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ON PILLAI'S PROBLEM WITH TRIBONACCI NUMBERS AND POWERS OF 2

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Abstract. The Tribonacci sequence $\{T_n\}_{n\geq 0}$ resembles the Fibonacci sequence in that it starts with the values $0, 1, 1$, and each term afterwards is the sum of the preceding three terms. In this paper, we find all integers c having at least two representations as a difference between a Tribonacci number and a power of 2. This paper continues the previous work [5].

1. Introduction

A perfect power is a positive integer of the form a^x where $a > 1$ and $x \geq 2$ are integers. Pillai wrote several papers on these numbers. In 1936 and again in 1945 (see [13]), he conjectured that for any given integer $c \geq 1$, the number of positive integer solutions (a, b, x, y), with $x \geq 2$ and $y \geq 2$, to the Diophantine equation

$$a^x - b^y = c \tag{1}$$

is finite. This conjecture which is still open for all $c \neq 1$, amounts to saying that the distance between two consecutive terms in the sequence of all perfect powers tends to infinity. The case $c = 1$ is Catalan's conjecture which predicted that the only consecutive perfect powers were 8 and 9 was solved by Mihăilescu [11].

The work started by Pillai was pursued in 1936 by A. Herschfeld [8, 9] who showed that if c is an integer with sufficiently large absolute value, then equation (1), in the special case $(a, b) = (3, 2)$, has at most one solution (x, y). For small $|c|$ this is not the case. Pillai [13, 14] extended Herschfeld's result to the more general exponential Diophantine equation (1) with fixed integers a, b, c with $\gcd(a, b) = 1$ and $a > b \geq 1$. Specifically, Pillai showed that there exists a positive integer $c_0(a, b)$ such that, for $|c| > c_0(a, b)$, equation (1) has at most one positive integer solution (x, y). Pillai's work (as well as Herschfeld's one)

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depended on Siegel's sharpening of Thue's inequality on the rational approximation of algebraic numbers [15] and the proof does not give any explicit value for $c_0(a,b)$. This was made effective by Stroeker and Tijdeman [16] (for the case $(a,b)=(3,2)$) and Mo De Ze and Tijdeman [12] for a more general case.

Recently, Ddamulira, Luca and Rakotomalala [5] considered the Diophantine equation

$$F_n - 2^m = c, \tag{2}$$

where c is a fixed integer and $\{F_n\}_{n\geq 0}$ is the sequence of Fibonacci numbers given by $F_0=0$, $F_1=1$ and $F_{n+2}=F_{n+1}+F_n$ for all $n\geq 0$. For equation (2), which can be seen as a variation of Pillai's problem, they proved that the only integers c having at least two representations of the form F_n-2^m are $c\in\{0,1,-3,5,-11,-30,85\}$ (here, $F_1=F_2=1$ are identified so representations involving F_1 or F_2 do not count as distinct). Moreover, they computed all the representations of the form (2) for all these values of c.

In this paper we study the similar problem with the Tribonacci sequence $\{T_n\}_{n\geq 0}$ given by $T_0=0$, $T_1=T_2=1$, and

$$T_{n+3}=T_{n+2}+T_{n+1}+T_n \quad \text{for all} \quad n\geq 0.$$

That is, we are interested in finding all positive integers c admitting at least two representations of the form T_n-2^m for some positive integers n and m. As in the Fibonacci case, we discard the situation when $n=1$ and just count the solutions for $n=2$, since $T_1=T_2$. The above is just a convention to avoid trivial parametric families such as $1-2^m=T_1-2^m=T_2-2^m$. Therefore, we always assume that $n\geq 2$.

We prove the following result.

Theorem 1. *The only integers c having at least two representations of the form T_n-2^m are $c\in\{0,-1,-3,5,-8\}$. Furthermore, for each c in the above set, all its representations of the form T_n-2^m with integers $n\geq 2$ and $m\geq 1$ are*

$$\begin{aligned}
0 &= 4-4=T_4-2^2=T_3-2^1=2-2,\\
-1 &= 7-8=T_5-2^3=T_2-2^1=1-2,\\
-3 &= 13-16=T_6-2^4=T_2-2^2=1-4,\\
5 &= 13-8=T_6-2^3=T_5-2^1=7-2,\\
-8 &= 504-512=T_{12}-2^9=T_7-2^5=24-32.
\end{aligned}$$

We note that in the recent paper [4], the authors have studied the same problem but involving both $\{F_n\}_{n\geq 0}$ and $\{T_m\}_{m\geq 0}$. That is, they found all integers c having two nontrivial representations as F_n-T_m for some positive integers m and n.

