sigma_2 &= (\mathrm{Tr}(\mathbf{Q})^2-\mathrm{Tr}(\mathbf{Q}^2)))/2,\\ \Gamma_{11} &= (\mathrm{Tr}(\mathbf{D}_0)^2-\mathrm{Tr}(\mathbf{D}_1)^2-\mathrm{Tr}(\mathbf{Q})^2-(\mathrm{Tr}(\mathbf{D}_0^2)\\ &\quad -\mathrm{Tr}(\mathbf{D}_1^2)-\mathrm{Tr}(\mathbf{Q}^2)))/2,\\ \Gamma_{12} &= \lambda_1\bar{q}_1+\lambda_2\bar{q}_2+\lambda_3\bar{q}_3,\\ \Gamma_{21} &= |\mathbf{D}_0|-|\mathbf{D}_1|-\Gamma_{12},\\ \Gamma_{22} &= \lambda_1(\lambda_{11}\bar{q}_1+\lambda_{21}\bar{q}_2+\lambda_{31}\bar{q}_3)+\lambda_2(\lambda_{12}\bar{q}_1+\lambda_{22}\bar{q}_2\\ &\quad +\lambda_{32}\bar{q}_3)+\lambda_3(\lambda_{13}\bar{q}_1+\lambda_{23}\bar{q}_2+\lambda_{33}\bar{q}_3)\end{aligned}$$

$\Lambda_0 = |\mathbf{D}_1|,$

$\Lambda_1 = (\mathrm{Tr}(\mathbf{D}_1)^2 - \mathrm{Tr}(\mathbf{D}_1^2))/2,$

$\Lambda_2 = \mathrm{Tr}(\mathbf{D}_1).$

In fact, the characteristic polynomials of the matrices $\mathbf{D}_0$, $\mathbf{D}_1$ and $\mathbf{Q}$ are given as

$$|s\mathbf{I} - \mathbf{D}_0| = s^3 + a_2 s^2 + a_1 s + a_0,$$
$$|s\mathbf{I} - \mathbf{D}_1| = s^3 - \Lambda_2 s^2 + \Lambda_1 s - \Lambda_0,$$
$$|s\mathbf{I} - \mathbf{Q}| = s^3 + \Sigma_2 s^2 + \Sigma_1 s.$$

where

$$(a_0, a_1, a_2) = (\Gamma_{12} + \Gamma_{21} + \Lambda_0,\ \Sigma_1 + \Gamma_{11} + \Lambda_1, \Sigma_2 + \Lambda_2). \tag{7}$$

The following identity is also given in [12].

$$(b_1, b_2) = (a_1 - r_1 a_0, \Gamma_{22}/\Gamma_{12}) \tag{8}$$

$$c_{11} = a_1 b_1 - a_0(\Sigma_2 + r_1(\Lambda_1 - \Sigma_1)) + (b_2 - a_2 + r_1 a_1)(\Lambda_0 - \Gamma_{12}), \tag{9}$$

$$c_{12} = b_2\Lambda_1 + 0.5(\Gamma_{11} b_2 + \Gamma_{33}/\Gamma_{12} + r_1(\Sigma_2\Lambda_0 - \Gamma_{21}\Lambda_2) + (\Gamma_{11}\Lambda_2 - 2\Lambda_0 - \Gamma_{21} - \Sigma_2\Lambda_1)), \tag{10}$$

$$c_{21} = b_2\Lambda_1 + 0.5(\Gamma_{11} b_2 - \Gamma_{33}/\Gamma_{12} + r_1(\Sigma_2\Lambda_0 - \Gamma_{21}\Lambda_2) + (\Gamma_{11}\Lambda_2 - 2\Lambda_0 - \Gamma_{21} - \Sigma_2\Lambda_1) \tag{11}$$

$$c_{22} = b_2\Lambda_2 - \Lambda_1 + r_1\Lambda_0$$

where Γ_{33} is auxiliary parameter which is given in terms of $(\boldsymbol{\Sigma}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda})$; see Kim [12] for details. Note that $\Gamma_{33}/\Gamma_{12} = c_{12} - c_{21}$ by Eqs. (10) and (11). It can be seen that $\boldsymbol{\pi} = \bar{q}/\Sigma_1$ and that $\lambda_A \equiv \boldsymbol{\pi}\mathbf{D}_1\boldsymbol{e} = \Gamma_{12}/\Sigma_1.$

