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QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN NORMED SPACES

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Abstract. In this paper, we solve the following quadratic ρ-functional inequalities

$$\begin{aligned}&\left\|f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)\right.\\&\qquad\qquad\left.-f(x)-f(y)-f(z)\right\|\\&\le\|\rho(f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)\\&\qquad\qquad-4f(x)-4f(y)-4f(z))\|,\end{aligned}\tag{0.1}$$

where ρ is a fixed non-Archimedean number with $|\rho| < \frac{1}{|4|}$, and

$$\begin{aligned}&\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)\\&\qquad\qquad-4f(x)-4f(y)-4f(z)\|\\&\le\left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)\right.\right.\\&\qquad\qquad\left.\left.-f(x)-f(y)-f(z)\right)\right\|,\end{aligned}\tag{0.2}$$

where ρ is a fixed non-Archimedean number with $|\rho| < |8|$.

Using the direct method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces and prove the Hyers-Ulam stability of quadratic ρ-functional equations associated with the quadratic ρ-functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

1. Introduction and Preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0, \infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the triangle inequality holds, i.e.,

$$|r + s| \le |r| + |s|, \qquad \forall r, s \in K.$$

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A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|, |s|\}, \qquad \forall r, s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1 ([11])**.** Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space.*

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space.*

The stability problem of functional equations originated from a question of Ulam [16] concerning the stability of group homomorphisms.
The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy equation.* In particular, every solution of the Cauchy equation is said to be an *additive mapping.* Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping.* The stability of quadratic functional equation was proved by Skof [15] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. See [3, 9, 10] for more functional equations.

The functional equation

$$2f\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) = f(x) + f(y)$$

is called a *Jensen type quadratic equation.*

In [6], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.2}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [14]. Gilányi [7] and Fechner [4] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [12] proved the Hyers-Ulam stability of additive functional inequalities.

In Section 3, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in non-Archimedean Banach spaces. We moreover prove the Hyers-Ulam stability of a quadratic ρ-functional equation associated with the quadratic ρ-functional inequality (0.1) in non-Archimedean Banach spaces.

In Section 4, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in non-Archimedean Banach spaces. We moreover prove the Hyers-Ulam stability of a quadratic ρ-functional equation associated with the quadratic ρ-functional inequality (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

2. Quadratic Functional Equations

Theorem 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies*

$$\begin{aligned} f\left(\frac{x+y+z}{2} + \frac{x-y-z}{2} + \frac{y-x-z}{2} + \frac{z-x-y}{2}\right) \\ = f(x) + f(y) + f(z) \end{aligned} \tag{2.1}$$

if and only if the mapping $f : X \to Y$ is a quadratic mapping.

Proof. Sufficiency. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = z = 0$ in (2.1), we have $4f(0) = 3f(0)$. So $f(0) = 0$.

Letting $y = z = 0$ in (2.1), we get

$$\begin{aligned} 2f\left(\frac{x}{2}\right) + 2f\left(-\frac{x}{2}\right) = f(x), \\ 2f\left(-\frac{x}{2}\right) + 2f\left(\frac{x}{2}\right) = f(-x) \end{aligned} \tag{2.2}$$

for all $x \in X$, which imply that $f(x) = f(-x)$ for all $x \in X$.

From this and (2.2), we obtain $4f\left(\frac{x}{2}\right) = f(x)$ or $f(2x) = 4f(x)$ for all $x \in X$.

Putting $z = 0$ in (2.1), we obtain

$$\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) = f(x) + f(y)$$

for all $x, y \in X$, which means that $f : X \to Y$ is a quadratic mapping.

Necessity. Assume that $f : X \to Y$ is quadratic.