2. Preliminaries

In this section we present some basic properties of the Tribonacci numbers and a lower bound for a nonzero linear form in logarithms of algebraic numbers. Additionally, we state a reduction lemma, which is an immediate variation of a result due to Dujella and Pethő from [7], and will be the key tool used in this paper to reduce some upper bounds. All these facts will be used in the proof of Theorem 1.

2.1. The Tribonacci sequence

The characteristic polynomial of the Tribonacci sequence $\{T_n\}_{n\geq 0}$ is

$$\Psi(x) = x^3 - x^2 - x - 1.$$

This polynomial, which is irreducible in $\mathbb{Q}[x]$, has a positive real root

$$\alpha = \frac{1}{3}\left(1 + (19 + 3\sqrt{33})^{1/3} + (19 - 3\sqrt{33})^{1/3}\right)$$

and two complex conjugate roots β and γ strictly inside the unit circle. Further, $|\beta| = |\gamma| = \alpha^{-1/2}$. A recent result of Dresden and Du [6] establishes a Binet-like formula for generalized Fibonacci sequences. For Tribonacci numbers it states that

$$T_n = C_\alpha \alpha^{n-1} + C_\beta \beta^{n-1} + C_\gamma \gamma^{n-1} \quad \text{for all} \quad n \geq 1, \tag{3}$$

where $C_X = (X-1)/(4X-6)$. Dresden and Du also showed that the contribution of the complex roots β and γ to the right–hand side of (3) is very small. More precisely,

$$|T_n - C_\alpha \alpha^{n-1}| < 1/2 \quad \text{for all} \quad n \geq 0. \tag{4}$$

It is also well-known (see [2]) that

$$\alpha^{n-2} \leq T_n \leq \alpha^{n-1} \quad \text{for all} \quad n \geq 1. \tag{5}$$

Let $\mathbb{L} := \mathbb{Q}(\alpha, \beta)$ be the splitting field of Ψ over $\mathbb{Q}$. Then $[\mathbb{L} : \mathbb{Q}] = 6$. Furthermore, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. The Galois group of $\mathbb{L}$ over $\mathbb{Q}$ is

$$G := Gal(\mathbb{L}/\mathbb{Q}) \cong \{(1), (\alpha\beta), (\alpha\gamma), (\beta\gamma), (\alpha\beta\gamma), (\alpha\gamma\beta)\} \cong S_3.$$

Here, we identify the automorphisms of G with the permutations of the roots of Ψ. For instance, the permutation $(\alpha\beta)$ corresponds to the automorphism $\sigma : \alpha \to \beta,\ \beta \to \alpha,\ \gamma \to \gamma$.

2.2. Linear forms in logarithms

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. The logarithmic height of η is given by

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

The following properties of the logarithmic height function $h(\cdot)$ will be used in the next sections:

$$\begin{aligned} h(\eta \pm \gamma) &\le h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\le h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta) \qquad (s \in \mathbb{Z}). \end{aligned}$$

Our main tool is a general lower bound for a linear form in logarithms of algebraic numbers given by the following result of Matveev (see [10] and Theorem 9.4 in [3]).

Theorem 2 (Matveev's theorem)**.** *Assume that $\gamma_1, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, \ldots, b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1,$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t\right),$$

where

$$B \ge \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \ge \max\{D h(\gamma_i), |\log \gamma_i|, 0.16\} \quad \text{for all} \quad i = 1, \ldots, t.$$

2.3. The reduction lemma

In the course of our calculations, we get some upper bounds on our variables which are very large, so we need to reduce them. To this end, we use some results of the theory of continued fractions. Specifically, for a nonhomogenenous linear forms in two integer variables, we will use a slight variation of a result due to Dujella and Pethő from [7], which itself is a generalization of a result of Baker–Davenport [1].