2.5 Moment Fitting based on Moments of Stationary Intervals

In [12], a moment fitting procedure is given for MAP(3)s based on moments of stationary intervals given in (1). First, moments are converted into $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ and $(\boldsymbol{\Sigma}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda})$. By Eq. (2), we have

$$a_0 = (r_2^3 + r_3^2 + r_1^2 r_4 - r_2(2r_1 r_3 + r_4))/r_b, \tag{12}$$

$$a_1 = (r_2^2 r_{3-r_1} r_3^2 + r_3 r_4 + r_1^2 r_5 - r_2(r_1 r_4 + r_5))/r_b, \tag{13}$$

$$(a_2, b_1, b_2) = \left(\frac{1 + a_1 r_2 - a_0 r_3}{r_1}, a_1 - a_0 r_1, a_0 r_2 - a_1 r_1 + a_2\right) \tag{14}$$

where $r_b = r_3^3 + r_1 r_4^2 + r_2^2 r_5 - r_3(2r_2 r_4 + r_1 r_5)$. Furthermore, the coefficient vector $\boldsymbol{c}$ is uniquely determined by Eqs. (3)~(6) in terms of $(\boldsymbol{a}, \boldsymbol{b})$ and four joint moments as follows.

$$c_{11} = a_0^2 r_{11} - a_1^2 + 2a_1 b_1, \tag{15}$$

$$c_{12} = a_2(b_1 - a_1) + a_1(b_2 + a_0 r_{11}) - a_0^2 r_{12}, \tag{16}$$

$$c_{21} = a_2(b_1 - a_1) + a_1(b_2 + a_0 r_{11}) - a_0^2 r_{21}, \tag{17}$$

$$c_{22} = a_0^2 r_{22} - a_2^2 + 2a_2 b_2 + a_1^2 r_{11} - a_1 a_0(r_{12} + r_{21}). \tag{18}$$

Second, $(\boldsymbol{\Sigma}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda})$ is converted into $(\mathbf{D}_0, \mathbf{D}_1)$. This procedure can be done in closed-form for MMPP(3)s. Otherwise, a non-linear system of equations needs to be solved by the definitions of $(\boldsymbol{\Sigma}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda})$. Numerical examples show that $(\mathbf{D}_0, \mathbf{D}_1)$.obtained in the second stage contains 9 rate parameters or less.

3. The Counting Process of MAP(3)s

3.1 Joint Laplace Transform and Moments of the Counting Process

The $(\boldsymbol{\Sigma}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda})$ representation can also be used for the Laplace transform of the counting process associated with MAP(3)s. Let N_t be the number of arrivals in $(0, t)$. The probability generating function of N_t and its Laplace transform are given as

$$g(z,t) \equiv \pi e^{(\mathbf{D}_0 + z\mathbf{D}_1)t}\boldsymbol{e},$$
$$\tilde{g}(z,s) = \pi(s\mathbf{I} - \mathbf{D}_0 - z\mathbf{D}_1)^{-1}\boldsymbol{e}$$

respectively; see Lucantoni [14] and Neuts [15]. Moreover, it can be easily verified that its Laplace transform is given as follows in terms of $(\Sigma, \Gamma, \Lambda)$

$$\begin{aligned}\tilde{g}(z,s) = & (\Sigma_1(s^2 + s\Sigma_2 + \Sigma_1) + (1-z)^2 \\ & (\Gamma_{22} - \Lambda_2\Gamma_{12} + \Sigma_1\Lambda_1) \\ & + (1-z)(\Sigma_1\Gamma_{11} - \Sigma_2\Gamma_{12} + s(\Lambda_2\Sigma_1 - \Gamma_{12})))/ \\ & (\Sigma_1(s^3 + s^2\Sigma_2 + s\Sigma_1 + (1-z)^3\Lambda_0 \\ & + (1-z)^2(s\Lambda_1 + \Gamma_{21}) \\ & + (1-z)(s^2\Lambda_2 + s\Gamma_{11} + \Gamma_{12})).\end{aligned}$$