By $f(x+y)+f(x-y) = 2f(x)+2f(y)$, one can easily get $f(0) = 0$, $f(x) = f(-x)$ and $f(2x) = 4f(x)$ for all $x \in X$. So

$$\begin{aligned} &f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) + f\Big(\frac{z-x-y}{2}\Big) \\ &= \Big[2f\Big(\frac{x}{2}\Big) + 2f\Big(\frac{y+z}{2}\Big)\Big] + \Big[2f\Big(-\frac{x}{2}\Big) + 2f\Big(\frac{y-z}{2}\Big)\Big] \\ &= 4f\left(\frac{x}{2}\right) + f\Big(\frac{y+z+y-z}{2}\Big) + f\Big(\frac{y+z-y+z}{2}\Big) \\ &= f(x) + f(y) + f(z) \end{aligned}$$

for all $x, y, z \in X$, which is the functional equation (2.1) and the proof is complete. □

Corollary 2.2. *Let X and Y be vector spaces. An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned} f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ = 4f(x) + 4f(y) + 4f(z) \end{aligned} \tag{2.3}$$

for all $x, y, z \in X$. Then the mapping $f : X \to Y$ is a quadratic mapping.

Proof. Assume that $f : X \to Y$ satisfies (2.3).

Letting $x = y = z = 0$ in (2.3), we have $4f(0) = 12f(0)$. So $f(0) = 0$.

Letting $z = 0$ in (2.3), we get

$$2f(x+y) + 2f(x-y) = 4f(x) + 4f(y)$$

and so $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. □

3. Quadratic ρ-functional Inequality (0.1)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < \frac{1}{|4|}$.

In this section, we solve and investigate the quadratic ρ-functional inequality (0.1) in non-Archimedean normed spaces.

Lemma 3.1. *An even mapping $f : X \to Y$ satisfies*

$$(3.1)\qquad \left\|f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right\| \le \|\rho(f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z))\|$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = z = 0$ in (3.1), we get

$$\|f(0)\| \le |\rho|\|8f(0)\|.$$

So $f(0) = 0$.

Letting $y = z = 0$ in (3.1), we get $\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le 0$ and so

$$(3.2)\qquad f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}&\left\|f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right\| \\ &\le \|\rho(f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z))\| \\ &= |\rho|\left\|4f\left(\frac{x+y+z}{2}\right)+4f\left(\frac{x-y-z}{2}\right)+4f\left(\frac{y-x-z}{2}\right)+4f\left(\frac{z-x-y}{2}\right)-4f(x)-4f(y)-4f(z)\right\| \\ &\le |4|\cdot|\rho|\left\|f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right\|\end{aligned}$$

and so

$$f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)=f(x)+f(y)+f(z)$$

for all $x, y, z \in X$.

The converse is obviously true. $\square$

Corollary 3.2. *An even mapping $f : X \to Y$ satisfies*

$$f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \tag{3.3}$$
$$= \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadratic.

The functional equation (3.3) is called a *quadratic ρ-functional equation.*

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (3.1) in non-Archimedean Banach spaces.

Theorem 3.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ and let $f : X \to Y$ be an even mapping such that*

$$\lim_{j\to\infty} |4|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) = 0, \tag{3.4}$$

$$\left\|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right\| \tag{3.5}$$
$$\leq \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\| + \varphi(x, y, z)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \sup_{j\geq 0} \left\{|4|^j \varphi\left(\frac{x}{2^j}, 0, 0\right)\right\} \tag{3.6}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (3.5), we get $\|f(0)\| \leq |\rho|\|8f(0)\|$. So $f(0) = 0$.

Letting $y = z = 0$ in (3.5), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(x, 0, 0) \tag{3.7}$$

for all $x \in X$. So

$$(3.8)\ \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|$$
$$\leq \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|\right\}$$
$$= \max\left\{|4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\|\right\}$$
$$\leq \sup_{j\geq l}\left\{|4|^j \varphi\left(\frac{x}{2^j}, 0, 0\right)\right\}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.6).

