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COUPLED COINCIDENCE POINT RESULTS FOR GENERALIZED SYMMETRIC MEIR-KEELER CONTRACTION ON PARTIALLY ORDERED METRIC SPACES WITH APPLICATION

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Abstract. We establish a coupled coincidence point theorem for generalized compatible pair of mappings $F, G : X \times X \to X$ under generalized symmetric Meir-Keeler contraction on a partially ordered metric space. We also deduce certain coupled fixed point results without mixed monotone property of $F : X \times X \to X$. An example supporting to our result has also been cited. As an application the solution of integral equations are obtain here to illustrate the usability of the obtained results. We improve, extend and generalize several known results.

1. Introduction and Preliminaries

Let (X, d) be a metric space and $T : X \to X$ a self mapping. If (X, d) is complete and T is a contraction, that is, there exists a constant $k \in [0, 1)$ such that

$$d(Tx,\ Ty) \leq kd(x,\ y),\ \text{for all } x,\ y \in X, \tag{1}$$

then, by Banach contraction mapping principle, which is a classical and powerful tool in nonlinear analysis, we know that T has a unique fixed point p and, for any $x_0 \in X$, the Picard iteration $\{T^n x_0\}$ converges to p. The Banach contraction mapping principle has been generalized in several directions, One of these generalizations, known as the Meir-Keeler fixed point theorem [13], has been obtained by replacing the contraction condition (1) by the following more general assumption: for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$x,\ y \in X,\ \varepsilon \leq d(x,\ y) < \varepsilon + \delta(\varepsilon) \Rightarrow d(Tx,\ Ty) < \varepsilon. \tag{2}$$

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Bhaskar and Lakshmikantham [3] introduced the notion of coupled fixed point, mixed monotone mappings in the setting of single-valued mappings and established some coupled fixed point theorems for a mapping with the mixed monotone property in the setting of partially ordered metric spaces.

In [3], Bhaskar and Lakshmikantham introduced the following:

Definition 1. Let $(X, \leq)$ be a partially ordered set and endow the product space $X \times X$ with the following partial order:

$$(u,\ v) \leq (x,\ y) \Leftrightarrow x \geq u \text{ and } y \leq v,\ \forall (u,\ v),\ (x,\ y) \in X \times X. \tag{3}$$

Definition 2. An element $(x, y) \in X \times X$ is called a *coupled fixed point* of the mapping $F : X \times X \to X$ if

$$F(x,\ y) = x \text{ and } F(y,\ x) = y. \tag{4}$$

Definition 3. Let $(X, \leq)$ be a partially ordered set. Suppose $F : X \times X \to X$ be a given mapping. We say that F has the *mixed monotone property* if for all $x, y \in X$, we have

$$x_1,\ x_2 \in X,\ x_1 \leq x_2 \implies F(x_1,\ y) \leq F(x_2,\ y) \tag{5}$$

and

$$y_1,\ y_2 \in X,\ y_1 \leq y_2 \implies F(x,\ y_1) \geq F(x,\ y_2). \tag{6}$$

F has the strict mixed monotone property if the strict inequality in the left-hand side of (5) and (6) implies the strict inequality in the right-hand side, respectively.

Lakshmikantham and Ciric [11] extended the notion of mixed monotone property to mixed $g-$monotone property and established coupled coincidence point results using a pair of commutative mappings, which generalized the results of Bhaskar and Lakshmikantham [3].

In [11], Lakshmikantham and Ciric introduced the following:

Definition 4. An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x,\ y) = g(x) \text{ and } F(y,\ x) = g(y). \tag{7}$$

Definition 5. An element $(x, y) \in X \times X$ is called a *common coupled fixed point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$x = F(x,\ y) = g(x) \text{ and } y = F(y,\ x) = g(y). \tag{8}$$

Definition 6. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be *commutative* if

$$g(F(x,\ y)) = F(g(x),\ g(y)), \text{ for all } (x,\ y) \in X \times X. \tag{9}$$

Definition 7. Let $(X, \leq)$ be a partially ordered set. Suppose $F : X \times X \to X$ and $g : X \to X$ are given mappings. We say that F has the *mixed $g-$monotone property* if for all $x, y \in X$, we have

$$x_1,\ x_2 \in X,\ g(x_1) \leq g(x_2) \implies F(x_1,\ y) \leq F(x_2,\ y) \tag{10}$$

and

$$y_1,\ y_2 \in X,\ g(y_1) \leq g(y_2) \implies F(x,\ y_1) \geq F(x,\ y_2). \tag{11}$$

If g is the identity mapping on X, then F satisfies the mixed monotone property.

Later, Choudhury and Kundu [5] introduced the following notion of compatibility in the context of coupled coincidence point and used this notion to improve the results of Lakshmikantham and Ciric [11] :

Definition 8. The mappings $F : X \times\ X \to X$ and $g : X \to X$ are said to be *compatible* if

$$\begin{aligned} \lim_{n\to\infty} d(gF(x_n,\ y_n),\ F(gx_n,\ gy_n)) &= 0, \\ \lim_{n\to\infty} d(gF(y_n,\ x_n),\ F(gy_n,\ gx_n)) &= 0, \end{aligned}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} gx_n = x, \\ \lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} gy_n = y, \text{ for some } x,\ y \in X. \end{aligned}$$

These results used to study the existence and uniqueness of solution for periodic boundary value problems.

Hussain et al. [10] introduced a new concept of generalized compatibility of a pair of mappings $F, G : X \times X \to X$ defined on a product space and proved some coupled coincidence point results. Hussain et al. [10] also deduce some coupled fixed point results without mixed monotone property.