By differentiating $\tilde{g}(z,s)$ with respect to z, we have

$$\left.\frac{\partial \tilde{g}(z,s)}{\partial z}\right|_{z=1} = \frac{\lambda_A}{s^2},$$

$$\left.\frac{\partial^2 \tilde{g}(z,s)}{\partial z^2}\right|_{z=1} = 2\frac{\lambda_A}{s^3}\left(\frac{b_2 s^2 + \psi_1 s + \Gamma_{12}}{s^2 + \Sigma_2 s + \Sigma_1}\right),$$

$$\left.\frac{\partial^3 \tilde{g}(z,s)}{\partial z^3}\right|_{z=1} = 3!\frac{\lambda_A}{s^4}\left(\frac{c_{22}s^4 + \psi_3 s^3 + \psi_2 s^2 + 2\Gamma_{12}\psi_1 s + \Gamma_{12}^2}{(s^2 + \Sigma_2 s + \Sigma_1)^2}\right)$$

where

$$\psi_1 = \Gamma_{11} - r_1\Gamma_{21},$$
$$\begin{aligned}\psi_2 = & \Gamma_{11}^2 + \Gamma_{12}\Lambda_2 + r_1(\Lambda_0\Sigma_1 - \Gamma_{11}\Gamma_{21}) \\ & + \Gamma_{22} - \Gamma_{21}\Sigma_2 - \Lambda_1\Sigma_1,\end{aligned}$$
$$\psi_3 = \Gamma_1(b_2 + \Lambda_2) - \Gamma_3 - \Lambda_1\Sigma_2 + r_1(\Lambda_0\Sigma_2 - \Gamma_3\Lambda_2).$$

The k-th factorial moments, $g^{(k)}(t) \equiv \mathrm{E}(N_t(N_t-1) \cdots (N_t - k + 1))$, can be obtained by the inverse Laplace transformation of $\partial^k \tilde{g}(z,s)/\partial z^k|_{z=1}$. That is, the first three factorial moments are obtained as follows

$$g^{(1)}(t) \equiv \boldsymbol{L}^{-1}\left[\left.\frac{\partial^1 \tilde{g}}{\partial z^1}\right|_{z=1}\right] = \mathrm{E}(N_t),$$

$$g^{(2)}(t) \equiv \boldsymbol{L}^{-1}\left[\left.\frac{\partial^2 \tilde{g}}{\partial z^2}\right|_{z=1}\right] = \mathrm{E}(N_t(N_t-1)),$$

$$g^{(3)}(t) \equiv \boldsymbol{L}^{-1}\left[\left.\frac{\partial^3 \tilde{g}}{\partial z^3}\right|_{z=1}\right] = \mathrm{E}(N_t(N_t-1)(N_t-2))$$

First, we have

$$\mathrm{E}(N_t) \equiv \boldsymbol{L}^{-1}\left[\frac{\lambda_A}{s^2}\right] = \lambda_A t.$$

For higher moments, we introduce the following notation to characterize the asymptotic variability of MAP(3)s.

$$\begin{aligned}M_a &= |\lambda_A\mathbf{I} - (-\mathbf{D}_0)| - |\lambda_A\mathbf{I} - \mathbf{D}_1| \\ &= (\lambda_A^3 - a_2\lambda_A^2 + a_1\lambda_A - a_0) - (\lambda_A^3 - \Lambda_2\lambda_A^2 + \Lambda_1\lambda_A - \Lambda_0) \\ &= -\Sigma_2\lambda_A^2 + \Gamma_{11}\lambda_A - \Gamma_{21}.\end{aligned}$$