For a real number X, we write $||X|| = \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 3. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let further $\epsilon = ||\mu q|| - M||\gamma q||$. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v and w with

$$u \le M \quad \textit{and} \quad w \ge \frac{\log(Aq/\epsilon)}{\log B}.$$

3. Proof of Theorem 1

Assume that $(n, m) \neq (n_1, m_1)$ are such that

$$T_n - 2^m = T_{n_1} - 2^{m_1}.$$

If $m = m_1$, then $T_n = T_{n_1}$ and since $\min\{n, n_1\} \ge 2$, we get that $n = n_1$, so $(n, m) = (n_1, m_1)$, contradicting our assumption. Thus, $m \neq m_1$, and we may assume that $m > m_1$. Since

$$T_n - T_{n_1} = 2^m - 2^{m_1}, \tag{6}$$

and the right–hand side is positive, we get that the left–hand side is also positive and so $n > n_1$. Thus, $n \ge 3$ and $n_1 \ge 2$.

On the other hand, since $n > n_1$ we have that $T_{n_1} \le T_{n-1}$ and therefore

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \ge T_{n-1} + T_{n-2} \ge T_{n_1} + T_{n-2}.$$

So, from the above and (5) and (6), we have

$$\begin{aligned} \alpha^{n-4} &\le T_{n-2} \le T_n - T_{n_1} = 2^m - 2^{m_1} < 2^m \quad \text{and} \\ \alpha^{n-1} &\ge T_n > T_n - T_{n_1} = 2^m - 2^{m_1} \ge 2^{m-1} \end{aligned} \tag{7}$$

leading to

$$1 + \left(\frac{\log 2}{\log \alpha}\right)(m-1) < n < \left(\frac{\log 2}{\log \alpha}\right) m + 4. \tag{8}$$

Note that the above inequality (8) in particular implies that $m < n$. In addition, if $n \le 250$, then $m < 220$. By a brute force computer enumeration in the range

$$2 \le n_1 < n \le 250 \quad \text{and} \quad 1 \le m_1 < m < 220$$

we found all solutions listed in Theorem 1. Thus, we may assume from now on that $n > 250$.

On the other hand, by (4) and (6), we get

$$\begin{aligned} \left|C_\alpha \alpha^{n-1} - 2^m\right| &= \left|(C_\alpha \alpha^{n-1} - T_n) + (T_{n_1} - 2^{m_1})\right| \\ &= \left|(C_\alpha \alpha^{n-1} - T_n) + (T_{n_1} - C_\alpha \alpha^{n_1 - 1}) + (C_\alpha \alpha^{n_1-1} - 2^{m_1})\right| \\ &< 1 + \frac{7}{10}\alpha^{n_1-1} + 2^{m_1} \\ &< \alpha^{n_1} + 2^{m_1} \\ &< 2\max\{\alpha^{n_1}, 2^{m_1}\}. \end{aligned}$$

In the above we also used that $C_\alpha = 0.6184\ldots$. Dividing by 2^m we get

$$\left|C_\alpha \alpha^{n-1} 2^{-m} - 1\right| < 2\max\left\{\frac{\alpha^{n_1}}{2^m}, 2^{m_1-m}\right\} < \max\{\alpha^{n_1-n+6}, 2^{m_1-m+1}\}, \tag{9}$$

where for the last right–most inequality above we used (7) and the fact that $2 < \alpha^2$. For the left–hand side above, we use Theorem 2 with the data

$$t = 3,\ \gamma_1 = C_\alpha,\ \gamma_2 = \alpha,\ \gamma_3 = 2,\ b_1 = 1,\ b_2 = n-1,\ b_3 = -m.$$

We take $\mathbb{K} = \mathbb{Q}(\alpha)$ for which $D = 3$. The minimal polynomial of C_α over $\mathbb{Z}$ is

$$44x^3 - 44x^2 + 12x - 1.$$

Since $|C_\alpha|, |C_\beta|, |C_\gamma| < 1$ we get that $h(C_\alpha) = (\log 44)/3$ so we can take $A_1 = \log 44$. We can also take $A_2 = 3h(\gamma_2) = \log\alpha$, $A_3 = 3h(\gamma_3) = 3\log 2$. Since $\max\{1, n-1, m\} = n-1$ we take $B = n$. Put

$$\Lambda = C_\alpha \alpha^{n-1} 2^{-m} - 1.$$

If $\Lambda = 0$, then we have that $C_\alpha\alpha^{n-1} = 2^m \in \mathbb{Z}$. Conjugating the above relation by the automorphism $(\alpha\beta)$, we obtain that $C_\beta\beta^{n-1} = 2^m$, which is false because $|C_\beta\beta^{n-1}| < 1$ while $2^m \geq 4$. Thus $\Lambda \neq 0$. Another way to see that $\Lambda \neq 0$ is by using the fact that the number C_α is not an algebraic integer.

