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APPROXIMATE BIHOMOMORPHISMS AND BIDERIVATIONS IN 3-LIE ALGEBRAS: REVISITED

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Abstract. Shokri et al. [14] proved the Hyers-Ulam stability of bihomomorphisms and biderivations by using the direct method.

It is easy to show that the definition of biderivations on normed 3-Lie algebras is meaningless and so the results of [14] are meaningless.

In this paper, we correct the definition of biderivations and the statements of the results in [14], and prove the corrected theorems.

1. Introduction and Preliminaries

A normed (Banach) Lie triple system is a normed (Banach) space $(A, \|\cdot\|)$ with a trilinear mapping $(x, y, z) \mapsto [x, y, z]$ from $A \times A \times A$ to A satisfying the following axioms:

$$\begin{aligned} [x,y,z] &= -[y,x,z], \\ [x,y,z] &= -[y,z,x] - [z,x,y], \\ [u,v,[x,y,z]] &= [[u,v,x],y,z] + [x,[u,v,y],z] + [x,y,[u,v,z]], \\ \|[x,y,z]\| &\leq \|x\|\|y\|\|z\| \end{aligned}$$

for all $u, v, x, y, z \in A$. The concept of Lie triple system was introduced by Lister [8] (see [6]).

Let A and B be normed Lie triple systems. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *homomorphism* if it satisfies

$$H([x,y,z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *derivation* if it satisfies

$$\delta([x,y,z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

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for all $x, y, z \in A$ (see [9, 14]).

Definition 1.1 ([14])**.** Let A be a normed Lie triple system. A $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is called a *biderivation* if it satisfies

$$\begin{aligned} \delta([x,y,z],w) &= [\delta(x,w),y,z] + [x,\delta(y,w),z] + [x,y,\delta(z,w)], \\ \delta(x,[y,z,w],w) &= [\delta(x,y),z,w] + [y,\delta(x,z),w] + [y,z,\delta(x,w)] \end{aligned}$$

for all $x, y, z, w \in A$.

The w-variable of the left side in the first equality is $\mathbb{C}$-linear and the x-variable of the left side in the second equality is $\mathbb{C}$-linear. But the w-variable of the right side in the first equality is not $\mathbb{C}$-linear and the x-variable of the right side in the second equality is not $\mathbb{C}$-linear. Thus we correct the definition of biderivation as follows.

Definition 1.2. Let A be a normed Lie triple system. A $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is called a *biderivation* if it satisfies

$$\begin{aligned} \delta([x,y,z],w) &= [\delta(x,w),y,z] + [x,\delta(y,w^*),z] + [x,y,\delta(z,w)], \\ \delta(x,[y,z,w],w) &= [\delta(x,y),z,w] + [y,\delta(x^*,z),w] + [y,z,\delta(x,w)] \end{aligned}$$

for all $x, y, z, w \in A$.

The stability problem of functional equations originated from a question of Ulam [15] concerning the stability of group homomorphisms. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [11] for linear mappings by considering an unbounded Cauchy difference. J.M. Rassias [10] followed the innovative approach of the Th.M. Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^p$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. The stability problems of various functional equations have been extensively investigated by a number of authors (see [3, 4, 5, 12, 13]).

All the mappings T and δ, given in [14, Sections 2 and 4], satisfy the bi-additive functional equation (1.2) in [14]. Letting $x = z = 0$ in (1.2), we get $f(y, w) = 0$ for all y, w. So the results of [14, Sections 2 and 4] are meaningless.

In this paper, we will replace the equation (1.2), given in [14], by

$$\begin{aligned} &f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) \\ &= 4f(x,z). \end{aligned} \tag{1}$$

Furthermore, we correct the statements of the results in [14, Sections 2 and 4], and prove the corrected theorems.

2. Hyers-Ulam Stability of the Functional Equation (1) in Banach Spaces

Throughout this section, assume that A is a normed space and B is a Banach space.

For a given mapping $f : A \times A \to B$, we define

$$\begin{aligned} E_{\lambda,\mu} f(x,y,z,w) &= f(\lambda x+\lambda y, \mu z+\mu w)+f(\lambda x+\lambda y, \mu z-\mu w) \\ &+f(\lambda x-\lambda y, \mu z+\mu w)+f(\lambda x-\lambda y, \mu z-\mu w)-4\lambda\mu f(x,z) \end{aligned}$$

for all $x, y, z, w \in A$ and all $\lambda, \mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : \ |\nu| = 1\}$.

From now on, assume that $f(0, z) = f(x, 0) = 0$ for all $x, z \in A$.

We need the following lemmas to obtain the main results.

