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COMPOSITE HURWITZ RINGS AS ARCHIMEDEAN RINGS

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Abstract. Let $D \subseteq E$ be an extension of integral domains with characteristic zero, I be a nonzero proper ideal of D, and let $\mathrm{H}(D, E)$ and $\mathrm{H}(D, I)$ (resp., $\mathrm{h}(D, E)$ and $\mathrm{h}(D, I)$) be composite Hurwitz series rings (resp., composite Hurwitz polynomial rings). In this article, we show that $\mathrm{H}(D, E)$ is an Archimedean ring if and only if $\mathrm{h}(D, E)$ is an Archimedean ring, if and only if $\bigcap_{n\geq 1} d^n E = (0)$ for each nonzero nonunit d in D. We also prove that $\mathrm{H}(D, I)$ is an Archimedean ring if and only if $\mathrm{h}(D, I)$ is an Archimedean ring, if and only if D is an Archimedean ring.

1. Introduction

1.1. Composite Hurwitz rings

Let R be a commutative ring with identity and let $\mathrm{H}(R)$ be the set of formal expressions of the form $\sum_{i=0}^{\infty} a_i X^i$, where $a_i \in R$. Define addition and $*$-product (or Hurwitz product) on $\mathrm{H}(R)$ as follows: for $f = \sum_{i=0}^{\infty} a_i X^i, g = \sum_{i=0}^{\infty} b_i X^i \in \mathrm{H}(R)$,

$$f + g = \sum_{i=0}^{\infty} (a_i + b_i) X^i \text{ and } f * g = \sum_{n=0}^{\infty} c_n X^n,$$

where $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$. Then $\mathrm{H}(R)$ becomes a commutative ring with identity under these two operations. More precisely, $\mathrm{H}(R) = (R[\![X]\!], +, *)$. We call it the *Hurwitz series ring* over R. (The Hurwitz series ring was first considered by Hurwitz [3].) The *Hurwitz polynomial ring* $\mathrm{h}(R)$ over R is a subring of $\mathrm{H}(R)$ consisting of formal expressions of the type $\sum_{i=0}^{n} a_i X^i$, *i.e.*, $\mathrm{h}(R) = (R[X], +, *)$.

Let $f = \sum_{i=0}^{\infty} a_i X^i \in \mathrm{H}(R)$. Then the *order* of f is the smallest nonnegative integer m such that $a_m \neq 0$ and is denoted by $\mathrm{ord}(f)$. For a nonnegative integer

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n, $f(n)$ denotes the coefficient of X^n in f. In order to prevent the confusion, we denote the nth Hurwitz power of f by $f^{(n)}$.

Let $D \subseteq E$ be an extension of commutative rings with identity, I be a nonzero proper ideal of D, and set $\mathrm{H}(D,E) := \{f \in \mathrm{H}(E) \mid f(0) \in D\}$, $\mathrm{h}(D,E) := \{f \in \mathrm{h}(E) \mid f(0) \in D\}$, $\mathrm{H}(D,I) := \{f \in \mathrm{H}(D) \mid f(n) \in I \text{ for all } n \geq 1\}$, and $\mathrm{h}(D,I) := \{f \in \mathrm{h}(D) \mid f(n) \in I \text{ for all } n \geq 1\}$. Then $D \subsetneq \mathrm{H}(D,I) \subsetneq \mathrm{H}(D) \subseteq \mathrm{H}(D,E) \subseteq \mathrm{H}(E)$ and $D \subsetneq \mathrm{h}(D,I) \subsetneq \mathrm{h}(D) \subseteq \mathrm{h}(D,E) \subseteq \mathrm{h}(E)$. The rings $\mathrm{H}(D,E)$ and $\mathrm{H}(D,I)$ are called *composite Hurwitz series rings* and the rings $\mathrm{h}(D,E)$ and $\mathrm{h}(D,I)$ are called *composite Hurwitz polynomial rings*. It was shown that $\mathrm{H}(D,E)$ is an integral domain if and only if $\mathrm{h}(D,E)$ is an integral domain, if and only if $D \subseteq E$ is an extension of integral domains with characteristic zero [5, Proposition 2.1]; and that $\mathrm{H}(D,I)$ is an integral domain if and only if $\mathrm{h}(D,I)$ is an integral domain, if and only if D is an integral domain with characteristic zero [5, Proposition 3.1].

For more on Hurwitz series rings, the readers can refer to [1] and [4].