In [10], Hussain et al. introduced the following:

Definition 9. Suppose that $F, G : X \times X \to X$ are two mappings. F is said to be $G-$*increasing* with respect to $\leq$ if for all $x, y, u, v \in X$, with $G(x, y) \leq G(u, v)$ we have $F(x, y) \leq F(u, v)$.

Example 10. Let $X = (0, +\infty)$ be endowed with the natural ordering of real numbers $\leq$. Define mappings $F, G : X \times X \to X$ by $F(x, y) = \ln(x + y)$ and $G(x, y) = x + y$ for all $(x, y) \in X \times X$. Note that F is $G-$increasing with respect to $\leq$.

Example 11. Let $X = \mathbb{N}$ endowed with the partial order defined by $x, y \in X \times X$, $x \leq y$ if and only if y divides x. Define the mappings $F, G : X \times X \to X$ by $F(x, y) = x^2y^2$ and $G(x, y) = xy$ for all $(x, y) \in X \times X$. Then F is $G-$*increasing* with respect to $\leq$.

Definition 12. An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of mappings $F, G : X \times X \to X$ if $F(x, y) = G(x, y)$ and $F(y, x) = G(y, x)$.

Example 13. Let $F, G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $F(x, y) = xy$ and $G(x, y) = \frac{2}{3}(x + y)$ for all $(x, y) \in X \times X$. Note that $(0, 0)$, $(1, 2)$ and $(2, 1)$ are coupled coincidence points of F and G.

Definition 14. Let $F, G : X \times X \to X$ be two mappings. We say that the pair $\{F, G\}$ is *commuting* if

$$F(G(x,\ y),\ G(y,\ x)) = G(F(x,\ y),\ F(y,\ x)),\ \forall x,\ y \in X. \tag{12}$$

Definition 15. Let $(X, \leq)$ be a partially ordered set, $F : X \times X \to X$ and $g : X \to X$. We say that F is $g-$*increasing* with respect to $\leq$ if for any $x, y \in X$,

$$gx_1 \leq gx_2 \text{ implies } F(x_1,\ y) \leq F(x_2,\ y), \tag{13}$$

and

$$gy_1 \leq gy_2 \text{ implies } F(x,\ y_1) \leq F(x,\ y_2). \tag{14}$$

Definition 16. Let $(X, \leq)$ be a partially ordered set, $F : X \times X \to X$. We say that F is *increasing* with respect to $\leq$ if for any $x, y \in X$,

$$x_1 \leq x_2 \text{ implies } F(x_1,\ y) \leq F(x_2,\ y), \tag{15}$$

and

$$y_1 \leq y_2 \text{ implies } F(x,\ y_1) \leq F(x,\ y_2). \tag{16}$$

Definition 17. Let $F, G : X \times X \to X$. We say that the pair $\{F, G\}$ is *generalized compatible* if

$$\begin{aligned}\lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0,\\ \lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n, x_n),\ F(x_n,\ y_n))) &= 0,\end{aligned}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{aligned}\lim_{n\to\infty} G(x_n,\ y_n) &= \lim_{n\to\infty} F(x_n,\ y_n) = x,\\ \lim_{n\to\infty} G(y_n,\ x_n) &= \lim_{n\to\infty} F(y_n,\ x_n) = y.\end{aligned}$$

Obviously, a commuting pair is generalized compatible but not conversely in general.

Recently Samet et al. [16] claimed that most of the coupled fixed point theorems on ordered metric spaces are consequences of well-known fixed point theorems. Coupled fixed point theory have developed literature, some of the instances of these works are [1, 2, 3, 4, 5, 6, 7, 8, 9, 14, 15, 17].

In [15], Samet established the coupled fixed points of mixed strict monotone generalized Meir-Keeler operators and also established the existence and uniqueness results for coupled fixed point. Berinde and Pecurar [2] obtained more general coupled fixed point theorems for mixed monotone operators $F : X \times X \to X$ satisfying a generalized symmetric Meir-Keeler contractive condition.

In this paper, we establish a coupled coincidence point theorem for generalized compatible pair of mappings $F, G : X \times X \to X$ under generalized symmetric Meir-Keeler contraction on a partially ordered metric space. We also deduce certain coupled fixed point results without mixed monotone property of F. We also give an example and an application to integral equation to support our results presented here. We extend and generalize the results of Berinde and Pecurar [2], Bhaskar and Lakshmikantham [3], Meir and Keeler [13], Samet [15] and many other results in the existing literature.

2. Main Results

Theorem 18. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F, G : X \times X \to X$ be two generalized compatible mappings such that F is $G-$increasing with respect to $\leq$, G is continuous and has the mixed monotone property, and there exist two elements $x_0, y_0 \in X$ with*

$$G(x_0,\ y_0) \leq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \geq F(y_0,\ x_0).$$

Suppose for each $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that*

$$\varepsilon \leq \frac{d(G(x,\ y),\ G(u,\ v)) + d(G(y,\ x),\ G(v,\ u))}{2} \leq \varepsilon + \delta(\varepsilon)$$

implies

$$\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2} < \varepsilon, \tag{17}$$

for all $x,\ y,\ u,\ v \in X$, *where* $G(x,\ y) \leq G(u,\ v)$ *and* $G(y,\ x) \geq G(v,\ u)$. *Suppose that for any* $x,\ y \in X$, *there exist* $u,\ v \in X$ *such that*

$$F(x,\ y) = G(u,\ v) \text{ and } F(y,\ x) = G(v,\ u). \tag{18}$$

Also suppose that either

(a) F *is continuous or*

(b) X *has the following properties:*

(i) *if a non-decreasing sequence* $\{x_n\} \to x$ *in* X *then* $x_n \leq x$, *for all* n,

(ii) *if a non-increasing sequence* $\{x_n\} \to x$ *in* X *then* $x \leq x_n$, *for all* n.