Then, the left–hand side of (9) is bounded below, by Theorem 2, as

$$\log|\Lambda| > -1.4 \times 30^6 \times 3^{4.5} \times 3^2 (1+\log 3)(1+\log n)(\log 44)(\log\alpha)(3\log 2).$$

Comparing with (9), we get

$$\min\{(n-n_1-6)\log\alpha, (m-m_1-1)\log 2\} < 1.3 \times 10^{13}(1+\log n),$$

which gives

$$\min\{(n-n_1)\log\alpha, (m-m_1)\log 2\} < 1.4 \times 10^{13}(1+\log n).$$

Now the argument splits into two cases.

Case 1. $\min\{(n-n_1)\log\alpha, (m-m_1)\log 2\} = (n-n_1)\log\alpha$.

In this case, we rewrite (6) as

$$\begin{aligned}\left|C_\alpha\alpha^{n-1} - C_\alpha\alpha^{n_1-1} - 2^m\right| &= \left|(C_\alpha\alpha^{n-1} - T_n) + (T_{n_1} - C_\alpha\alpha^{n_1-1}) - 2^{m_1}\right| \\ &< 1 + 2^{m_1} \leq 2^{m_1+1},\end{aligned}$$

so

$$\left|C_\alpha(\alpha^{n-n_1} - 1)\alpha^{n_1-1}2^{-m} - 1\right| < 2^{m_1-m+1}. \tag{10}$$

Let us introduce

$$\Lambda_1 = C_\alpha(\alpha^{n-n_1} - 1)\alpha^{n_1-1}2^{-m} - 1.$$

We apply again Theorem 2. We take $t = 3$ and

$$\gamma_1 = C_\alpha(\alpha^{n-n_1} - 1),\ \gamma_2 = \alpha,\ \gamma_3 = 2,\ b_1 = 1,\ b_2 = n_1 - 1,\ b_3 = -m.$$

We begin by noticing that the three numbers $\gamma_1, \gamma_2, \gamma_3$ belong to $\mathbb{K} = \mathbb{Q}(\alpha)$, so we can take $D = 3$. Clearly, $\Lambda_1 \neq 0$, for if $\Lambda_1 = 0$, then

$$C_\alpha(\alpha^{n-n_1} - 1)\alpha^{n_1-1} = 2^m,$$

and conjugating this last relation by the automorphism $(\alpha\beta)$, we obtain that $C_\beta(\beta^{n-n_1}-1)\beta^{n_1-1}=2^m$. But this not possible since $|C_\beta(\beta^{n-n_1}-1)\beta^{n_1-1}|<1$ while $2^m\geq 4$.

Since

$$\begin{aligned} h(\gamma_1) &\leq h(C_\alpha)+h(\alpha^{n-n_1}-1)\\ &\leq \frac{\log 44}{3}+(n-n_1)\frac{\log\alpha}{3}+\log 2 \end{aligned} \tag{11}$$

it follows that $3h(\gamma_1)\leq \log 352+(n-n_1)\log\alpha<\log 352+1.4\times 10^{13}(1+\log n)$. So, we can take $A_1=1.5\times 10^{13}(1+\log n)$. Further, as before, we can take $A_2=\log\alpha$ and $A_3=3\log 2$. Finally, by recalling that $m<n$, we can take $B=n$.

We then get that

$$\begin{aligned} &\log|\Lambda_1|\\ &>-1.4\times 30^6\times 3^{4.5}\times 3^2(1+\log 3)(1+\log n)(1.5\times 10^{13}(1+\log n))(\log\alpha)(3\log 2). \end{aligned}$$

Thus,

$$\log|\Lambda_1|>-5.2\times 10^{25}(1+\log n)^2.$$

Comparing this with (10), we get that

$$(m-m_1)\log 2<5.3\times 10^{25}(1+\log n)^2.$$

Case 2. $\min\{(n-n_1)\log\alpha,(m-m_1)\log 2\}=(m-m_1)\log 2$.

