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ON SUFFICIENCY AND DUALITY FOR ROBUST OPTIMIZATION PROBLEMS INVOLVING (V,ρ)-INVEX FUNCTIONS

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Abstract. In this paper, we formulate a sufficient optimality theorem for the robust optimization problem (UP) under (V,ρ)-invexity assumption. Moreover, we formulate a Mond-Weir type dual problem for the robust optimization problem (UP) and show that the weak and strong duality hold between the primal problems and the dual problems.

1. Introduction

Consider a standard nonlinear programming problem with inequality constraints

$$\text{(P)}\quad \inf_{x\in\mathbb{R}^n}\{f(x) : g_i(x)\leq 0,\ i=1,\cdots,m\},$$

where $f:\mathbb{R}^n\to\mathbb{R}$ and $g_i:\mathbb{R}^n\to\mathbb{R}$ are continuously differentiable functions. The problem in the face of data uncertainty in the constraints and the objective function can be captured by the following nonlinear programming problem:

$$\text{(UP)}\quad \inf_{x\in\mathbb{R}^n}\{f(x,u) : g_i(x,v_i)\leq 0,\ i=1,\cdots,m\},$$

where u, v_i are uncertain parameters and $u\in U$, $v_i\in V_i, i=1,\cdots,m$ for some convex compact sets $U\subset\mathbb{R}^p, V_i\subset\mathbb{R}^q, i=1,\cdots,m$, respectively and $f:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$, $g_i:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}, i=1,\cdots,m$ are continuously differentiable. Sometimes, $f(x,u)$ in (UP) can be $f(x)$ without the uncertain parameter $u\in U$. Robust optimization, which has emerged as a powerful deterministic approach for studying mathematical programming under uncertainty ([1] – [4], [6]), associates with the uncertain program (UP) its robust counterpart [5],

$$\text{(RP)}\quad \inf_{x\in\mathbb{R}^n}\{\max_{u\in U} f(x,u) : g_i(x,v_i)\leq 0,\ \forall v_i\in V_i,\quad i=1,\cdots,m\},$$

where the uncertain constraints are enforced for every possible value of the parameters within their prescribed uncertainty sets U, V_i, $i=1,\cdots,m$.

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Recently, Lee and Kim [8] established a necessary optimality theorem and a sufficient optimality theorem for the problem (UP) under convexity. And they give Wolfe type dual problem for the problem (UP).

In this paper, we give a sufficient optimality theorem for the robust optimization problem (UP) under (V,ρ)-invexity assumption. Moreover, we formulate a Mond-Weir type dual problem for the robust optimization problem (UP) and show that the weak and strong duality hold between the primal problems and the dual problems.

2. Optimality Theorems

In this section, we provide necessary and sufficient optimality conditions for the uncertain optimization problem (UP). To begin with, we recall that the robust feasible set F is defined by

$$F := \{x \in \mathbb{R}^n \ : \ g_i(x,v_i) \leq 0, \ \forall v_i \in V_i, \quad i = 1, \cdots, m\}.$$

We say that x^* is a robust solution of (UP) if x^* is a minimizer of (UP), that is, $x^* \in F$ and $f(x,u) \geq f(x^*,u)$ $\forall x \in F, \ u \in U$. Denote that $\nabla_1 g$ is the derivative of g with respect to the first variables.

We introduce the following definition due to Kuk et al. [7]

Definition 1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be (V,ρ)-invex at $u \in \mathbb{R}^n$ with respect to the function η and $\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ if there exists $\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ and $\rho \in \mathbb{R}$ such that for any $x \in \mathbb{R}^n$

$$\alpha(x,u)\big[f(x) - f(u)\big] \geqq \nabla f(u)^T \eta(x,u) + \rho\|\theta(x,u)\|^2.$$

Definition 2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be η-invex at $u \in \mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$

$$f(x) - f(u) \geqq \nabla f(u)^T \eta(x,u).$$

Now we give a necessary optimality theorem for a solution of (UP).