Then F *and* G *have a coupled coincidence point.*

Proof. By hypothesis, there exist $x_0,\ y_0 \in X$ such that

$$G(x_0,\ y_0) \leq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \geq F(y_0,\ x_0).$$

From (18), we can choose $x_1,\ y_1 \in X$ such that

$$G(x_1,\ y_1) = F(x_0,\ y_0) \text{ and } G(y_1,\ x_1) = F(y_0,\ x_0).$$

Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$G(x_{n+1},\ y_{n+1}) = F(x_n,\ y_n) \text{ and } G(y_{n+1},\ x_{n+1}) = F(y_n,\ x_n), \text{ for all } n \geq 0. \tag{19}$$

We shall show that

$$G(x_n,\ y_n) \leq G(x_{n+1},\ y_{n+1}) \text{ and } G(y_n,\ x_n) \geq G(y_{n+1},\ x_{n+1}), \text{ for all } n \geq 0. \tag{20}$$

We shall use the mathematical induction. Let $n = 0$, since

$$\begin{aligned} G(x_0,\ y_0) &\leq F(x_0,\ y_0) = G(x_1,\ y_1), \\ G(y_0,\ x_0) &\geq F(y_0,\ x_0) = G(y_1,\ x_1), \end{aligned}$$

we have

$$G(x_0,\ y_0) \leq G(x_1,\ y_1) \text{ and } G(y_0,\ x_0) \geq G(y_1,\ x_1).$$

Thus (20) hold for $n = 0$. Suppose now that (20) hold for some fixed $n \in \mathbb{N}$. Then since

$$G(x_n,\ y_n) \leq G(x_{n+1},\ y_{n+1}) \text{ and } G(y_n,\ x_n) \geq G(y_{n+1},\ x_{n+1}),$$

and as F is $G-$increasing with respect to $\leq$, from (19), we have

$$\begin{aligned} G(x_{n+1},\ y_{n+1}) &= F(x_n,\ y_n) \leq F(x_{n+1},\ y_{n+1}) = G(x_{n+2},\ y_{n+2}), \\ G(y_{n+1},\ x_{n+1}) &= F(y_n,\ x_n) \geq F(y_{n+1},\ x_{n+1}) = G(y_{n+2},\ x_{n+2}). \end{aligned}$$

Thus by the mathematical induction we conclude that (20) hold for all $n \geq 0$. Therefore

$$G(x_0,\ y_0) \leq G(x_1,\ y_1) \leq ... \leq G(x_n,\ y_n) \leq G(x_{n+1},\ y_{n+1}) \leq ...$$

and

$$G(y_0,\ x_0) \geq G(y_1,\ x_1) \geq ... \geq G(y_n,\ x_n) \geq G(y_{n+1},\ x_{n+1}) \geq ...$$

Now, by (17), for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \frac{d(G(x,\ y),\ G(u,\ v)) + d(G(y,\ x),\ G(v,\ u))}{2} \leq \varepsilon + \delta(\varepsilon)$$

implies

$$\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2} < \varepsilon. \tag{21}$$

Condition (21) implies the strict contractive condition

$$\begin{aligned} &\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2} \\ &< \frac{d(G(x,\ y),\ G(u,\ v)) + d(G(y,\ x),\ G(v,\ u))}{2}, \end{aligned} \tag{22}$$

for $G(x, y) \leq G(u, v)$ and $G(y, x) \geq G(v, u)$. Thus, by (22), we have

$$\begin{aligned} &\frac{d(G(x_{n+1},\ y_{n+1}),\ G(x_n,\ y_n)) + d(G(y_{n+1},\ x_{n+1}),\ G(y_n,\ x_n))}{2} \\ &= \frac{d(F(x_n,\ y_n),\ F(x_{n-1},\ y_{n-1})) + d(F(y_n,\ x_n),\ F(y_{n-1},\ x_{n-1}))}{2} \\ &< \frac{d(G(x_n,\ y_n),\ G(x_{n-1},\ y_{n-1})) + d(G(y_n,\ x_n),\ G(y_{n-1},\ x_{n-1}))}{2}, \end{aligned}$$

which shows that the sequence of nonnegative numbers $\{\alpha_n\}_{n=0}^{\infty}$ given by

$$\alpha_n = \frac{d(G(x_n,\ y_n),\ G(x_{n-1},\ y_{n-1})) + d(G(y_n,\ x_n),\ G(y_{n-1},\ x_{n-1}))}{2}, \tag{23}$$

is non-increasing. Therefore, there exists some $\varepsilon \geq 0$ such that

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{1}{2} \left[\begin{array}{c} d(G(x_n,\ y_n),\ G(x_{n-1},\ y_{n-1})) \\ +d(G(y_n,\ x_n),\ G(y_{n-1},\ x_{n-1})) \end{array} \right] = \varepsilon.$$

We shall prove that $\varepsilon = 0$. Suppose, to the contrary, that $\varepsilon > 0$. Then there exists a positive integer p such that

$$\varepsilon < \alpha_p < \varepsilon + \delta(\varepsilon),$$

which, by (21), implies

$$\frac{d(F(x_p,\ y_p),\ F(x_{p-1},\ y_{p-1})) + d(F(y_p,\ x_p),\ F(y_{p-1},\ x_{p-1}))}{2} < \varepsilon,$$

it follows, by (19), that

$$\alpha_{p+1} = \frac{d(G(x_{p+1},\ y_{p+1}),\ G(x_p,\ y_p)) + d(G(y_{p+1},\ x_{p+1}),\ G(y_p,\ x_p))}{2} < \varepsilon,$$

which is a contradiction. Thus $\varepsilon = 0$ and hence

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{1}{2}\left[\begin{array}{c} d(G(x_n,\ y_n),\ G(x_{n-1},\ y_{n-1})) \\ +d(G(y_n,\ x_n),\ G(y_{n-1},\ x_{n-1})) \end{array}\right] = 0. \tag{24}$$