In this case, we rewrite (6) as

$$\begin{aligned} \left|C_\alpha\alpha^{n-1}-2^m+2^{m_1}\right| &= \left|(C_\alpha\alpha^{n-1}-T_n)+(T_{n_1}-C_\alpha\alpha^{n_1-1})+C_\alpha\alpha^{n_1-1}\right|\\ &< 1+\frac{7}{10}\alpha^{n_1-1}<\alpha^{n_1} \end{aligned}$$

so

$$\left|C_\alpha(2^{m-m_1}-1)^{-1}\alpha^{n-1}2^{-m_1}-1\right|<\frac{\alpha^{n_1}}{2^m-2^{m_1}}\leq\frac{2\alpha^{n_1}}{2^m}<\alpha^{n_1-n+6}. \tag{12}$$

Inequality (12) suggests once again studying a lower bound for the absolute value of

$$\Lambda_2=C_\alpha(2^{m-m_1}-1)^{-1}\alpha^{n-1}2^{-m_1}-1.$$

We take $t=3$ and

$$\gamma_1=C_\alpha(2^{m-m_1}-1)^{-1},\ \gamma_2=\alpha,\ \gamma_3=2,\ b_1=1,\ b_2=n-1,\ b_3=-m_1.$$

In this application of Matveev's theorem we take $B=n$ and $\mathbb{K}=\mathbb{Q}(\alpha)$ and so $D=3$. Note that, if $\Lambda_2=0$, then $C_\alpha=(\alpha^{-1})^{n-n_1}\cdot 2^{m_1}\cdot(2^{m-m_1}-1)$ implying that C_α is an algebraic integer, which is not the case. Thus, $\Lambda_2\neq 0$.

Now, we note that

$$\begin{aligned} h(\gamma_1) &\leq h(C_\alpha)+h(2^{m-m_1}-1)=\frac{\log 44}{3}+\log(2^{m-m_1}-1)\\ &<\log(2^{m-m_1+2}). \end{aligned}$$

Thus, $h(\gamma_1) < (m - m_1 + 2)\log 2 < 1.5 \times 10^{13}(1 + \log n)$ and so we take $A_1 = 4.5 \times 10^{13}(1 + \log n)$. As before, we can take $A_2 = \log\alpha$ and $A_3 = 3\log 2$.

It then follows from Theorem 2, after some calculations, that

$$\log|\Lambda_2| > -1.6 \times 10^{26}(1 + \log n)^2.$$

From this and (12), we obtain that

$$(n - n_1)\log\alpha < 1.7 \times 10^{26}(1 + \log n)^2.$$

Thus, in both Case 1 and Case 2, we have

$$\begin{aligned} \min\{(n-n_1)\log\alpha, (m-m_1)\log 2\} &< 1.4 \times 10^{13}(1+\log n) \\ \max\{(n-n_1)\log\alpha(m-m_1)\log 2\} &< 1.7 \times 10^{26}(1+\log n)^2. \end{aligned} \tag{13}$$

We now finally rewrite equation (6) as

$$\left|C_\alpha\alpha^{n-1} - C_\alpha\alpha^{n_1-1} - 2^m + 2^{m_1}\right| = \left|(C_\alpha\alpha^{n-1} - T_n) + (T_{n_1} - C_\alpha\alpha^{n_1-1})\right| < 1.$$

We divide both sides above by $2^m - 2^{m_1}$ getting

$$\begin{aligned} \left|\frac{C_\alpha(\alpha^{n-n_1}-1)}{2^{m-m_1}-1}\alpha^{n_1-1}2^{-m_1} - 1\right| &< \frac{1}{2^m - 2^{m_1}} \le \frac{2}{2^m} \\ &< \frac{\alpha^2}{\alpha^{n-4}} = \alpha^{6-n}, \end{aligned} \tag{14}$$

because $2 < \alpha^2$ and $\alpha^{n-4} < 2^m$. To find a lower-bound on the left–hand side above, we use again Theorem 2 with the data $t = 3$ and

$$\gamma_1 = \frac{C_\alpha(\alpha^{n-n_1}-1)}{2^{m-m_1}-1},\ \gamma_2 = \alpha,\ \gamma_3 = 2,\ b_1 = 1,\ b_2 = n_1 - 1,\ b_3 = -m_1.$$