Theorem 2.1. [8] *Let $\bar{x} \in F$ be a robust solution of (UP). Suppose that $f(\bar{x},\cdot)$ is concave on U and $g_j(\bar{x},\cdot)$ are concave on $V_j, j = 1, \cdots, m$. Then there exist $\mu_j \geq 0, \ j = 1, \cdots, m, \ \bar{u} \in U$ and $\bar{v}_j \in V_j, \ j = 1, \cdots, m$ such that*

$$\begin{aligned}
&\lambda \nabla_1 f(\bar{x},\bar{u}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x},\bar{v}_j) = 0,\\
&f(\bar{x},\bar{u}) = \max_{u \in U} f(\bar{x},u),\\
&\mu_j g_j(\bar{x},\bar{v}_j) = 0, \ j = 1, \cdots, m.
\end{aligned}$$

Moreover, if we assume that the Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) holds at $\bar{x}$, then

$$\begin{aligned}
&\nabla_1 f(\bar{x},\bar{u}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x},\bar{v}_j) = 0,\\
&f(\bar{x},\bar{u}) = \max_{u\in U} f(\bar{x},u),\\
&\mu_j g_j(\bar{x},\bar{v}_j) = 0,\ j = 1,\cdots,m.
\end{aligned}$$

Now we give sufficient optimality theorems for the uncertain optimization problem (UP) by using its robust counterpart (RP).

Theorem 2.2. *Let $\bar{x} \in F$ and $f(\bar{x},\cdot)$ is concave on U and $g_j(\bar{x},\cdot)$ are concave on $V_j, j = 1,\cdots,m$. Suppose that there exist $\mu_j \geq 0,\ j = 1,\cdots,m$, $\bar{u} \in U$ and $\bar{v}_j \in V_j,\ j = 1,\cdots,m$ such that*

$$\nabla_1 f(\bar{x},\bar{u}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x},\bar{v}_j) = 0, \tag{1}$$

$$f(\bar{x},\bar{u}) = \max_{u\in U} f(\bar{x},u), \tag{2}$$

$$\mu_j g_j(\bar{x},\bar{v}_j) = 0,\ j = 1,\cdots,m.$$

If $f(\cdot,\bar{u})$ is (V,ρ)-invex and $g_j(\cdot,\bar{v}_j),\ j = 1,\cdots,m$, is η-invex at $\bar{x}$ with respect to the same η, and $\rho\|\theta(x,\bar{x})\|^2 \geqq 0$, then $\bar{x} \in F$ is a solution of (UP).

Proof. Suppose that $\bar{x} \in F$ is not a robust solution of (UP). Then there exist a feasible solution x of (UP) such that

$$\max_{u\in U} f(x,u) < \max_{u\in U} f(\bar{x},u).$$

From (2), $f(x,\bar{u}) < f(\bar{x},\bar{u})$. Since $\alpha(x,u) > 0$,

$$\alpha(x,u)[f(x,\bar{u}) - f(\bar{x},\bar{u})] < 0.$$

By the (V,ρ)-invexity of $f(\cdot,\bar{u})$, we have

$$\nabla_1 f(\bar{x},\bar{u})^T \eta(x,\bar{x}) + \rho\|\theta(x,\bar{x})\|^2 < 0.$$

Since $\rho\|\theta(x,\bar{x})\|^2 \geq 0$,

$$\nabla_1 f(\bar{x},\bar{u})^T \eta(x,\bar{x}) < 0,$$

and so, it follows from (1) that $\sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x},\bar{v}_j)^T \eta(x,\bar{x}) > 0$. Then, by the η-invexity of $g_j(\cdot,\bar{v}_j)$, we have

$$\mu_j g_j(x,\bar{v}_j) > \mu_j g_j(\bar{x},\bar{v}_j).$$

Since $\sum_{j=1}^{m} \mu_j g_j(\bar{x},\bar{v}_j) = 0$, we have $\sum_{j=1}^{m} \mu_j g_j(x,\bar{v}_j) > 0$, which is contradiction, since $\mu_j \geq 0,\ j = 1,\cdots,m$ and x is a feasible solution of (UP). Consequently, $\bar{x}$ is a robust solution of (UP). ☐

3. Duality Theorems

In this section, we establish Mond-Weir type robust duality between (UP) and (MD).

$$\begin{aligned} \text{(MD)} \qquad \text{maximize} \qquad & f(x,u) \\ \text{subject to} \qquad & \nabla_1 f(x,u) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(x,v_j) = 0, \qquad (3) \\ & \sum_{j=1}^{m} \mu_j g_j(x,v_j) = 0, \\ & \mu_j \geq 0,\ u \in U,\ v_j \in V_j,\ j = 1,\cdots,m. \end{aligned}$$

Let $V = V_1 \times \cdots \times V_m$.