Let now $\varepsilon > 0$ be arbitrary and $\delta(\varepsilon)$ the corresponding value from the hypothesis of our theorem. By (24), there exists a positive integer k such that

$$\alpha_{k+1} = \frac{1}{2}\left[\begin{array}{c} d(G(x_{k+1},\ y_{k+1}),\ G(x_k,\ y_k)) \\ +d(G(y_{k+1},\ x_{k+1}),\ G(y_k,\ x_k)) \end{array}\right] < \delta(\varepsilon). \tag{25}$$

For this fixed number k, consider now the set $A_k = \{(G(x,\ y),\ G(y,\ x)) : G(x_k,\ y_k) \leq G(x,\ y),\ G(y_k,\ x_k) \geq G(y,\ x),\ \frac{1}{2}[d(G(x_k,\ y_k),\ G(x,\ y)) + d(G(y_k,\ x_k),\ G(y,\ x))]\} < \varepsilon + \delta(\varepsilon)$. By (25), $A_k \neq \phi$. We claim that

$$(G(x,\ y),\ G(y,\ x)) \in A_k \Rightarrow (F(x,\ y),\ F(y,\ x)) \in A_k. \tag{26}$$

Let $(G(x,\ y), G(y,\ x)) \in A_k$. Then

$$\frac{d(G(x_k,\ y_k),\ G(x,\ y)) + d(G(y_k,\ x_k),\ G(y,\ x))}{2} < \varepsilon + \delta(\varepsilon), \tag{27}$$

which, by (17), implies

$$\frac{d(F(x_k,\ y_k),\ F(x,\ y)) + d(F(y_k,\ x_k),\ F(y,\ x))}{2} < \varepsilon. \tag{28}$$

Now, by (25) and (28), we have

$$\begin{aligned}
&\frac{d(G(x_k,\ y_k),\ F(x,\ y)) + d(G(y_k,\ x_k),\ F(y,\ x))}{2} \\
&\leq \frac{d(G(x_k,\ y_k),\ F(x_k,\ y_k)) + d(G(y_k,\ x_k),\ F(y_k,\ x_k))}{2} \\
&\quad + \frac{d(F(x_k,\ y_k),\ F(x,\ y)) + d(F(y_k,\ x_k),\ F(y,\ x))}{2} \\
&\leq \frac{d(G(x_k,\ y_k),\ G(x_{k+1},\ y_{k+1})) + d(G(y_k,\ x_k),\ G(y_{k+1},\ x_{k+1}))}{2} \\
&\quad + \frac{d(F(x_k,\ y_k),\ F(x,\ y)) + d(F(y_k,\ x_k),\ F(y,\ x))}{2} \\
&< \varepsilon + \delta(\varepsilon).
\end{aligned}$$

Thus $(F(x, y), F(y, x)) \in A_k$. Again

$$\begin{aligned}
&\frac{d(G(x_k,\ y_k),\ G(x_{k+1},\ y_{k+1})) + d(G(y_k,\ x_k),\ G(y_{k+1},\ x_{k+1}))}{2} \\
&\leq \frac{d(G(x_k,\ y_k),\ F(x,\ y)) + d(G(y_k,\ x_k),\ F(y,\ x))}{2} \\
&\quad + \frac{d(F(x,\ y),\ G(x_{k+1},\ y_{k+1})) + d(F(y,\ x),\ G(y_{k+1},\ x_{k+1}))}{2} \\
&< 2(\varepsilon + \delta(\varepsilon)).
\end{aligned}$$

Thus $(G(x_{k+1}, y_{k+1}), G(y_{k+1}, x_{k+1})) \in A_k$ and by induction,

$$(G(x_n,\ y_n),\ G(y_n,\ x_n)) \in A_k, \text{ for all } n > k.$$

This implies that for all $n, m > k$, we have

$$\begin{aligned}
&\frac{d(G(x_n,\ y_n),\ G(x_m,\ y_m)) + d(G(y_n,\ x_n),\ G(y_m,\ x_m))}{2} \\
&\leq \frac{d(G(x_n,\ y_n),\ G(x_k,\ y_k)) + d(G(y_n,\ x_n),\ G(y_k,\ x_k))}{2} \\
&\quad + \frac{d(G(x_k,\ y_k),\ G(x_m,\ y_m)) + d(G(y_k,\ x_k),\ G(y_m,\ x_m))}{2} \\
&< 2(\varepsilon + \delta(\varepsilon)) = 4\varepsilon.
\end{aligned}$$

This shows that $\{G(x_n, y_n)\}_{n=0}^{\infty}$ and $\{G(y_n, x_n)\}_{n=0}^{\infty}$ are Cauchy sequences in X. Since X is complete, there is some $x, y \in X$ such that

(29)
$$\begin{aligned}
&\lim_{n\to\infty} G(x_n,\ y_n) = \lim_{n\to\infty} F(x_n,\ y_n) = x, \\
&\lim_{n\to\infty} G(y_n,\ x_n) = \lim_{n\to\infty} F(y_n,\ x_n) = y.
\end{aligned}$$

Since the pair $\{F, G\}$ satisfies the generalized compatibility, from (29), we get

(30)
$$\lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) = 0,$$

and

(31)
$$\lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) = 0.$$

Now suppose that assumption (a) holds. Then

$$\begin{aligned}
&d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(x,\ y)) \\
&\leq d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) \\
&\quad + d(G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ G(x,\ y)).
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, using (29), (30) and the fact that F and G are continuous, we have

$$F(x,\ y) = G(x,\ y).$$

Similarly we can show that

$$F(y,\ x) = G(y,\ x).$$

Thus (x, y) is a coupled coincidence point of F and G.