We also take $B = n$ and we have $\mathbb{K} = \mathbb{Q}(\alpha)$ with $D = 3$. By using properties of the logarithmic height function, we have

$$\begin{aligned} 3h(\gamma_1) &\le 3\left(h(C_\alpha) + h(\alpha^{n-n_1}-1) + h(2^{m-m_1}-1)\right) \\ &< \log 352 + (n-n_1)\log\alpha + 3(m-m_1)\log 2 \\ &< 5.2 \times 10^{26}(1+\log n)^2, \end{aligned}$$

where in the above chain of inequalities we used the argument from (11) as well as the bounds (13). So, we can take $A_1 = 5.2 \times 10^{26}(1 + \log n)^2$, and certainly $A_2 = \log\alpha$ and $A_3 = 3\log 2$. We need to show that if we put

$$\Lambda_3 = \frac{C_\alpha(\alpha^{n-n_1}-1)}{2^{m-m_1}-1}\alpha^{n_1-1}2^{-m_1} - 1,$$

then $\Lambda_3 \neq 0$. But $\Lambda_3 = 0$ leads to

$$C_\alpha(\alpha^{n-n_1}-1)\alpha^{n_1-1} = 2^m - 2^{m_1},$$

which upon conjugation by the automorphism $(\alpha\beta)$ and taking absolute value leads to a contradiction as before. Thus, $\Lambda_3 \neq 0$. Theorem 2 gives

$$\begin{aligned} &\log|\Lambda_3| \\ &> -1.4 \times 30^6 \times 3^{4.5} \times 3^2(1+\log 3)(1+\log n)(5.2\times 10^{26}(1+\log n)^2)(\log\alpha)(3\log 2), \end{aligned}$$

which together with (14) gives

$$(n-6)\log\alpha < 1.8\times 10^{39}(1+\log n)^3.$$

Thus, $n < 4\times 10^{45}$.

We now need to reduce the above bound for n and to do so several times we make use of Lemma 3. To begin with, we return to (9) and put

$$z = (n-1)\log\alpha - m\log 2 + \log C_\alpha.$$

For technical reason we assume that $\min\{n-n_1, m-m_1\} \geq 20$. In the case that this condition fails we consider one of the following inequalities instead:

(i) if $n-n_1 < 20$ but $m-m_1 \geq 20$, we consider (10);
(ii) if $n-n_1 \geq 20$ but $m-m_1 < 20$, we consider (12);
(iii) if both $n-n_1 < 20$ and $m-m_1 < 20$, we consider (14).

Let us start by considering (9). Note that $z \neq 0$; thus, we distinguish the following cases. If $z > 0$, then $e^z - 1 > 0$, so from (9) we obtain

$$0 < z < e^z - 1 < \max\{\alpha^{n_1-n+6}, 2^{m_1-m+1}\}.$$

Suppose now that $z < 0$. Since $|\Lambda| = |e^z - 1| < 1/2$, we get that $e^{|z|} < 2$. Therefore

$$0 < |z| \leq e^{|z|} - 1 = e^{|z|}|e^z - 1| < 2\max\{\alpha^{n_1-n+6}, 2^{m_1-m+1}\}.$$

In any case, we have that the inequality

$$0 < |z| < 2\max\{\alpha^{n_1-n+6}, 2^{m_1-m+1}\} \tag{15}$$

always holds. Replacing z in the above inequality by its formula and dividing through by $\log 2$, we conclude that

$$0 < \left|(n-1)\left(\frac{\log\alpha}{\log 2}\right) - m + \frac{\log C_\alpha}{\log 2}\right| < \max\{190\cdot\alpha^{-(n-n_1)}, 6\cdot 2^{-(m-m_1)}\}.$$

We apply Lemma 3 with

$$\tau = \frac{\log\alpha}{\log 2},\quad \mu = \frac{\log C_\alpha}{\log 2},\quad (A,B) = (190,\alpha),\quad \text{or}\quad (6,2).$$

We let $\tau = [a_0, a_1, a_2, \ldots] = [0, 1, 7, 3, 1, 1, 1, 4, \ldots]$ be the continued fraction of τ. We take $M = 4\times 10^{45}$ which is an upper bound on n. We find that the convergent $p/q = p_{86}/q_{86}$ is such that $q > 6M$. By using this we have $\varepsilon > 0.43$, therefore either

$$n - n_1 < \frac{\log(190q/0.43)}{\log\alpha} < 186,\quad \text{or}\quad m - m_1 < \frac{\log(6q/0.43)}{\log 2} < 159.$$

Thus, we have that either $n - n_1 \leq 190$, or $m - m_1 \leq 160$.