It is shown below that $M_a = 0$ for Poisson process. The counterpart of M_a for MAP(2) can be found in Kim [11]. Let $R_1 = \sqrt{\Sigma_2^2 - 4\Sigma_1}$. For higher moments, we consider the following two cases for the inverse transformation, i.e. $\Sigma_2^2 - 4\Sigma_1 \neq 0$ and $\Sigma_2^2 - 4\Sigma_1 = 0$. For the case of $\Sigma_2^2 - 4\Sigma_1 \neq 0$, it can be shown that

$$\begin{aligned}g^2(t) = & \lambda_A^2 t^2 + \frac{2M_a}{\Sigma_1}t + \frac{e^{-t(R_1+\Sigma_2)/2} + e^{-t(R_1-\Sigma_2)/2} - 2}{\Sigma_1} \\ & \left(\frac{\Sigma_2 M_a}{\Sigma_1} + \lambda_A(\lambda_A - b_2)\right) - \frac{e^{-t(R_1+\Sigma_2)/2} - e^{-t(R_1-\Sigma_2)/2}}{\Sigma_1} \\ & \left(M_a\left(\frac{R_1}{\Sigma_1} + \frac{2}{R_1}\right) + \lambda_A(\lambda_A - b_2)\frac{\Sigma_2}{R_1}\right).\end{aligned}$$

Note that $g^2(t)$ is real-valued even if $\Sigma_2^2 - 4\Sigma_1$ is negative. The index of dispersion for counts (IDC), $I(t) \equiv \mathrm{Var}(N_t)/\mathrm{E}(N_t)$ is given as follows

$$\begin{aligned}I(t) = & 1 + \frac{2M_a}{\Gamma_{12}} + \frac{e^{-t(R_1+\Sigma_2)/2} + e^{-t(R_1-\Sigma_2)/2} - 2}{t\Gamma_{12}} \\ & \left(\frac{\Sigma_2 M_a}{\Sigma_1} + \lambda_A(\lambda_A - b_2)\right) - \frac{e^{-t(R_1+\Sigma_2)/2} - e^{-t(R_1-\Sigma_2)/2}}{t\Gamma_{12}} \\ & \left(M_a\left(\frac{R_1}{\Sigma_1} + \frac{2}{R_1}\right) + \lambda_A(\lambda_A - b_2)\frac{\Sigma_2}{R_1}\right). \qquad (19)\end{aligned}$$

If $\Sigma_2^2 - 4\Sigma_1 = 0$, then

$$g^2(t) = \lambda_A^2 t^2 + \frac{2M_a}{\Sigma_1} t - 2\left(\frac{M_a \Sigma_2}{\Sigma_1^2} + \frac{\lambda_A(\lambda_A - b_2)}{\Sigma_1}\right) + \frac{e^{-t\Sigma_2/2}}{\Sigma_1}\left(2M_a\left(\frac{\Sigma_2}{\Sigma_1} + t\right) + \lambda_A(\lambda_a - b_2)(2 + t\Sigma_2)\right)$$

and

$$I(t) = 1 + \frac{2M_a}{\Gamma_{12}} - \frac{2}{t\Gamma_{12}}\left(\frac{M_a \Sigma_2}{\Sigma_1} + \lambda_A(\lambda_A - b_2)\right) + \frac{2e^{-t\Sigma_2/2}}{t\Gamma_{12}}\left(2M_a\left(\frac{\Sigma_2}{\Sigma_1} + t\right) + \lambda_A(\lambda_A - b_2)(2 + t\Sigma_2)\right). \quad (20)$$

For both (19) and (20), the asymptotic variability is

$$I(\infty) \equiv \lim_{t \to \infty} I(t) = 1 + \frac{2M_a}{\Gamma_{12}}.$$

3.2 Dependence among c^2, $\mathrm{Cov}(T_1, T_2)$, and $I(\infty)$

Let T_1 and T_2 be the two consecutive stationary intervals. Also, let c^2 be the squared coefficient of variation of T_1, i.e. $c^2 \equiv \mathrm{E}(T^2)/\mathrm{E}(T)^2 - 1$. We introduce another simplifying notation M_d as follows

$$M_d \equiv (b_2 - a_2)\lambda_A^2 + a_1\lambda_A - a_0 = M_a + (b_2 - \Lambda_2)\lambda_A^2 + \Lambda_1\lambda_A - \Lambda_0.$$