Now, suppose that (b) holds. By (20) and (29), we have $\{G(x_n, y_n)\}$ is a non-decreasing sequence, $G(x_n, y_n) \to x$ and $\{G(y_n, x_n)\}$ is a non-increasing sequence, $G(y_n, x_n) \to y$ as $n \to \infty$. Thus for all n, we have

$$G(x_n,\ y_n) \leq x \text{ and } G(y_n,\ x_n) \geq y. \tag{32}$$

Since G is continuous, by (29), (30) and (31), we have

$$\begin{aligned}
&\lim_{n\to\infty} G(G(x_n,\ y_n),\ G(y_n,\ x_n)) \\
&= G(x,\ y) \\
&= \lim_{n\to\infty} G(F(x_n,\ y_n),\ F(y_n,\ x_n)) \\
&= \lim_{n\to\infty} F(G(x_n,\ y_n),\ G(y_n,\ x_n))
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} G(G(y_n,\ x_n),\ G(x_n,\ y_n)) \\
&= G(y,\ x) \\
&= \lim_{n\to\infty} G(F(y_n,\ x_n),\ F(x_n,\ y_n)) \\
&= \lim_{n\to\infty} F(G(y_n,\ x_n),\ G(x_n,\ y_n)).
\end{aligned} \tag{34}$$

Since G has the mixed monotone property, it follows from (32) that $G(G(x_n, y_n), G(y_n, x_n)) \leq G(x, y)$ and $G(G(y_n, x_n), G(x_n, y_n)) \geq G(y, x)$. Now using (22), we get

$$\frac{d\left(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ F(x,\ y)\right) + d\left(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ F(y,\ x)\right)}{2}$$
$$< \frac{d\left(G(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(x,\ y)\right) + d\left(G(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(y,\ x)\right)}{2}.$$

Letting $n \to \infty$ in the above inequality, by using (33) and (34), we get

$$d\left(G(x,\ y),\ F(x,\ y)\right) = 0 \text{ and } d(G(y,\ x),\ F(y,\ x)),$$

it follows that

$$G(x,\ y) = F(x,\ y) \text{ and } G(y,\ x) = F(y,\ x),$$

that is, $(x,\ y)$ is a coupled coincidence point of F and G. □

Corollary 19. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F,\ G : X \times X \to X$ be two commuting mappings such that F is $G-$increasing with respect to $\leq$, G is continuous and has the mixed monotone property, and there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0,\ y_0) \leq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \geq F(y_0,\ x_0).$$

Suppose that the inequalities (17) and (18) hold and either

(a) F is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$ in X then $x_n \leq x$, for all n,

(ii) if a non-increasing sequence $\{x_n\} \to x$ in X then $x \leq x_n$, for all n.

Then F and G have a coupled coincidence point.

Now we deduce the results without mixed $g-$monotone property of F.

Corollary 20. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F is $g-$increasing with respect to $\leq$, and for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \leq \frac{d(gx,\ gu) + d(gy,\ gv)}{2} \leq \varepsilon + \delta(\varepsilon)$$

implies

$$\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2} < \varepsilon, \tag{35}$$

for all $x, y, u, v \in X$, where $g(x) \leq g(u)$ and $g(y) \geq g(v)$. Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and monotone increasing with respect to $\leq$ and the pair $\{F, g\}$ is compatible. Also suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$ in X then $x_n \leq x$, for all n,

(ii) if a non-increasing sequence $\{x_n\} \to x$ in X then $x \leq x_n$, for all n.

If there exist two elements $x_0,\ y_0 \in X$ with

$$gx_0 \leq F(x_0,\ y_0) \text{ and } gy_0 \geq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point.

Corollary 21. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F*

is $g-$increasing with respect to $\leq$ and satisfying (35). *Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and monotone increasing with respect to $\leq$ and the pair $\{F, g\}$ is commuting. Also suppose that either*

(a) F is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$ in X then $x_n \leq x$, for all n,

(ii) if a non-increasing sequence $\{x_n\} \to x$ in X then $x \leq x_n$, for all n.

If there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \leq F(x_0, y_0) \text{ and } gy_0 \geq F(y_0, x_0).$$

Then F and g have a coupled coincidence point.

Now, we deduce the result without mixed monotone property of F.

Corollary 22. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X \times X \to X$ be an increasing mapping with respect to $\leq$ and for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \leq \frac{d(x, u) + d(y, v)}{2} \leq \varepsilon + \delta(\varepsilon)$$

implies

$$\frac{d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))}{2} < \varepsilon, \tag{36}$$

for all $x, y, u, v \in X$, where $x \leq u$ and $y \geq v$. Also suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$ in X then $x_n \leq x$, for all n,

(ii) if a non-increasing sequence $\{x_n\} \to x$ in X then $x \leq x_n$, for all n.

If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0).$$

Then F has a coupled fixed point.