1.2. Archimedean rings

Let R be a commutative ring with identity. We say that R is an *Archimedean ring* if $\bigcap_{n\geq 1} a^n R = (0)$ for any nonzero nonunit $a \in R$. It is clear that an integral domain satisfying the ascending chain condition on principal ideals is Archimedean [2, Remark 1.1]. In [5], the authors characterized when composite Hurwitz series rings $\mathrm{H}(D,E)$ and $\mathrm{H}(D,I)$ and composite Hurwitz polynomial rings $\mathrm{h}(D,E)$ and $\mathrm{h}(D,I)$ satisfy the ascending chain condition on principal ideals, where $D \subseteq E$ is an extension of integral domains with characteristic zero and I is a nonzero proper ideal of D. In fact, it was shown that $\mathrm{H}(D,E)$ satisfies the ascending chain condition on principal ideals if and only if $\mathrm{h}(D,E)$ satisfies the ascending chain condition on principal ideals, if and only if $\bigcap_{n\geq 1} d_1 \cdots d_n E = (0)$ for each infinite sequence $(d_n)_{n\geq 1}$ consisting of nonzero nonunits of D [5, Theorem 2.4]; and that $\mathrm{H}(D,I)$ satisfies the ascending chain condition on principal ideals if and only if $\mathrm{h}(D,I)$ satisfies the ascending chain condition on principal ideals, if and only if D satisfies the ascending chain condition on principal ideals [5, Theorem 3.4].

In this article, we study when composite Hurwitz series rings $\mathrm{H}(D,E)$ and $\mathrm{H}(D,I)$ and composite Hurwitz polynomial rings $\mathrm{h}(D,E)$ and $\mathrm{h}(D,I)$ are Archimedean rings, where $D \subseteq E$ is an extension of integral domains with characteristic zero and I is a nonzero proper ideal of D. More precisely, we show that $\mathrm{H}(D,E)$ is an Archimedean ring if and only if $\mathrm{h}(D,E)$ is an Archimedean ring, if and only if $\bigcap_{n\geq 1} d^n E = (0)$ for any nonzero nonunit d of D (Theorem 3). We also prove that $\mathrm{H}(D,I)$ is an Archimedean ring if and only if $\mathrm{h}(D,I)$ is an Archimedean ring, if and only if D is an Archimedean ring (Theorem 6).

2. Main results

We start this section with a characterization of units in the composite Hurwitz series ring $\mathrm{H}(D,E)$ and the composite Hurwitz polynomial ring $\mathrm{h}(D,E)$.

Lemma 1. ([5, Lemma 2.2]) *Let $D \subseteq E$ be an extension of commutative rings with identity. Then the following assertions hold.*

(1) *A Hurwitz series $f \in H(D,E)$ is a unit if and only if $f(0)$ is a unit in D.*
(2) *A Hurwitz polynomial $f \in h(D,E)$ is a unit if and only if $f(0)$ is a unit in D and for each $n \geq 1$, $f(n)$ is nilpotent or some power of $f(n)$ is with torsion.*

To study when composite Hurwitz rings $\mathrm{H}(D,E)$ and $\mathrm{h}(D,E)$ are Archimedean, we need the following lemma.

Lemma 2. *Let $D \subseteq E$ be an extension of integral domains with characteristic zero and let f be a nonzero nonunit of $H(D,E)$ (resp., $h(D,E)$). If f has the positive order (resp., positive degree), then $\bigcap_{n\geq 1} f^{(n)} * H(D,E) = (0)$ (resp., $\bigcap_{n\geq 1} f^{(n)} * h(D,E) = (0)$).*

Proof. We first consider the composite Hurwitz series ring case. Let f be a nonzero nonunit of $\mathrm{H}(D,E)$ which has the positive order and let $g \in \bigcap_{n\geq 1} f^{(n)} * \mathrm{H}(D,E)$. Then for each $n \geq 1$, there exists a suitable element $h_n \in \mathrm{H}(D,E)$ such that $g = f^{(n)} * h_n$. Since $\mathrm{H}(D,E)$ is an integral domain, $\mathrm{ord}(g) \geq n{\cdot}\mathrm{ord}(f)$ for all $n \geq 1$; so $\mathrm{ord}(g) = \infty$. Hence $g = 0$, and thus $\bigcap_{n\geq 1} f^{(n)} * \mathrm{H}(D,E) = (0)$.

We next consider the composite Hurwitz polynomial ring case. Let f be a nonzero nonunit of $\mathrm{h}(D,E)$ which has the positive degree and choose any $g \in \bigcap_{n\geq 1} f^{(n)} * \mathrm{h}(D,E)$. Then for each $n \geq 1$, we can find an element $h_n \in \mathrm{h}(D,E)$ such that $g = f^{(n)} * h_n$. Since $\mathrm{h}(D,E)$ is an integral domain, $\deg(g) \geq n{\cdot}\deg(f)$ for all $n \geq 1$; so $\deg(g) = \infty$. Hence $g = 0$, and thus $\bigcap_{n\geq 1} f^{(n)} * \mathrm{h}(D,E) = (0)$. □

We are now ready to characterize Archimedean rings in terms of composite Hurwitz rings $\mathrm{H}(D,E)$ and $\mathrm{h}(D,E)$.