Lemma 2.1 ([2]). *Let $f : A\times A \to B$ be a bi-additive mapping such that $f(\lambda x, \mu y) = \lambda\mu f(x, y)$ for all $x, y \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. Then the mapping $f : A \times A \to B$ is $\mathbb{C}$-bilinear.*

Lemma 2.2. *Let $f : A \times A \to B$ be a mapping satisfying $E_{\lambda,\mu} f(x, y, z, w) = 0$ for all $x, y, z, w \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. Then the mapping $f : A \times A \to B$ is $\mathbb{C}$-bilinear.*

Proof. Let $\lambda = \mu = 1$, $y = x$ and $w = 0$. Then $2f(2x, z) = 4f(x, z) = 0$ for all $y, z \in A$. Let $\lambda = \mu = 1$ and $w = 0$. Then $2f(x+y, z)+2f(x-y, z) = 4f(x, z) = 0$ for all $x, y, z \in A$. Let $x_1 = x + y$ and $x_2 = x - y$. Then $2f(x_1, z) + 2f(x_2, z) = 4f\left(\frac{x_1+x_2}{2}, z\right) = 2f(x_1 + x_2, z)$ and so f is additive in the first variable.

Similarly, one can show that f is additive in the second variable.

Letting $y = w = 0$, we get $4f(\lambda x, \mu z) = 4\lambda\mu f(x, z)$ for all $x, z \in A$. By Lemma 2.1, the mapping f is $\mathbb{C}$-bilinear. □

Theorem 2.3. *Let $f : A \times A \to B$ be a mapping for which there exists a function $\varphi : A^4 \to [0,\infty)$ such that*

$$\|E_{\lambda,\mu} f(x,y,z,w)\| \leq \varphi(x,y,z,w), \tag{2}$$

$$\Phi(x,y,z,w) : = \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{4^n}\varphi(2^n x, 2^n y, 2^n z, 2^n w) < \infty$$

for all $\lambda, \mu \in \mathbb{T}^1$ *and all* $x, y, z, w \in A$. *Then there exists a unique* $\mathbb{C}$*-bilinear mapping* $T : A \times A \to B$ *such that*

$$\|f(x,z) - T(x,z)\| \le \Phi(x,x,z,z) \tag{3}$$

for all $x, z \in A$.

Proof. Letting $\lambda = \mu = 1$, $y = x$ and $w = z$ in (2), we get

$$\|f(2x,2z) - 4f(x,z)\| \le \varphi(x,x,z,z) \tag{4}$$

and so

$$\left\|f(x,z) - \frac{1}{4}f(2x,2z)\right\| \le \frac{1}{4}\varphi(x,x,z,z) \tag{5}$$

for all $x, z \in A$.

It follows from (5) that

$$\left\|\frac{1}{4^l}f(2^l x, 2^l z) - \frac{1}{4^m}f(2^m x, 2^m z)\right\| \le \sum_{j=l}^{m-1} \frac{1}{4^{j+1}}\varphi\left(2^j x, 2^j x, 2^j z, 2^j z\right) \tag{6}$$

for all $x, z \in A$ and all nonnegative integers m, l with $m > l$. This implies that the sequence $\left\{\frac{1}{4^n}f(2^n x, 2^n z)\right\}$ is a Cauchy sequence for all $x, z \in A$. Since A is complete, the sequence $\left\{\frac{1}{4^n}f(2^n x, 2^n z)\right\}$ converges. Thus one can define the mapping $T : A \times A \to A$ by

$$T(x,z) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n z)$$

for all $x, z \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (6), we get (3).

By the definition of the mapping T, we have

$$\begin{aligned} \|E_{\lambda,\mu}T(x,y,z,w)\| &= \lim_{n\to\infty} \frac{1}{4^n}\|E_{\lambda,\mu}f(2^n x, 2^n y, 2^n z, 2^n w)\| \\ &\le \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y, 2^n z, 2^n w) = 0 \end{aligned}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. By Lemma 2.2, the mapping $T : A \times A \to A$ is $\mathbb{C}$-bilinear.

Let $T' : A \times A \to A$ be another $\mathbb{C}$-bilinear mapping satisfying (3). Then we have

$$\begin{aligned} \|T(x,z) - T'(x,z)\| &= \frac{1}{4^n}\|T(2^n x, 2^n z) - T'(2^n x, 2^n z)\| \\ &\leq \frac{1}{4^n}\|T(2^n x, 2^n z) - f(2^n x, 2^n z)\| \\ &\quad + \frac{1}{4^n}\|f(2^n x, 2^n z) - T'(2^n x, 2^n z)\| \\ &\leq 2\frac{1}{4^n}\Phi(2^n x, 2^n x, 2^n z, 2^n z), \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x, z \in A$. This proves the uniqueness of T.