Theorem 3.1. *(Weak Duality) Let $x \in \mathbb{R}^n$ be feasible for (UP) and $(\bar{x},\bar{u},\bar{v},\bar{\mu}) \in \mathbb{R}^n \times U \times V \times \mathbb{R}^m$ be feasible for (MD), and assume that $f(\bar{x},\cdot)$ is concave on U and $g_j(\bar{x},\cdot),\ j = 1,\cdots,m$ are concave on V_j. Suppose that $f(\cdot,\bar{u})$ is (V,ρ)-invex and $g_j(\cdot,\bar{v}_j),\ j = 1,\cdots,m$ are η-invex at $\bar{x}$ with respect to the same η and $\rho\|\theta(x,\bar{x})\|^2 \geqq 0$, then*

$$\max_{u\in U} f(x,u) \geq f(\bar{x},\bar{u}).$$

Proof. Let x be feasible for (UP) and $(\bar{x},\bar{u},\bar{v},\bar{\mu})$ be feasible for (MD). Then we have

$$\sum_{j=1}^{m} \mu_j g_j(x,\bar{v}_j) \leqq \sum_{j=1}^{m} \mu_j g_j(\bar{x},\bar{v}_j).$$

By the η-invexity of $g_j(\cdot,\bar{v}_j),\ j = 1,\cdots,m$, we have

$$\sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x},\bar{v}_j)^T \eta(x,\bar{x}) \leqq 0.$$

Using (3), we obtain

$$\nabla_1 f(x,u)^T \eta(x,\bar{x}) \geqq 0. \qquad (4)$$

Now suppose that

$$\max_{u\in U} f(x,u) < f(\bar{x},\bar{u}).$$

Then $f(x,\bar{u}) < f(\bar{x},\bar{u})$. Since $\alpha(x,\bar{x}) > 0$,

$$\alpha(x,\bar{x})\,[f(x,\bar{u}) - f(\bar{x},\bar{u})] < 0.$$

By the (V,ρ)-invexity of $f(\cdot,\bar{u})$ at $\bar{x}$,

$$\nabla_1 f(\bar{x},\bar{u})^T \eta(x,\bar{x}) + \rho\|\theta(x,\bar{x})\|^2 < 0.$$

Since $\rho\|\theta(x,\bar{x})\|^2 \geqq 0$, we have

$$\nabla_1 f(\bar{x},\bar{u})^T \eta(x,\bar{x}) < 0$$

which contradicts (4). Hence the result holds. □

Theorem 3.2. *(Strong Duality) Let $\bar{x}$ be a solution of (UP). Assume that the Extended Mangasarian-Fromovitz constraint qualification holds at $\bar{x}$. Then, there exist $(\bar{u}, \bar{v}, \bar{\mu})$ such that $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu})$ is feasible for (MD). Moreover, if the weak duality holds, then $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu})$ is a solution of (MD).*

Proof. Since $\bar{x}$ be a solution of (UP) at which the Extended Mangasarian-Fromovitz constraint qualification is satisfied, then by Theorem 2.1, there exists $\bar{\mu}_j \geq 0,\ j = 1, \cdots, m,\ \bar{u} \in U$ and $\bar{v}_j \in V_j,\ \ j = 1, \cdots, m$ such that

$$\begin{aligned}
&\nabla_1 f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0, \\
&f(\bar{x}, \bar{u}) = \max_{u \in U} f(\bar{x}, u), \\
&\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0,\ j = 1, \cdots, m.
\end{aligned}$$

Thus $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu})$ is a feasible for (MD). On the other hand, by weak duality (Theorem 3.1), $\max_{u \in U} f(\bar{x}, u) = f(\bar{x}, \bar{u}) \geqq f(\widetilde{x}, \widetilde{u})$ for and (MD) feasible solution $(\widetilde{x}, \widetilde{u}, \widetilde{v}, \widetilde{\mu})$. Hence $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu})$ is a solution of (MD). □

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