It follows from (3.4) and (3.5) that

$$\left\|h\left(\frac{x+y+z}{2}\right) + h\left(\frac{x-y-z}{2}\right) + h\left(\frac{y-x-z}{2}\right) + h\left(\frac{z-x-y}{2}\right) - h(x) - h(y) - h(z)\right\|$$
$$= \lim_{n\to\infty} |4|^n \left\|f\left(\frac{x+y+z}{2^{n+1}}\right) + f\left(\frac{x-y-z}{2^{n+1}}\right) + f\left(\frac{y-x-z}{2^{n+1}}\right) + f\left(\frac{z-x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\|$$
$$\leq \lim_{n\to\infty} |4|^n |\rho| \left\|f\left(\frac{x+y+z}{2^n}\right) + f\left(\frac{x-y-z}{2^n}\right) + f\left(\frac{y-x-z}{2^n}\right) + f\left(\frac{z-x-y}{2^n}\right) - 4f\left(\frac{x}{2^n}\right) - 4f\left(\frac{y}{2^n}\right) - 4f\left(\frac{z}{2^n}\right)\right\| + \lim_{n\to\infty} |4|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)$$
$$= \|\rho(h(x+y+z) + h(x-y-z) + h(y-x-z) + h(z-x-y) - 4h(x) - 4h(y) - 4h(z))\|$$

for all $x, y, z \in X$. So

$$\begin{aligned}
&\left\| h\left(\frac{x+y+z}{2}\right) + h\left(\frac{x-y-z}{2}\right) + h\left(\frac{y-x-z}{2}\right) + h\left(\frac{z-x-y}{2}\right) \right. \\
&\qquad\qquad \left. - h(x) - h(y) - h(z) \right\| \\
&\le \|\rho(h(x+y+z) + h(x-y-z) + h(y-x-z) + h(z-x-y) \\
&\qquad\qquad -4h(x) - 4h(y) - 4h(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 3.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (3.6). Then we have

$$\begin{aligned}
\|h(x) - T(x)\| &= \left\| 4^q h\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\| \\
&\le \max\left\{ \left\| 4^q h\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\|, \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \right\} \\
&\le \sup_{j\ge 0} \left\{ |4|^{q+j} \varphi\left(\frac{x}{2^{q+j}}, 0, 0\right) \right\},
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (3.6). □

Corollary 3.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned}
(3.9)\qquad &\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\
&\qquad \left. - f(x) - f(y) - f(z) \right\| \\
&\le \left\| \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \right. \\
&\qquad\qquad \left. -4f(x) - 4f(y) - 4f(z)) \right\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \theta\|x\|^r \tag{3.10}$$

for all $x \in X$.

Theorem 3.5. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying* (3.5) *and*

$$\lim_{j\to\infty}\frac{1}{|4|^j}\varphi(2^jx,2^jy,2^jz)=0 \tag{3.11}$$

for all $x,y,z\in X$. *Then there exists a unique quadratic mapping* $h:X\to Y$ *such that*

$$\|f(x)-h(x)\|\le\sup_{j\ge1}\left\{\frac{1}{|4|^j}\varphi(2^jx,0,0)\right\} \tag{3.12}$$

for all $x\in X$.

Proof. It follows from (3.7) that

$$\left\|f(x)-\frac{1}{4}f(2x)\right\|\le\frac{1}{|4|}\varphi(2x,0,0)$$

for all $x\in X$. Hence

$$\begin{aligned}&\left\|\frac{1}{4^l}f(2^lx)-\frac{1}{4^m}f(2^mx)\right\| \\ &\le\max\left\{\left\|\frac{1}{4^l}f\left(2^lx\right)-\frac{1}{4^{l+1}}f\left(2^{l+1}x\right)\right\|,\cdots,\left\|\frac{1}{4^{m-1}}f\left(2^{m-1}x\right)-\frac{1}{4^m}f\left(2^mx\right)\right\|\right\} \\ &=\max\left\{\frac{1}{|4|^l}\left\|f\left(2^lx\right)-\frac{1}{4}f\left(2^{l+1}x\right)\right\|,\cdots,\frac{1}{|4|^{m-1}}\left\|f\left(2^{m-1}x\right)-\frac{1}{4}f\left(2^mx\right)\right\|\right\} \\ &\le\sup_{j\ge l+1}\left\{\frac{1}{|4|^j}\varphi(2^jx,0,0)\right\}\end{aligned} \tag{3.13}$$

for all nonnegative integers m and l with $m>l$ and all $x\in X$. It follows from (3.13) that the sequence $\{\frac{1}{4^n}f(2^nx)\}$ is a Cauchy sequence for all $x\in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^nx)\}$ converges. So one can define the mapping $h:X\to Y$ by

$$h(x):=\lim_{n\to\infty}\frac{1}{4^n}f(2^nx)$$

for all $x\in X$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (3.13), we get (3.12).