Then, c^2, $\mathrm{Cov}(T_1, T_2)$, and $I(\infty)$ can be written in terms of M_a and M_d as follows

$$c^2 = 1 + \frac{2M_d}{a_0},$$

$$I(\infty) = 1 + \frac{2M_a}{\Gamma_{12}},$$

$$\mathrm{Cov}(T_1, T_2) = \frac{r_1^2}{a_0}\left(M_a - \frac{(\Gamma_{12} - \Lambda_0)M_d}{a_0}\right)$$

which reduces to the following identity in terms of c^2, $\mathrm{Cov}(T_1, T_2)$, and $I(\infty)$

$$\mathrm{Cov}(T_1, T_2) = \frac{r_1^2}{2a_0}\left(\Gamma_{12}(I(\infty) - c^2) + \Lambda_0(c^2 - 1)\right). \quad (21)$$

Therefore, if c^2 should be matched in a moment fitting, then there is only one more degree of freedom left for $I(\infty)$ and $\mathrm{Cov}(T_1, T_2)$.

4. MAP(3) Moment Fittings with the Counting Process

In this section, we present moment fitting procedures for MAP(3)s that account for the following set of moments including $I(t)$ and/or $I(\infty)$.

- $(r_1, r_2, r_3, r_4, r_5, r_{12}, r_{21}, r_{22})$, and $I(\infty)$
- $(r_1, r_2, r_3, r_4, r_5, r_{11}, r_{21}, r_{12})$, and $I(t)$
- $(r_1, r_2, r_3, r_4, r_5, r_{12}, r_{21})$, $I(t)$ and $I(\infty)$

As in [12], moment fitting procedures are done in two steps. First, moments are converted into $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and $(\Sigma, \Gamma, \Lambda)$. Whenever first five marginal moments are used for fitting, the coefficients $(a_0, a_1, a_2, b_1, b_2)$ are exactly determined by Eqs. (12)~(14). However, the procedure based on four joint moments in [12] needs to be modified if $I(t)$ and/or $I(\infty)$ should be matched instead of one or more of joint moments. Note that r_{11} needs to be replaced whenever r_2 and $I(\infty)$ are matched by the dependence given in Eq. (21). We choose to replace r_{22} whenever $I(t)$ is matched.

Second, $(\Sigma, \Gamma, \Lambda)$ is converted into $(\mathbf{D}_0, \mathbf{D}_1)$ by solving a non-linear system of equations by the definitions of $(\Sigma, \Gamma, \Lambda)$ given in Section 2.4. Closed-form formula is available for MMPP(3)s.

4.1 Fitting based on $(r_1, r_2, r_3, r_4, r_5, r_{12}, r_{21}, r_{22})$, and $I(\infty)$

If the asymptotic variability $I(\infty)$ should be matched instead of r_{11}, then the moment fitting procedure in Kim [12] based on nine moments of stationary intervals (1) can be modified accordingly by the dependence among c^2, Cov (T_1, T_2), and $I(\infty)$ given in (21). That is, $I(\infty)$ can be exactly matched if we set

$$r_{11}=r_1^2\left(\frac{\Gamma_{12}(I(\infty)-c^2)+\Lambda_0(c^2-1)}{2a_0}+1\right). \tag{22}$$

Note that the coefficients $(a_0, a_1, a_2, b_1, b_2)$ are exactly determined by Eqs. (12)~(14). By solving the following system of equations and (22) for r_{11} and $(\Gamma_{12}, \Gamma_{21}, \Lambda_0)$.

$$\Gamma_{12}=\frac{r_3^2+(r_4-r_{22})(r_{11}-r_2)+r_{12}r_{21}-r_3(r_{12}+r_{21})}{r_b},$$

$$\Gamma_{21}=a_0-\Gamma_{12}-\Lambda_0,$$

$$\Lambda_0=\Gamma_{12}-(r_3^2-r_2r_4+(r_4-r_2^2)r_{11}+(r_1r_2-r_3)(r_{12}+r_{21}) \\ +(r_2-r_1^2)r_{22})/r_b$$

we get

$$r_{11}=(r_c(I(\infty)-c^2)+(1-I(\infty))r_{12}r_{21} \\ +(1-c^2)r_1(r_1r_{22}-r_2(r_{12}+r_{21}))-2r_a)/ \\ ((c^2-1)r_2^2+(I(\infty)-c^2)r_4 \\ +(1-I(\infty))r_{22})-2r_a/r_1^2)$$

where $r_a=r_2^3+r_3^2+r_1^2r_4-r_2(2r_1r_3+r_4)$ and $r_c=r_2(r_4-r_{22})+r_3(r_{12}+r_{21})-r_3^2$. The rest of the parameters can be determined as follows