Theorem 3. *Let $D \subseteq E$ be an extension of integral domains with characteristic zero. Then the following statements are equivalent.*

(1) *$H(D,E)$ is an Archimedean ring.*
(2) *$h(D,E)$ is an Archimedean ring.*
(3) *$\bigcap_{n\geq 1} d^n E = (0)$ for each nonzero nonunit d in D.*

Proof. (1) $\Rightarrow$ (2) Let f be a nonzero nonunit of $\mathrm{h}(D,E)$. If $f(0)$ is a nonunit in D, then Lemma 1(1) says that f is a nonunit of $\mathrm{H}(D,E)$. Since $\mathrm{H}(D,E)$ is an Archimedean ring, we obtain

$$\begin{aligned}\bigcap_{n\geq 1} f^{(n)} * \mathrm{h}(D,E) &\subseteq \bigcap_{n\geq 1} f^{(n)} * \mathrm{H}(D,E)\\ &= (0).\end{aligned}$$

We next suppose that $f(0)$ is a unit in D. Then by Lemma 1(2), the degree of f is positive; so by Lemma 2, $\bigcap_{n\geq 1} f^{(n)} * \mathrm{h}(D,E) = (0)$. Thus $\mathrm{h}(D,E)$ is an Archimedean ring.

(2) $\Rightarrow$ (3) Let d be a nonzero nonunit of D and let $e \in \bigcap_{n\geq 1} d^n E$. Note that by Lemma 1(2), d is a nonzero nonunit of $\mathrm{h}(D,E)$. Since $\mathrm{h}(D,E)$ is an Archimedean ring, we obtain

$$\begin{aligned}eX &\in \bigcap_{n\geq 1} d^n * \mathrm{h}(D,E)\\ &= (0).\end{aligned}$$

Thus $e = 0$, which indicates that $\bigcap_{n\geq 1} d^n E = (0)$.

(3) $\Rightarrow$ (1) Suppose that $\bigcap_{n\geq 1} d^n E = (0)$ for each nonzero nonunit d in D, and let f be a nonzero nonunit of $\mathrm{H}(D,E)$. If f has the positive order, then the result comes directly from Lemma 2; so we assume that the order of f is zero. Let $g \in \bigcap_{n\geq 1} f^{(n)} * \mathrm{H}(D,E)$. Then for each $n \geq 1$, we can find an element $h_n \in \mathrm{H}(D,E)$ such that $g = f^{(n)} * h_n$; so $g(\mathrm{ord}(g)) = f(0)^n h_n(\mathrm{ord}(h_n))$ for all $n \geq 1$. Hence $g(\mathrm{ord}(g)) \in \bigcap_{n\geq 1} f(0)^n E$. Note that by Lemma 1(1), $f(0)$ is a nonunit in D; so $\bigcap_{n\geq 1} f(0)^n E = (0)$ by the assumption. Thus $g = 0$, which shows that $\mathrm{H}(D,E)$ is an Archimedean ring. □

We next characterize when the composite Hurwitz series ring $\mathrm{H}(D,I)$ and the composite Hurwitz polynomial ring $\mathrm{h}(D,I)$ are Archimedean. To do this, we need the following two lemmas.

Lemma 4. ([5, Lemma 3.2]) *Let D be a commutative ring with identity and I be a nonzero proper ideal of D. Then the following assertions hold.*

(1) *A Hurwitz series $f \in H(D,I)$ is a unit if and only if $f(0)$ is a unit in D.*
(2) *A Hurwitz polynomial $f \in h(D,I)$ is a unit if and only if $f(0)$ is a unit in D and for each $n \geq 1$, $f(n)$ is nilpotent or some power of $f(n)$ is with torsion.*

Lemma 5. *Let D be an integral domain with characteristic zero, I be a nonzero proper ideal of D, and let f be a nonzero nonunit of $H(D,I)$ (resp., $h(D,I)$). If f has the positive order (resp., positive degree), then $\bigcap_{n\geq 1} f^{(n)} * H(D,I) = (0)$ (resp., $\bigcap_{n\geq 1} f^{(n)} * h(D,I) = (0)$).*

Proof. While the proof can be done by a simple modification of that of Lemma 2, we insert it for the sake of completeness.