Therefore, the mapping $T : A \times A \to A$ is a unique $\mathbb{C}$-bilinear mapping satisfying (3), as desired. □

Theorem 2.4. *Let $f : A \times A \to B$ be a mapping for which there exists a function $\varphi : A^4 \to [0, \infty)$ satisfying (2) and*

$$\Phi(x,y,z,w) : = \frac{1}{4}\sum_{n=1}^{\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right) < \infty$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $T : A \times A \to B$ such that

$$\|f(x,z) - T(x,z)\| \leq \Phi(x,x,z,z)$$

for all $x, z \in A$.

Proof. It follows from (4) that

$$\left\|f(x,z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{z}{2}\right)$$

for all $x, z \in A$.

The rest of the proof is similar to the proof of Theorem 2.3. □

3. Hyers-Ulam Stability of Biderivations on Banach Lie Triple Systems

Throughout this section, assume that A is a Banach Lie triple system.

In this section, we prove the Hyers-Ulam stability of biderivations on Banach Lie triple systems.

Theorem 3.1. *Let θ and p be positive real numbers with $p < \frac{1}{2}$ and let $f : A \times A \to A$ be a mapping such that*

$$\|E_{\lambda,\mu} f(x,y,z,w)\| \leq \theta\|x\|^p\|y\|^p\|z\|^p\|w\|^p, \tag{7}$$

$$\begin{aligned}(8)\qquad &\|f([x,y,z],w)-[f(x,w),y,z]-[x,f(y,w^*),z]-[x,y,f(z,w)]\|\\ &+\|f(x,[y,z,w])-[f(x,y),z,w]-[y,f(x^*,z),w]-[y,z,f(x,w)]\|\\ &\le \theta\|x\|^p\|y\|^p\|z\|^p\|w\|^p\end{aligned}$$

for all $\lambda,\mu\in\mathbb{T}^1$ and all $x,y,z,w\in A$. If the mapping $f:A\times A\to A$ satisfies

$$(9)\qquad \lim_{n\to\infty}\frac{1}{4^n}f(2^nx,2^nz)=\lim_{n\to\infty}\frac{1}{16^n}f(8^nx,2^nz)=\lim_{n\to\infty}\frac{1}{16^n}f(2^nx,8^nz)$$

for all $x,z\in A$, then there exists a unique biderivation $\delta:A\times A\to A$ such that

$$(10)\qquad \|f(x,z)-\delta(x,z)\|\le\frac{\theta}{4-4^{2p}}\|x\|^{2p}\|z\|^{2p}$$

for all $x,z\in A$.

Proof. Letting $\varphi(x,y,z,w):=\theta\|x\|^p\|y\|^p\|z\|^p\|w\|^p$ in Theorem 2.3, we get a unique $\mathbb{C}$-bilinear mapping $\delta:A\times A\to A$ given by $\delta(x,z):=\lim_{n\to\infty}\frac{1}{4^n}f(2^nx,2^nz)$ satisfying (10).

It follows from (8) and (9) that

$$\begin{aligned}&\|\delta([x,y,z],w)-[\delta(x,w),y,z]-[x,\delta(y,w^*),z]-[x,y,\delta(z,w)]\|\\ &\quad+\|\delta(x,[y,z,w])-[\delta(x,y),z,w]-[y,\delta(x^*,z),w]-[y,z,\delta(x,w)]\|\\ &\quad=\lim_{n\to\infty}\frac{1}{16^n}(\|f(8^n[x,y,z],2^nw)-[f(2^nx,2^nw),2^ny,2^nz]\\ &\qquad\quad-[2^nx,f(2^ny,2^nw^*),2^nz]-[2^nx,2^ny,f(2^nz,2^nw)]\|\\ &\qquad+\|f(2^nx,8^n[y,z,w])-[f(2^nx,2^ny),2^nz,2^nw]\\ &\qquad\quad-[2^ny,f(2^nx^*,2^nz),2^nw]-[2^ny,2^nz,f(2^nx,2^nw)]\|)\\ &\quad\le\lim_{n\to\infty}\frac{16^{pn}\theta}{16^n}\|x\|^p\|y\|^p\|z\|^p\|w\|^p=0\end{aligned}$$

for all $x,y,z,w\in A$. So

$$\delta([x,y,z],w)=[\delta(x,w),y,z]+[x,\delta(y,w^*),z]+[x,y,\delta(z,w)]$$

and

$$\delta(x,[y,z,w])=[\delta(x,y),z,w]+[y,\delta(x^*,z),w]+[y,z,\delta(x,w)]$$

for all $x,y,z,w\in A$.