The rest of the proof is similar to the proof of Theorem 3.3. □

Corollary 3.6. *Let* $r>2$ *and* θ *be positive real numbers, and let* $f:X\to Y$ *be an even mapping satisfying* (3.9). *Then there exists a unique quadratic mapping* $h:X\to Y$ *such that*

$$\|f(x)-h(x)\|\le\frac{|2|^r\theta}{|4|}\|x\|^r \tag{3.14}$$

for all $x\in X$.

Let

$$\begin{aligned} A(x,y,z) &:= \left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) \right. \\ &\qquad \left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \right\|, \\ B(x,y,z) &:= \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ &\qquad -4f(x) - 4f(y) - 4f(z))\| \end{aligned}$$

for all $x, y, z \in X$.

For $x, y, z \in X$ with $\|A(x,y,z)\| \le \|B(x,y,z)\|$,

$$\|A(x,y,z)\| - \|B(x,y,z)\| \le \|A(x,y,z) - B(x,y,z)\|.$$

For $x, y, z \in X$ with $\|A(x,y,z)\| > \|B(x,y,z)\|$,

$$\begin{aligned} \|A(x,y,z)\| &= \|A(x,y,z) - B(x,y,z) + B(x,y,z)\| \\ &\le \max\{\|A(x,y,z) - B(x,y,z)\|, \|B(x,y,z)\|\} \\ &= \|A(x,y,z) - B(x,y,z)\| \\ &\le \|A(x,y,z) - B(x,y,z)\| + \|B(x,y,z)\|, \end{aligned}$$

since $\|A(x,y,z)\| > \|B(x,y,z)\|$. So we have

$$\begin{aligned} &\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\ &\qquad \left. - f(x) - f(y) - f(z) \right\| - \|\rho(f(x+y+z) + f(x-y-z) \\ &\qquad + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\| \\ &\le \left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\ &\qquad - f(x) - f(y) - f(z) - \rho(f(x+y+z) + f(x-y-z) \\ &\qquad \left. + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)) \right\|. \end{aligned}$$

As corollaries of Theorems 3.3 and 3.5, we obtain the Hyers-Ulam stability results for the quadratic ρ-functional equation (3.3) in non-Archimedean Banach spaces.

Corollary 3.7. *Let $\varphi : X^3 \to [0,\infty)$ be a function with $\varphi(0,0,0) = 0$ and let $f : X \to Y$ be an even mapping satisfying* (3.4) *and*

$$\begin{aligned} \Big\| f\left(\frac{x+y+z}{2}\right) &+ f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \\ &-f(x) - f(y) - f(z) - \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) \\ &+f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\Big\| \le \varphi(x,y,z) \end{aligned} \tag{3.15}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying (3.6)*.*

Corollary 3.8. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned} \Big\| f\left(\frac{x+y+z}{2}\right) &+ f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \\ &-f(x) - f(y) - f(z) - \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) \\ &+f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\Big\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{3.16}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying (3.10)*.*

Corollary 3.9. *Let $\varphi : X^3 \to [0,\infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying* (3.11) *and* (3.15)*. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (3.12)*.*

Corollary 3.10. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.16)*. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (3.14)*.*

4. QUADRATIC ρ-FUNCTIONAL INEQUALITY (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |8|$.

In this section, we solve and investigate the quadratic ρ-functional inequality (0.2) in non-Archimedean normed spaces.