Example 23. Suppose that $X = \mathbb{R}$, equipped with the usual metric $d : X \times X \to [0, +\infty)$. Let $F, G : X \times X \to X$ be defined as

$$F(x, y) = \begin{cases} \frac{x^2-y^2}{3}, \text{ if } x \geq y \\ 0, \text{ if } x < y, \end{cases}$$

and

$$G(x, y) = \begin{cases} x^2 - y^2, \text{ if } x \geq y \\ 0, \text{ if } x < y. \end{cases} .$$

First, we shall show that the mappings F and G satisfy the condition (17). Let x, y, u, $v \in X$ such that $G(x, y) \leq G(u, v)$ and $G(y, x) \geq G(v, u)$, such that

$$\varepsilon \leq \frac{d(G(x,\ y),\ G(u,\ v)) + d(G(y,\ x),\ G(v,\ u))}{2} \leq \varepsilon + \delta(\varepsilon),$$

that is,

$$\varepsilon \leq \frac{1}{2}\left[\left|x^2 - y^2\right| + \left|u^2 - v^2\right|\right] \leq \varepsilon + \delta(\varepsilon).$$

Then

$$\begin{aligned}
&\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2} \\
=\ & \frac{1}{2}\left[\left|\frac{x^2-y^2}{3} - \frac{u^2-v^2}{3}\right| + \left|\frac{y^2-x^2}{3} - \frac{v^2-u^2}{3}\right|\right] \\
=\ & \left|\frac{x^2-y^2}{3} - \frac{u^2-v^2}{3}\right| \\
\leq\ & \frac{1}{3}\left[\left|x^2 - y^2\right| + \left|u^2 - v^2\right|\right] \\
\leq\ & \frac{2}{3}(\varepsilon + \delta(\varepsilon)) < \varepsilon.
\end{aligned}$$

Thus the contractive condition (17) is satisfied for all x, y, u, $v \in X$. In addition, like in [10], all the other conditions of Theorem 18 are satisfied and $z = (0, 0)$ is a coincidence point of F and G.

Now we prove the uniqueness of the coupled coincidence point. Note that if $(X, \leq)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order relation, for all $(x, y), (u, v) \in X \times X$:

$$(x,\ y) \leq (u,\ v) \Longleftrightarrow G(x,\ y) \leq G(u,\ v) \text{ and } G(y,\ x) \geq G(v,\ u),$$

where $G : X \times X \to X$ is one-one.

Theorem 24. *In addition to the hypotheses of Theorem 18, suppose that for every (x, y), (x^*, y^*) in $X \times X$, there exists another (u, v) in $X \times X$ which is comparable to (x, y) and (x^*, y^*), then F and G have a unique coupled coincidence point.*

Proof. From Theorem 18, the set of coupled coincidence points of F and G is non-empty. Assume that (x, y), $(x^*, y^*) \in X \times X$ are two coupled coincidence points of F and G, that is,

$$\begin{aligned}
F(x,\ y) &= G(x,\ y) \text{ and } F(y,\ x) = G(y,\ x), \\
F(x^*,\ y^*) &= G(x^*,\ y^*) \text{ and } F(y^*,\ x^*) = G(y^*,\ x^*).
\end{aligned}$$

We shall prove that $G(x, y) = G(x^*, y^*)$ and $G(y, x) = G(y^*, x^*)$. By assumption, there exists $(u, v) \in X \times X$, that is, comparable to (x, y) and (x^*, y^*). We define the sequences $\{G(u_n, v_n)\}$ and $\{G(v_n, u_n)\}$ as follows, with $u_0 = u$, $v_0 = v$:

$$G(u_{n+1},\ v_{n+1}) = F(u_n,\ v_n),\ G(v_{n+1},\ u_{n+1}) = F(v_n,\ u_n),\ n \geq 0.$$

Since (u, v) is comparable to (x, y), we may assume that $(x, y) \leq (u, v) = (u_0, v_0)$, which implies that $G(x, y) \leq G(u_0, v_0)$ and $G(y, x) \geq G(v_0, u_0)$. We suppose that $(x, y) \leq (u_n, v_n)$ for some n. We prove that $(x, y) \leq (u_{n+1}, v_{n+1})$. Since F is $G-$increasing, we have $G(x, y) \leq G(u_n, v_n)$ implies $F(x, y) \leq F(u_n, v_n)$ and $G(y, x) \geq G(v_n, u_n)$ implies $F(y, x) \geq F(v_n, u_n)$. Therefore

$$G(x,\ y) = F(x,\ y) \leq F(u_n,\ v_n) = G(u_{n+1},\ v_{n+1})$$

and

$$G(y,\ x) = F(y,\ x) \geq F(v_n,\ u_n) = G(v_{n+1},\ u_{n+1}).$$

Thus, we have

$$(x,\ y) \leq (u_{n+1},\ v_{n+1}),\ \text{for all } n.$$

Now, by (22), we have

$$\begin{aligned} & \frac{d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})) + d(G(y,\ x),\ G(v_{n+1},\ u_{n+1}))}{2} \\ = \ & \frac{d(F(x,\ y),\ F(u_n,\ v_n)) + d(F(y,\ x),\ F(v_n,\ u_n))}{2} \\ < \ & \frac{d(G(x,\ y),\ G(u_n,\ v_n)) + d(G(y,\ x),\ G(v_n,\ u_n))}{2}, \end{aligned}$$

which shows that the sequence of nonnegative numbers $\{\beta_n\}_{n=0}^{\infty}$ given by

$$\beta_n = \frac{d(G(x,\ y),\ G(u_n,\ v_n)) + d(G(y,\ x),\ G(v_n,\ u_n))}{2},$$

is non-increasing. Therefore, there exists some $\varepsilon \geq 0$ such that

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \frac{d(G(x,\ y),\ G(u_n,\ v_n)) + d(G(y,\ x),\ G(v_n,\ u_n))}{2} = \varepsilon.$$