First, let us assume that $n - n_1 \leq 190$. In this case we consider inequality (10) and assume that $m - m_1 \geq 20$. We put

$$z_1 = (n_1-1)\log\alpha - m\log 2 + \log\left(C_\alpha(\alpha^{n-n_1} - 1)\right).$$

By the same arguments used for proving (15), from (10) we get that

$$0 < |z_1| < \frac{4}{2^{m-m_1}},$$

and so

$$0 < \left|(n_1 - 1)\left(\frac{\log\alpha}{\log 2}\right) - m + \frac{\log(C_\alpha(\alpha^{n-n_1} - 1))}{\log 2}\right| < 6 \cdot 2^{-(m-m_1)}. \tag{16}$$

We keep the same $\tau,\ M,\ q,\ (A, B) = (6, 2)$ and put

$$\mu_k = \frac{\log(C_\alpha(\alpha^k - 1))}{\log 2}, \qquad k = 1, 2, \ldots, 190.$$

We now apply Lemma 3 to inequality (16) for the values of $k \in [1, 190]$. A computer search with *Mathematica* revealed that if $k \in [1, 190]$, then the maximum value of $\log(Aq/\epsilon_k)/\log B$ is < 170. Hence, $m - m_1 \le 170$.

Now let us assume that $m - m_1 \le 160$ and let us consider inequality (12). We write

$$z_2 = (n - 1)\log\alpha - m_1 \log 2 + \log(C_\alpha(2^{m-m_1} - 1))$$

and we assume that $n - n_1 \ge 20$. Then

$$0 < |z_2| < \frac{2\alpha^6}{\alpha^{n-n_1}}.$$

Replacing z_2 in the above inequality by its formula and dividing through by $\log 2$, we finally arrive at

$$0 < \left|(n - 1)\left(\frac{\log\alpha}{\log 2}\right) - m_1 + \frac{\log(C_\alpha(2^{m-m_1} - 1))}{\log 2}\right| < 112 \cdot \alpha^{-(n-n_1)}.$$

We apply again Lemma 3 with the same $\tau,\ q,\ M,\ (A, B) = (112, \alpha)$ and

$$\mu_k = \frac{\log(C_\alpha(2^k - 1)))}{\log 2} \quad \text{for} \quad k = 1, 2, \ldots, 160.$$

As before, a computer search with *Mathematica* revealed that if $k \in [1, 160]$, then the maximum value of $\log(Aq/\epsilon_k)/\log B$ is < 192. Hence, $n - n_1 \le 200$.

To conclude the above computations, we first got that either $n - n_1 \le 190$ or $m - m_1 \le 160$. If $n - n_1 \le 190$, then $m - m_1 \le 170$, and if $m - m_1 \le 160$, then $n - n_1 \le 200$. In conclusion, we always have that

$$n - n_1 \le 200 \quad \text{and} \quad m - m_1 \le 170.$$

Finally, we go to (14). We put

$$z_3 = (n_1 - 1)\log\alpha - m_1 \log 2 + \log\left(\frac{C_\alpha(\alpha^{n-n_1} - 1)}{2^{m-m_1} - 1}\right).$$

Since $n > 250$, from (14) we deduce that

$$0 < |z_3| < \frac{2\alpha^6}{\alpha^n}.$$

Hence,

$$0 < \left|(n_1-1)\left(\frac{\log\alpha}{\log 2}\right) - m_1 + \frac{\log\left(C_\alpha(\alpha^k-1)/(2^\ell-1)\right)}{\log 2}\right| < 112\cdot\alpha^{-n},$$

where $(k,\ell) := (n-n_1, m-m_1)$. We apply again Lemma 3 with the same τ, M, q, $(A,B)=(112,\alpha)$ and

$$\mu_{k,\ell} = \frac{\log\left(C_\alpha(\alpha^k-1)/(2^\ell-1)\right)}{\log 2} \quad \text{for} \quad 1\le k\le 200,\ 1\le l\le 170.$$

With the help of Mathematica we find that if $k\in[1,200]$ and $\ell\in[1,170]$, then the maximum value of $\log(112q/\epsilon)/\log\alpha$ is < 205. Thus, $n < 205$, which contradicts our assumption that $n > 250$. Theorem 1 is therefore proved.

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