$$\Sigma_1=r_1\Gamma_{12},$$

$$\Sigma_2=-a_2+\frac{1}{r_b}((r_4^2-r_3r_5)+r_{11}(r_1r_5-r_3^2) \\ +(r_2r_3-r_1r_4)(r_{12}+r_{21})+r_{22}(r_1r_3-r_2^2)).$$

$$\Gamma_{11}=-2\Sigma_2+\frac{1}{r_b}(r_3r_4-r_2r_5+(r_1r_4-2r_2r_3+r_5)r_{11} \\ +(r_2^2-r_4)(r_{12}+r_{21})+(r_3-r_1r_2)r_{22}),$$

$$\Gamma_{22}=a_4\Gamma_{12},$$

$$\Lambda_1=a_1-\Sigma_1-\Gamma_{11},$$

$$\Lambda_2=a_2-\Sigma_2.$$

4.2 Fitting based on $(r_1, r_2, r_3, r_4, r_5, r_{11}, r_{12}, r_{21})$, and $I(t)$

The coefficients $(a_0, a_1, a_2, b_1, b_2)$ and (c_{11}, c_{12}, c_{21}) are exactly determined by Eqs. (12)~(14) and (15)~(17) respectively. Then, we write $(\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22}, \Lambda_0, \Lambda_1\Lambda_2)$ in terms of Σ_1 and Σ_2 for which we solve a system of two equations. That is,

$$\Gamma_{12}=\Sigma_1/r_1,$$

$$\Gamma_{22}=b_2\Sigma_1/r_1,$$

$$\Lambda_2=a_2-\Sigma_2.$$

Furthermore, $(\Gamma_{11}, \Gamma_{21}, \Lambda_1, \Lambda_0)$ can also be written in terms of Σ_1 and Σ_2 by simultaneously solving Eqs. for $(a_0, a_1, c_{11}, c_{12})$ in (7), (9), and (10). Note that Γ_{33} can be written in terms of Σ_1 and Σ_2 since it is given in terms of $(\boldsymbol{\Sigma}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda})$; see Appendix of [12] for the details.

Finally, Eqs. (19) or (20) along with Γ_{33} equated with $(c_{12}-c_{21})\Sigma_1/r_1$ can be numerically solved together for Σ_1 and Σ_2.

4.3 Fitting based on $(r_1, r_2, r_3, r_4, r_5, r_{12}, r_{21})$, $I(\infty)$ and $I(t)$

The coefficients $(a_0, a_1, a_2, b_1, b_2)$ are exactly determined by Eqs. (12)~(14). However, the coefficients (c_{11}, c_{12}, c_{21}) in Eqs. (15)~(17) are not determined directly by the moments since r_{11} is not used for moment matching. Instead, (c_{11}, c_{12}, c_{21}) are expressed in terms of Γ_{12} and Λ_0 by Eq. (22). The rest of the procedure is the same as in Section 4.2. That is, $(\Gamma_{12}, \Gamma_{22}, \Lambda_2)$ can be written in terms of Σ_1 and Σ_2. By Eqs. (7) and (8), $(\Gamma_{11}, \Gamma_{21}, \Lambda_1, \Lambda_0)$ can be written in terms of Σ_1 and Σ_2 which, in turn, can be numerically solved for by Eqs. (19) or (20) along with definition of Γ_{33}.

5. A Numerical Example of MAP(3) Moment Fittings

As a numerical example, consider the MMPP(4) given in Balcıoḡlu et al. [1] given as follows

$$\mathbf{Q} = \begin{bmatrix} -0.32 & 0.1 & 0.1 & 0.12 \\ 0.04 & -0.2 & 0.06 & 0.1 \\ 0.05 & 0.1 & -0.25 & 0.1 \\ 0 & 0.01 & 0.01 & -0.02 \end{bmatrix},$$

$$\mathbf{D}_1 = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2.36 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}, \qquad (23)$$

for which the reduced moments of stationary intervals are given as

$$(r_1, r_2, r_3, r_4, r_5) = (1.25, 4.89, 22.04, 100.44, 457.96),$$
$$(r_{11}, r_{21}, r_{12}, r_{22}) = (4.53, 20.31, 20.29, 91.98),$$
$$c^2 = 5.25,$$
$$I(20) = 28.40,$$
$$I(\infty) = 43.71.$$