We shall prove that $\varepsilon = 0$. Suppose, to the contrary, that $\varepsilon > 0$. Then there exists a positive integer p such that

$$\varepsilon < \beta_p < \varepsilon + \delta(\varepsilon),$$

which, by (17), implies

$$\frac{d(F(x,\ y),\ F(x_p,\ y_p)) + d(F(y,\ x),\ F(y_p,\ x_p))}{2} < \varepsilon,$$

it follows, by (19), that

$$\beta_{p+1} = \frac{d(G(x,\ y),\ G(x_{p+1},\ y_{p+1})) + d(G(y,\ x),\ G(y_{p+1},\ x_{p+1}))}{2} < \varepsilon,$$

which is a contradiction. Thus $\varepsilon = 0$ and hence

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \frac{d(G(x,\ y),\ G(u_n,\ v_n)) + d(G(y,\ x),\ G(v_n,\ u_n))}{2} = 0. \tag{37}$$

it follows that

$$G(x,\ y) = \lim_{n\to\infty} G(u_n,\ v_n) \text{ and } G(y,\ x) = \lim_{n\to\infty} G(v_n,\ u_n).$$

Similarly, we can show that

$$G(x^*,\ y^*) = \lim_{n\to\infty} G(u_n,\ v_n) \text{ and } G(y^*,\ x^*) = \lim_{n\to\infty} G(v_n,\ u_n)).$$

Thus $G(x,\, y) = G(x^*,\, y^*)$ and $G(y,\, x) = G(y^*,\, x^*)$. □

3. Application to Integral Equations

As an application of the results established in section 2 of our paper, we study the existence of the solution to a Fredholm nonlinear integral equation. We shall consider the following integral equation

$$x(p) = \int_a^b (K_1(p,\ q) + K_2(p,\ q))\,[f(q,\ x(q)) + g(q,\ x(q))]\,dq + h(p), \tag{38}$$

for all $p \in I = [a,\, b]$.

Let Θ denote the set of all functions $\theta : [0,\, +\infty) \to [0,\, +\infty)$ satisfying
(i_θ) θ is non-decreasing,
(ii_θ) $\theta(p) \leq p$.

Assumption 25. We assume that the functions $K_1,\ K_2,\ f,\ g$ fulfill the following conditions:

(i) $K_1(p,\, q) \geq 0$ and $K_2(p,\, q) \leq 0$ for all $p,\, q \in I$,

(ii) There exist the positive numbers λ, μ and $\theta \in \Theta$ such that for all $x,\, y \in \mathbb{R}$ with $x \geq y$, the following conditions hold:

$$0 \leq f(q,\ x) - f(q,\ y) \leq \lambda\theta(x - y), \tag{39}$$

and

$$-\mu\theta(x - y) \leq g(q,\ x) - g(q,\ y) \leq 0, \tag{40}$$

(*iii*)

$$\max\{\lambda,\ \mu\} \sup_{p\in I} \int_a^b [K_1(p,\ q) - K_2(p,\ q)]dq \leq \frac{1}{6}. \tag{41}$$

Definition 26 ([12]). A pair $(\alpha,\ \beta) \in X^2$ with $X = C(I,\ \mathbb{R})$, where $C(I,\ \mathbb{R})$ denote the set of all continuous functions from I to $\mathbb{R}$, is called a *coupled lower-upper solution* of (38) if, for all $p \in I$,

$$\begin{aligned}\alpha(p) &\leq \int_a^b K_1(p,\ q)\,[f(q,\ \alpha(q)) + g(q,\ \beta(q))]\,dq \\ &\quad + \int_a^b K_2(p,\ q)\,[f(q,\ \beta(q)) + g(q,\ \alpha(q))]\,dq + h(p),\end{aligned}$$

and

$$\begin{aligned}\beta(p) &\geq \int_a^b K_1(p,\ q)\,[f(q,\ \beta(q)) + g(q,\ \alpha(q))]\,dq \\ &\quad + \int_a^b K_2(p,\ q)\,[f(q,\ \alpha(q)) + g(q,\ \beta(q))]\,dq + h(p).\end{aligned}$$

Theorem 27. *Consider the integral equation* (38) *with* $K_1,\ K_2 \in C(I \times I,\ \mathbb{R})$, f, $g \in C(I \times \mathbb{R},\ \mathbb{R})$ *and* $h \in C(I,\ \mathbb{R})$. *Suppose that there exists a coupled lower-upper solution* $(\alpha,\ \beta)$ *of* (38) *and Assumption 25 is satisfied. Then the integral equation* (38) *has a solution in* $C(I,\ \mathbb{R})$.

Proof. Consider $X = C(I,\ \mathbb{R})$, the natural partial order relation, that is, for x, $y \in C(I,\ \mathbb{R})$,

$$x \leq y \Longleftrightarrow x(p) \leq y(p),\ \forall p \in I.$$

It is well known that X is a complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{p\in I} |x(p) - y(p)|\,.$$

Now define on $X \times X$ the following partial order: for $(x,\ y)$, $(u,\ v) \in X \times X$,

$$(x,\ y) \leq (u,\ v) \Longleftrightarrow x(p) \leq u(p) \text{ and } y(p) \geq v(p),\ \forall p \in I.$$

Obviously, for any $(x,\ y) \in X \times X$, the functions $\max\{x,\ y\}$ and $\min\{x,\ y\}$ are the upper and lower bounds of x and y respectively. Therefore for every $(x,\ y)$, $(u,\ v) \in X \times X$, there exists the element $(\max\{x,\ u\},\ \min\{y,\ v\})$ which is comparable to $(x,\ y)$ and $(u,\ v)$. Define now the mapping $F : X \times X \to X$ by

$$\begin{aligned} F(x,\ y)(p) \ &= \ \int_a^b K_1(p,\ q)\,[f(q,\ x(q)) + g(q,\ y(q))]\,dq \\ &\quad + \int_a^b K_2(p,\ q)\,[f(q,\ y(q)) + g(q,\ x(q))]\,dq + h(p), \end{aligned}$$

for all $p \in I$. We can prove, like in [10], that F is increasing. Let $x,\ y,\ u,\ v \in X$ with $x \le u$ and $y \ge v$, such that

$$\varepsilon \le \frac{d(x,\ u) + d(y,\ v)}{2} \le \varepsilon + \delta(\varepsilon). \tag{42}$$

