East Asian Math. J.
Vol. 33 (2017), No. 3, pp. 291–293
http://dx.doi.org/10.7858/eamj.2017.021

TWO IDENTITIES INVOLVING THETA FUNCTIONS

Mahendra Pal Chaudhary, Getachew Abiye Salilew, and Junesang Choi*

Abstract. We aim to present two identities which reveal certain interesting relationships among three fundamental theta functions arising from the Jacobi's triple product in an elementary way.

1. Introduction and preliminaries

Jacobi [7] initiated the theory of theta functions which has a long history and many applications in a variety of research fields such as number theory (for example, quadratic forms and elliptic functions) and quantum physics. The Jacobi triple product, which an infinite series is expressed as an infinite product, is given as follows (see [7]):

$$\sum_{k=-\infty}^{\infty} y^k x^{k^2} = \prod_{k=1}^{\infty} (1+yx^{2k-1}) \left(1+\frac{1}{y}x^{2k-1}\right)(1-x^{2k}) \tag{1.1}$$

$$(|x|<1,\ y \neq 0).$$

From (1.1), it is easy to define the following three fundamental theta functions:

$$f(-x) := \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(3n-1)}{2}} = \prod_{k=1}^{\infty} (1-x^k)\,; \tag{1.2}$$

$$\phi(x) := \sum_{n=-\infty}^{\infty} x^{n^2} = \prod_{k=1}^{\infty} (1+x^{2k+1})(1+x^{2k-1})(1-x^{2k}); \tag{1.3}$$

$$\psi(x) := \sum_{n=0}^{\infty} x^{\frac{n(n+1)}{2}} = \prod_{k=1}^{\infty} \frac{(1-x^{2k})}{(1-x^{2k-1})}. \tag{1.4}$$

In this note, we present two identities which reveal certain interesting relationships among three fundamental theta functions arising from the Jacobi's triple product in an elementary way. For more details and results, the interested reader may be referred to the works [1, 2, 3, 4, 5, 6, 8, 9, 10] and the references therein.

Received March 17, 2017; Accepted April 12, 2017.
2010 *Mathematics Subject Classification.* Primary 5A30, 11F27 ; Secondary 11P83.
Key words and phrases. Jacobi's triple product; Three fundamental theta functions.
* Corresponding author.

(pISSN 1226-6973, eISSN 2287-2833)

To do this, we express the functions $f(-x)$, $\phi(x)$, and $\psi(x)$ in the rising powers:

$$\begin{aligned} f(-x) &= 1+\sum_{n=1}^{\infty}(-1)^n\left(x^{\frac{n(3n-1)}{2}}+x^{\frac{n(3n+1)}{2}}\right)\\ &= 1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-\cdots; \end{aligned} \tag{1.5}$$

$$\phi(x) = 1+2\sum_{n=1}^{\infty}x^{n^2} = 1+2x+2x^4+2x^9+2x^{16}+\cdots; \tag{1.6}$$

$$\psi(x) = 1+\sum_{n=1}^{\infty}x^{\frac{n(n+1)}{2}} = 1+x+x^3+x^6+x^{10}+x^{15}+\cdots. \tag{1.7}$$

2. Main results

Here, we present two identities that reveal certain interesting relationships among the three functions $f(-x)$, $\phi(x)$, and $\psi(x)$, which are asserted by the following theorem.

Theorem 2.1. *Let $f(-x)$, $\phi(x)$, and $\psi(x)$ be the functions in* (1.2), (1.3), *and* (1.4), *respectively. Then*

$$\begin{aligned} f(-x^3)f(-x^9) =&\phi(x^{54})[\psi(x^4)-x^4\psi(x^{36})]+x^{13}\psi(x^{108})[\phi(x^2)-\phi(x^{18})]\\ &-x^3\psi(x^{27})[\psi(x)-x\psi(x^9)] \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &8x^3[\psi(x^4)-x^4\psi(x^{36})][\psi(x^{20})-x^{20}\psi(x^{180})]\\ &=[\phi(x^3)\phi(x^{15})-\phi(-x^3)\phi(-x^{15})]+3[\phi(-x^{27})\phi(-x^{135})-\phi(x^{27})\phi(x^{135})]\\ &\quad+[\phi(x^3)\phi(x^{135})-\phi(-x^3)\phi(-x^{135})]+[\phi(x^{27})\phi(x^{15})-\phi(-x^{27})\phi(-x^{15})]. \end{aligned} \tag{2.2}$$

Proof. For (2.1), let $\mathcal{L}(x)$ and $\mathcal{R}(x)$ be the left- and right-sides of (2.1), respectively. By using the expansions in (1.5), (1.6), and (1.7), we can see that

$$\begin{aligned} \mathcal{L}(x) &= 1-x^3-x^6-x^9+x^{12}+2x^{15}-x^{18}+2x^{21}-x^{30}\\ &-x^{33}-x^{36}-x^{39}+x^{45}-x^{48}-x^{51}+2x^{54}+x^{60}+2x^{63}\\ &+x^{66}-x^{69}-x^{75}+2x^{78}-x^{81}-x^{87}-x^{90}-x^{96}-x^{99}\\ &-x^{105}-2x^{108}+2x^{111}+2x^{114}-x^{120}+x^{123}+x^{129}\\ &-x^{135}+2x^{138}+2x^{141}+x^{144}-2x^{150}+2x^{153}-x^{156}\\ &-x^{162}-x^{165}-x^{168}+x^{171}-\cdots\\ &=\mathcal{R}(x). \end{aligned}$$

This completes the proof of (2.1). Similarly, we can prove (2.2). We omit the details.

□

References

[1] M. P. Chaudhary, On q-product identities, *Pacific J. Appl. Math.* **5**(2) (2013), 123–129.
[2] M. P. Chaudhary and J. Choi, Note on modular relations for Roger Ramanujan type identities and representation for Jacobi identities, *East Asian Math. J.* **31**(5) (2015), 659–665.
[3] M. P. Chaudhary and G. A. Salilew, Elementary proof for theta function identities, submitted, 2017.
[4] Z. Cao, Integer matrix exact covering sytems and product identities for theta functions, *Inter. Math. Res. Notices* **19** (2011), 4471–4514.
[5] S. Cooper, The quinntuble product identity, *Int. J. Number Theory* **2**(1) (2006), 115–161.
[6] M.D. Hirschorn, A simple proof of an identity of Ramanujan, *J. Austral. Math. Soc. Ser. A* **44**(1)(1988), 42–45.
[7] G. J. Jacobi, *Fundamental Nova Theoriae Functionum Ellipticarum*, Sumptibus Fratrum Borntrger, Regiomonti, 1829.
[8] H. Schroter, De Aequationibus Modularibus, Disserrtatio Inaugurolis, Albertino, Litterarum University, Regiomonti, 1854.
[9] J. L. C. Villacorto, Problems on theta function identities.
[10] J. L. C. Villacorto and M. P. Chaudhary, On Certain theta function identities, submitted, (2017).

Mahendra Pal Chaudhary
Department of Mathematics, International Scientific Research and Welfare Organization, New Delhi 110018, India
E-mail address: dr.m.p.chaudhary@gmail.com

Getachew Abiye Salilew
Department of Mathematics, College of Natural and Computational Science, Madda Walabu University, Bale Robe, Ethiopia
E-mail address: Getachewabye@gmail.com

Junesang Choi
Department of Mathematics, Dongguk University, Gyeongju 38066, Republic of Korea
E-mail address: junesang@mail.dongguk.ac.kr