Lemma 4.1. *An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned}(4.1)\qquad &\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)\\ &\qquad -4f(x)-4f(y)-4f(z)\|\\ &\le \left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right.\right.\\ &\qquad \left.\left.+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\right\|\end{aligned}$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (4.1).

Letting $x = y = z = 0$ in (4.1), we get

$$\|8f(0)\| \le |\rho|\|f(0)\|.$$

So $f(0) = 0$.

Letting $x = y, z = 0$ in (4.1), we get

$$(4.2)\qquad \|2f(2x) - 8f(x)\| \le 0$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (4.1) and (4.2) that

$$\begin{aligned}&\big\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)\\ &\qquad -4f(y)-4f(z)\big\|\\ &\le \left\|\rho\Big(f\Big(\frac{x+y+z}{2}\Big)+f\Big(\frac{x-y-z}{2}\Big)+f\Big(\frac{y-x-z}{2}\Big)\right.\\ &\qquad \left.+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\Big)\right\|\\ &= \left\|\rho\left(\frac{1}{4}f(x+y+z)+\frac{1}{4}f(x-y-z)+\frac{1}{4}f(y-x-z)\right.\right.\\ &\qquad \left.\left.+\frac{1}{4}f(z-x-y)-f(x)-f(y)-f(z)\right)\right\|\\ &= \|\rho\|\frac{1}{|4|}\big\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)\\ &\qquad -4f(x)-4f(y)-4f(z)\big\|\end{aligned}$$

and so

$$\begin{aligned}&f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)\\ &\qquad = 4f(x)+4f(y)+4f(z)\end{aligned}$$

for all $x, y, z \in X$.

The converse is obviously true. □

Corollary 4.2. *An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned}(4.3)\qquad & f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x) \\ & \quad -4f(y)-4f(z) \\ & = \rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right. \\ & \qquad \left. +f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\end{aligned}$$

for all $x, y, z \in X$ and only if $f : X \to Y$ is quadratic.

The functional equation (4.3) is called a *quadratic ρ-functional equation.*

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (4.1) in non-Archimedean Banach spaces.

Theorem 4.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function with $\varphi(0,0,0) = 0$ and let $f : X \to Y$ be an even mapping satisfying*

$$(4.4)\qquad \lim_{j\to\infty} |4|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) = 0,$$

$$\begin{aligned}(4.5)\quad & \|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x) \\ & \quad -4f(y)-4f(z)\| \\ & \leq \left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)\right.\right. \\ & \qquad \left.\left. -f(x)-f(y)-f(z)\right)\right\| + \varphi(x,y,z)\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$(4.6)\qquad \|f(x)-h(x)\| \leq \sup_{j\geq 0}\left\{|2|^{2j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right)\right\}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (4.5), we get $\|8f(0)\| \leq |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $x = y, z = 0$ in (4.5), we get

$$(4.7)\qquad \left\|4f\left(\frac{x}{2}\right)-f(x)\right\| \leq \frac{1}{|2|}\varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)$$

for all $x \in X$. So

$$\begin{aligned}(4.8)\quad \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\le \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &= \max\left\{|4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\le \sup_{j\ge l}\left\{|2|^{2j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right)\right\}\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.8) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.6).

The rest of the proof is similar to the proof of Theorem 3.3. □

Corollary 4.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned}(4.9)\quad &\|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) \\ &\qquad -4f(y) - 4f(z)\| \\ &\le \left\|\rho\left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right)\right.\right. \\ &\qquad \left.\left. +f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right)\right\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$(4.10)\qquad \|f(x) - h(x)\| \le \frac{2\theta}{|2|^{r+1}}\|x\|^r$$

for all $x \in X$.

Theorem 4.5. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying* (4.5) *and*

$$(4.11)\qquad \sum_{j\to\infty} \frac{1}{|4|^j}\varphi(2^j x, 2^j y, 2^j z) = 0$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{1}{|8|} \sup_{j \geq 0} \left\{ \frac{1}{|4|^j} \varphi(2^j x, 2^j x, 0) \right\} \tag{4.12}$$

for all $x \in X$.