Now, by using (39) and (40), we have

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ = \ & \int_a^b K_1(p,\ q)\,[f(q,\ x(q)) + g(q,\ y(q))]\,dq \\ & + \int_a^b K_2(p,\ q)\,[f(q,\ y(q)) + g(q,\ x(q))]\,dq \\ & - \int_a^b K_1(p,\ q)\,[f(q,\ u(q)) + g(q,\ v(q))]\,dq \\ & - \int_a^b K_2(p,\ q)\,[f(q,\ v(q)) + g(q,\ u(q))]\,dq \\ = \ & \int_a^b K_1(p,\ q)\,[f(q,\ x(q)) - f(q,\ u(q)) + g(q,\ y(q)) - g(q,\ v(q))]\,dq \\ & + \int_a^b K_2(p,\ q)\,[f(q,\ y(q)) - f(q,\ v(q)) + g(q,\ x(q)) - g(q,\ u(q))]\,dq \\ = \ & \int_a^b K_1(p,\ q)\,[(f(q,\ x(q)) - f(q,\ u(q))) - (g(q,\ v(q)) - g(q,\ y(q)))]\,dq \\ & - \int_a^b K_2(p,\ q)\,[(f(q,\ v(q)) - f(q,\ y(q))) - (g(q,\ x(q)) - g(q,\ u(q)))]\,dq \end{aligned}$$

Thus

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ & \le \int_a^b K_1(p,\ q)\,[\lambda\theta\,(x(q) - u(q)) + \mu\theta\,(v(q) - y(q))]\,dq \\ & \quad - \int_a^b K_2(p,\ q)\,[\lambda\theta\,(v(q) - y(q)) + \mu\theta\,(x(q) - u(q))]\,dq. \end{aligned} \tag{43}$$

Since the function θ is non-decreasing and $x \geq u$, $y \leq v$, we have

$$\theta\left(x(q) - u(q)\right) \leq \theta\left(\sup_{q\in I}|x(q) - u(q)|\right) = \theta(d(x,\ u)),$$
$$\theta\left(v(q) - y(q)\right) \leq \theta\left(\sup_{q\in I}|v(q) - y(q)|\right) = \theta(d(y,\ v)).$$

Hence by (43), in view of the fact that $K_2(p, q) \leq 0$, we obtain

$$\begin{aligned}
&|F(x,\ y)(p) - F(u,\ v)(p)| \\
&\leq \int_a^b K_1(p,\ q)\left[\lambda\theta(d(x,\ u)) + \mu\theta(d(y,\ v))\right] dq \\
&\quad - \int_a^b K_2(p,\ q)\left[\lambda\theta(d(y,\ v)) + \mu\theta(d(x,\ u))\right] dq, \\
&\leq \int_a^b K_1(p,\ q)\left[\max\{\lambda,\ \mu\}\theta(d(x,\ u)) + \max\{\lambda,\ \mu\}\theta(d(y,\ v))\right] dq \\
&\quad - \int_a^b K_2(p,\ q)\left[\max\{\lambda,\ \mu\}\theta(d(y,\ v)) + \max\{\lambda,\ \mu\}\theta(d(x,\ u))\right] dq,
\end{aligned}$$

as all the quantities on the right hand side of (43) are non-negative. Now, taking the supremum with respect to p we get, by using (41),

$$\begin{aligned}
&d(F(x,\ y),\ F(u,\ v)) \\
&\leq \max\{\lambda,\ \mu\}\sup_{p\in I}\int_a^b (K_1(p,\ q) - K_2(p,\ q))\, dq. \left[\theta(d(x,\ u)) + \theta(d(y,\ v))\right] \\
&\leq \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{6}.
\end{aligned}$$

Thus

$$d(F(x,\ y),\ F(u,\ v)) \leq \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{6}.$$

Similarly

$$d(F(y,\ x),\ F(v,\ u)) \leq \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{6}.$$

Combining them, we get

$$\begin{aligned}
&\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2} \\
&\leq \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{6}.
\end{aligned} \tag{44}$$

Now, since θ is non-decreasing, we have

$$\theta(d(x,\ u)) \leq \theta\left(d(x,\ u) + d(y,\ v)\right),$$
$$\theta(d(y,\ v)) \leq \theta\left(d(x,\ u) + d(y,\ v)\right),$$

which implies, by (ii_θ), that

$$\begin{aligned}\frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{2} &\leq \theta\left(d(x,\ u) + d(y,\ v)\right)\\ &\leq d(x,\ u) + d(y,\ v).\end{aligned}$$

Hence

$$\frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{6} \leq \frac{1}{3}\left[d(x,\ u) + d(y,\ v)\right]. \tag{45}$$

Thus by (42), (44) and (45), we have

$$\begin{aligned}&\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2}\\ &\leq \frac{1}{3}\left[d(x,\ u) + d(y,\ v)\right]\\ &\leq \frac{2}{3}(\varepsilon + \delta(\varepsilon)) < \varepsilon,\end{aligned}$$

which is the contractive condition (36) in Corollary 22. Now, let $(\alpha,\ \beta) \in X \times X$ be a coupled upper-lower solution of (38), then we have $\alpha(p) \leq F(\alpha,\ \beta)(p)$ and $\beta(p) \geq F(\ \beta,\ \alpha)(p)$, for all $p \in I$, which shows that all hypothesis of Corollary 22 are satisfied. This proves that F has a coupled fixed point $(x,\ y) \in X \times X$ which is the solution in $X = C(I,\ \mathbb{R})$ of the integral equation (38). $\square$

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