Therefore, the mapping $\delta:A\times A\to A$ is a biderivation satisfying (10), as desired. □

Theorem 3.2. *Let θ and p be positive real numbers with $p > 1$ and let $f : A \times A \to A$ be a mapping satisfying (7) and (8). If the mapping $f : A \times A \to A$ satisfies*

$$\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{z}{2^n}\right) = \lim_{n\to\infty} 16^n f\left(\frac{x}{8^n}, \frac{z}{2^n}\right) = \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}, \frac{z}{8^n}\right) \tag{11}$$

for all $x, z \in A$, then there exists a unique biderivation $\delta : A \times A \to A$ such that

$$\|f(x,z) - \delta(x,z)\| \leq \frac{\theta}{4^{2p} - 4}\|x\|^{2p}\|z\|^{2p} \tag{12}$$

for all $x, z \in A$.

Proof. Letting $\varphi(x,y,z,w) := \theta\|x\|^p\|y\|^p\|z\|^p\|w\|^p$ in Theorem 2.4, we get a unique $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ given by $\delta(x,z) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{z}{2^n}\right)$ satisfying (12).

It follows from (8) that

$$\begin{aligned}
&\|\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w^*),z] - [x,y,\delta(z,w)]\| \\
&\quad + \|\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x^*,z),w] - [y,z,\delta(x,w)]\| \\
&\quad = \lim_{n\to\infty} 16^n \left(\left\| f\left(\frac{[x,y,z]}{8^n}, \frac{w}{2^n}\right) - \left[f\left(\frac{x}{2^n}, \frac{w}{2^n}\right), \frac{y}{2^n}, \frac{z}{2^n}\right] \right.\right. \\
&\qquad \left. - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n}, \frac{w^*}{2^n}\right), \frac{z}{2^n}\right] - \left[\frac{x}{2^n}, \frac{y}{2^n}, f\left(\frac{z}{2^n}, \frac{w}{2^n}\right)\right] \right\| \\
&\qquad + \left\| f\left(\frac{x}{2^n}, \frac{[y,z,w]}{8^n}\right) - \left[f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \frac{z}{2^n}, \frac{w}{2^n}\right] - \right. \\
&\qquad \left.\left. \left[\frac{y}{2^n}, f\left(\frac{x^*}{2^n}, \frac{z}{2^n}\right), \frac{w}{2^n}\right] - \left[\frac{y}{2^n}, \frac{z}{2^n}, f\left(\frac{x}{2^n}, \frac{w}{2^n}\right)\right] \right\| \right) \\
&\quad \leq \lim_{n\to\infty} \frac{16^n \theta}{16^{pn}} \|x\|^p\|y\|^p\|z\|^p\|w\|^p = 0
\end{aligned}$$

for all $x, y, z, w \in A$. So

$$\delta([x,y,z],w) = [\delta(x,w),y,z] + [x,\delta(y,w^*),z] + [x,y,\delta(z,w)]$$

and

$$\delta(x,[y,z,w]) = [\delta(x,y),z,w] + [y,\delta(x^*,z),w] + [y,z,\delta(x,w)]$$

for all $x, y, z, w \in A$.

Therefore, the mapping $\delta : A \times A \to A$ is a biderivation satisfying (12), as desired. □

Theorem 3.3. *Let θ and p be positive real numbers with $p < 2$ and let $f : A \times A \to A$ be a mapping satisfying* (9) *and*

$$\|E_{\lambda,\mu} f(x,y,z,w)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p), \tag{13}$$

$$\begin{aligned} &\|f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w^*),z] - [x,y,f(z,w)]\| \\ &+\|f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x^*,z),w] - [y,z,f(x,w)]\| \\ &\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) \end{aligned} \tag{14}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique biderivation $\delta : A \times A \to A$ such that

$$\|f(x,z) - \delta(x,z)\| \leq \frac{2\theta}{4 - 2^p}(\|x\|^p + \|z\|^p) \tag{15}$$

for all $x, z \in A$.

Proof. Letting $\varphi(x,y,z,w) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ in Theorem 2.3, we get a unique $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ given by $\delta(x,z) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n z)$ satisfying (15).

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.4. *Let θ and p be positive real numbers with $p > 4$ and let $f : A \times A \to A$ be a mapping satisfying* (11), (13) *and* (14). *Then there exists a unique biderivation $\delta : A \times A \to A$ such that*

$$\|f(x,z) - \delta(x,z)\| \leq \frac{2\theta}{2^p - 4}(\|x\|^p + \|z\|^p) \tag{16}$$

for all $x, z \in A$.

Proof. Letting $\varphi(x,y,z,w) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ in Theorem 2.4, we get a unique $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ given by $\delta(x,z) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{z}{2^n}\right)$ satisfying (16).

The rest of the proof is similar to the proof of Theorem 3.2. □

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