Proof. It follows from (4.7) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{|8|} \varphi(x, x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \\ &\leq \max\left\{ \left\| \frac{1}{4^l} f\left(2^l x\right) - \frac{1}{4^{l+1}} f\left(2^{l+1} x\right) \right\|, \cdots, \left\| \frac{1}{4^{m-1}} f\left(2^{m-1} x\right) - \frac{1}{4^m} f\left(2^m x\right) \right\| \right\} \\ &= \max\left\{ \frac{1}{|4|^l} \left\| f\left(2^l x\right) - \frac{1}{4} f\left(2^{l+1} x\right) \right\|, \cdots, \frac{1}{|4|^{m-1}} \left\| f\left(2^{m-1} x\right) - \frac{1}{4} f\left(2^m x\right) \right\| \right\} \\ &\leq \frac{1}{|8|} \sup_{j \geq l} \left\{ \frac{1}{|4|^j} \varphi(2^j x, 2^j x, 0) \right\} \end{aligned} \tag{4.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.13) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.13), we get (4.12).

The rest of the proof is similar to the proof of Theorems 3.3. □

Corollary 4.6. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying* (4.9). *Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{2\theta}{|8|} \|x\|^r \tag{4.14}$$

for all $x \in X$.

Let

$$\begin{aligned} A(x,y,z) &:= \|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y) \\ &\qquad -4f(x)-4f(y)-4f(z)\| \\ B(x,y,z) &:= \left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right.\right. \\ &\qquad \left.\left.+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\right\| \end{aligned}$$

for all $x,y,z\in X$.

For $x,y,z\in X$ with $\|A(x,y,z)\|\le\|B(x,y,z)\|$,

$$\|A(x,y,z)\|-\|B(x,y,z)\|\le\|A(x,y,z)-B(x,y,z)\|.$$

For $x,y,z\in X$ with $\|A(x,y,z)\|>\|B(x,y,z)\|$,

$$\begin{aligned} \|A(x,y,z)\| &= \|A(x,y,z)-B(x,y,z)+B(x,y,z)\| \\ &\le \max\{\|A(x,y,z)-B(x,y,z)\|,\|B(x,y,z)\|\} \\ &= \|A(x,y,z)-B(x,y,z)\| \\ &\le \|A(x,y,z)-B(x,y,z)\|+\|B(x,y,z)\|, \end{aligned}$$

since $\|A(x,y,z)\|>\|B(x,y,z)\|$. So we have

$$\begin{aligned} &\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y) \\ &\qquad -4f(x)-4f(y)-4f(z)\| \\ &-\left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right.\right. \\ &\qquad \left.\left.+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\right\| \\ &\le \left\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)\right. \\ &\qquad -4f(x)-4f(y)-4f(z)-\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)\right. \\ &\qquad \left.\left.+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\right\|. \end{aligned}$$

As corollaries of Theorems 4.3 and 4.5, we obtain the Hyers-Ulam stability results for the quadratic ρ-functional equation (4.3) in non-Archimedean Banach spaces.

Corollary 4.7. *Let $\varphi : X^3 \to [0,\infty)$ be a function with $\varphi(0,0,0) = 0$ and let $f : X \to Y$ be an even mapping satisfying* (4.4) *and*

$$\begin{aligned}(4.15)\quad \Big\| & f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) \\ & -4f(y) - 4f(z) - \rho\left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right)\right. \\ & \left. +f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right)\Big\| \le \varphi(x,y,z)\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying (4.6)*.*

Corollary 4.8. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\begin{aligned}(4.16)\quad \Big\| & f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) \\ & -4f(y) - 4f(z) - \rho\left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right)\right. \\ & \left. +f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right)\Big\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r)\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying (4.10)*.*

Corollary 4.9. *Let $\varphi : X^3 \to [0,\infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying* (4.11) *and* (4.15) *Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (4.12)*.*

Corollary 4.10. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (4.16)*. Then there exists a unique quadratic mapping $h : X \to Y$ satisfying* (4.14)*.*

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