We first consider the composite Hurwitz series ring case. Let f be a nonzero nonunit of $\mathrm{H}(D,I)$ which has the positive order. If $g \in \bigcap_{n\geq 1} f^{(n)} * \mathrm{H}(D,I)$, then for each $n \geq 1$, there exists an element $h_n \in \mathrm{H}(D,I)$ such that $g = f^{(n)} * h_n$. Note that $\mathrm{H}(D,I)$ is an integral domain; so $\mathrm{ord}(g) \geq n{\cdot}\mathrm{ord}(f)$ for all $n \geq 1$. Hence $\mathrm{ord}(g) = \infty$, which means that $g = 0$. Thus $\bigcap_{n\geq 1} f^{(n)} * \mathrm{H}(D,I) = (0)$.

We next consider the composite Hurwitz polynomial ring case. Let f be a nonzero nonunit of $\mathrm{h}(D,I)$ which has the positive degree, and let $g \in \bigcap_{n\geq 1} f^{(n)} * \mathrm{h}(D,I)$. Then for each $n \geq 1$, we can find an element $h_n \in \mathrm{h}(D,I)$ such that $g = f^{(n)} * h_n$. Since $\mathrm{h}(D,I)$ is an integral domain, $\deg(g) \geq n{\cdot}\deg(f)$ for all $n \geq 1$; so $\deg(g) = \infty$. Hence $g = 0$, and thus $\bigcap_{n\geq 1} f^{(n)} * \mathrm{h}(D,I) = (0)$. □

We are closing this article with a characterization of Archimedean rings via the composite Hurwitz series ring $\mathrm{H}(D,I)$ and the composite Hurwitz polynomial ring $\mathrm{h}(D,I)$.

Theorem 6. *Let D be an integral domain with characteristic zero and I be a nonzero proper ideal of D. Then the following statements are equivalent.*

(1) *$H(D,I)$ is an Archimedean ring.*
(2) *$h(D,I)$ is an Archimedean ring.*
(3) *D is an Archimedean ring.*
(4) *$H(D)$ is an Archimedean ring.*
(5) *$h(D)$ is an Archimedean ring.*

Proof. (1) ⇒ (2) Let f be a nonzero nonunit of $\mathrm{h}(D,I)$. If $f(0)$ is a nonunit in D, then by Lemma 4(1), f is a nonunit of $\mathrm{H}(D,I)$. Since $\mathrm{H}(D,I)$ is an Archimedean ring, we obtain

$$\begin{aligned}\bigcap_{n\geq 1} f^{(n)} * \mathrm{h}(D,I) &\subseteq \bigcap_{n\geq 1} f^{(n)} * \mathrm{H}(D,I)\\ &= (0).\end{aligned}$$

We next assume that $f(0)$ is a unit in D. Note that by Lemma 4(2), f is a nonconstant; so by Lemma 5, $\bigcap_{n\geq 1} f^{(n)} * \mathrm{h}(D,I) = (0)$. Thus $\mathrm{h}(D,I)$ is an Archimedean ring.

(2) ⇒ (3) Let d be a nonzero nonunit of D. Then by Lemma 4(2), d is a nonzero nonunit of $\mathrm{h}(D,I)$. Since $\mathrm{h}(D,I)$ is Archimedean, we obtain

$$\begin{aligned}\bigcap_{n\geq 1} d^n D &\subseteq \bigcap_{n\geq 1} d^n * \mathrm{h}(D,I)\\ &= (0),\end{aligned}$$

which shows that D is an Archimedean ring.

(3) ⇒ (1) Let f be a nonzero nonunit of $\mathrm{H}(D,I)$. If f has the positive order, then by Lemma 5, $\bigcap_{n\geq 1} f^{(n)} * \mathrm{H}(D,I) = (0)$. Now, we suppose that the order of f is zero, and let $g \in \bigcap_{n\geq 1} f^{(n)} * \mathrm{H}(D,I)$. Then for each $n \geq 1$, there exists an element $h_n \in \mathrm{H}(D,I)$ such that $g = f^{(n)} * h_n$; so $g(\mathrm{ord}(g)) = f(0)^n h_n(\mathrm{ord}(h_n))$

for all $n \geq 1$. Hence $g(\text{ord}(g)) \in \bigcap_{n\geq 1} f(0)^n D$. Note that by Lemma 4(1), $f(0)$ is a nonunit in D; so $\bigcap_{n\geq 1} f(0)^n D = (0)$ because D is Archimedean. Hence $g = 0$, and thus $\mathrm{H}(D, I)$ is an Archimedean ring.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) These equivalences can be obtained by applying Theorem 3 to the case when $D = E$. □

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