From the marginal moments, the following coefficients are obtained

$$(a_0, a_1, a_2, b_1, b_2) = (3.90, 19.75, 9.29, 14.88, 3.62),$$

By the moment fitting procedure described in based on $(r_1, r_2, r_3, r_4, r_5, r_{12}, r_{21}, r_{22})$, and $I(\infty)$, we get

$$(\Sigma, \Gamma, \Lambda) = (0.0384, 0.4346, 2.005, 0.0307, 0.6693, 0.1113, 3.196, 17.71, 8.853)$$

along with $\Gamma_{33} = 0.0066155$ for which a MAP(3) can be obtained by the identities between $(\Sigma, \Gamma, \Lambda)$ and $(\mathbf{D}_0, \mathbf{D}_1)$ given in Section 2.4. The MMPP(4) in (23) can be approximated as an MMPP(3) as follows

$$\mathbf{D}_0 = \begin{bmatrix} -0.220 & 0.016 & 0.004 \\ 0.099 & -2.848 & 0.078 \\ 0.111 & 0.088 & -6.180 \end{bmatrix},$$

$$\mathbf{D}_1 = \begin{bmatrix} 0.200 & 0 & 0 \\ 0 & 2.671 & 0 \\ 0 & 0 & 5.982 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} -0.020 & 0.016 & 0.004 \\ 0.099 & -0.177 & 0.078 \\ 0.111 & 0.088 & -0.199 \end{bmatrix}.$$

The conversion formula from $(\Sigma, \Gamma, \Lambda)$ to $(\mathbf{D}_0, \mathbf{D}_1)$ for MMPP(3)s is given in [12].

By the moment fitting procedure based on $(r_1, r_2, r_3, r_4, r_5, r_{11}, r_{12}, r_{21})$, and $I(t)$, we get

$$(\Sigma, \Gamma, \Lambda) = (0.0337, 0.3953, 1.843, 0.0269, 0.6412, 0.0975, 3.228, 17.88, 8.892)$$

for which the following MMPP(3) can be obtained as follows

$$\mathbf{D}_0 = \begin{bmatrix} -0.22 & 0.016 & 0.004 \\ 0.1 & -2.883 & 0.094 \\ 0.107 & 0.114 & -6.224 \end{bmatrix},$$

$$\mathbf{D}_1 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 2.689 & 0 \\ 0 & 0 & 6.003 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} -0.02 & 0.016 & 0.004 \\ 0.1 & -0.194 & 0.094 \\ 0.107 & 0.114 & -0.221 \end{bmatrix}.$$

By the moment fitting procedure based on $(r_1, r_2, r_3, r_4, r_5, r_{12}, r_{21})$, $I(t)$ and $I(\infty)$, we get

$$(\Sigma, \Gamma, \Lambda) = (0.0293, 0.3589, 1.678, 0.0235, 0.6112, 0.0850, 3.261, 18.05, 8.929)$$

for which the following MMPP(3) can be obtained

$$\mathbf{D}_0 = \begin{bmatrix} -0.22 & 0.012 & 0.008 \\ 0.123 & -2.864 & 0.034 \\ 0.061 & 0.122 & -6.204 \end{bmatrix},$$

$$\mathbf{D}_1 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 2.708 & 0 \\ 0 & 0 & 6.021 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} -0.02 & 0.012 & 0.008 \\ 0.123 & -0.157 & 0.034 \\ 0.161 & 0.122 & -0.182 \end{bmatrix}.$$

6. Conclusion

In this paper, we proposed a minimal representation for the LT of the counting process associated with MAP(3)s. It is shown that the minimal representation for the LST of the stationary intervals can be used for LT of the counting process. Since the higher moments of the counting process are given as exponential functions, moment fitting procedure is more complicated than when moments of stationary intervals are used. We derived the second moment of the counting process and developed moment fitting procedures based on moments of both stationary intervals and counting process. Moment fittings including the third and higher-order moment of the counting process could be a direction of the future research.

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