

HOMOGENEOUS MULTILINEAR FUNCTIONS ON HYPERGRAPH CLIQUES

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ABSTRACT. Motzkin and Straus established a close connection between the maximum clique problem and a solution (namely graph-Lagrangian) to the maximum value of a class of homogeneous quadratic multilinear functions over the standard simplex of the Euclidean space in 1965. This connection and its extensions were successfully employed in optimization to provide heuristics for the maximum clique problem in graphs. It is useful in practice if similar results hold for hypergraphs. In this paper, we develop a homogeneous multilinear function based on the structure of hypergraphs and their complement hypergraphs. Its maximum value generalizes the graph-Lagrangian. Specifically, we establish a connection between the clique number and the generalized graph-Lagrangian of 3-uniform graphs, which supports the conjecture posed in this paper.

1. Introduction

In 1941, Turán [24] provided an answer to the following question: What is the maximum number of edges in a graph with n vertices not containing a complete subgraph of order k , for a given k ? This is the well-known Turán theorem. Later, in another classical paper, Motzkin and Straus [12] provided a new proof of Turán theorem based on the continuous characterization of the clique number of a graph using graph-Lagrangians of graphs.

The Motzkin-Straus result basically says that the graph-Lagrangian of a graph which is the maximum of a homogeneous quadratic multilinear function (determined by the graph) over the standard simplex of the Euclidean plane is connected to the clique number of this graph (the precise statement is given in Theorem 2.2). This result provides a solution to the optimization problem for a class of homogeneous quadratic multilinear functions over the standard simplex of an Euclidean plane. The Motzkin-Straus result and its extension

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were successfully employed in optimization to provide heuristics for the maximum clique problem [1, 2, 6, 9]. It has been also generalized to vertex-weighted graphs [9] and edge-weighted graphs with applications to pattern recognition in image analysis [1, 2, 5, 6, 9, 14, 15]. In addition, the graph-Lagrangian of a hypergraph has been a useful tool in hypergraph extremal problems. For example, Sidorenko [19] and Frankl-Füredi [7] applied graph-Lagrangians of hypergraphs in finding Turán densities of hypergraphs. Frankl and Rödl [8] applied it in disproving Erdős long standing jumping constant conjecture. More applications of graph-Lagrangians can be found in [4], [11], and [20].

An attempt to generalize the Motzkin-Straus theorem to hypergraphs is due to Sós and Straus [20]. Recently, in [3, 4] Rota Bulò and Pelillo generalized the Motzkin and Straus' result to r -graphs in some way using a continuous characterization of maximal cliques other than graph-Lagrangians of hypergraphs. The obvious generalization of Motzkin and Straus' result to hypergraphs is false. In fact, there are many examples of hypergraphs that do not achieve their graph-Lagrangian on any proper subhypergraph.

In this paper, we develop a type of homogeneous multilinear function based on the structure of hypergraphs and their complementary hypergraphs. Its maximum value (called generalized graph-Lagrangian) generalizes the graph-Lagrangian. The main results provide solutions to the polynomial programming over the standard simplex of the Euclidean space. Specifically, we establish a connection between the clique number and the generalized graph-Lagrangians for 3-uniform graphs. The results presented in this paper also provide substantial evidence for five conjectures posed in this paper and extend some known results in the literature [11, 18]. If the proposed five Conjectures in this paper hold, then they can be used to provide heuristics for the maximum clique problem.

The rest of the paper is organized as follows. In Section 2, we state a few definitions, related problems, and some conjectures. In Section 3, we describe some preliminary results. Our main results are given in Section 4. Concluding remarks are given in Section 5.

2. Definitions and problems

A hypergraph $H = (V, E)$ consists of a vertex set V and an edge set E , where every edge in E is a subset of V . If necessary, we use $E(H)$ and $V(H)$ to denote the edge set and vertex set of H respectively. If all edges have the same cardinality, then H is a uniform hypergraph, otherwise H is a non-uniform hypergraph. If all the edges have cardinality r , then H is an r -uniform hypergraph (or an r -graph). A 2-uniform graph is a graph. An edge $\{i_1, i_2, \dots, i_r\}$ in a hypergraph is simply written as $i_1 i_2 \cdots i_r$ throughout the paper.

For a positive integer n , let $[n]$ denote the set $\{1, 2, \dots, n\}$. For a finite set V and a positive integer i , let $\binom{V}{i}$ denote the family of all i -subsets of V . The

complete hypergraph K_t^r is a hypergraph on t vertices with edge set $\binom{[t]}{r}$. A complete r -graph on t vertices is also called a clique with order t . A clique is said to be maximum if it has maximum cardinality. The clique number of an r -graph H is defined as the cardinality of the maximum clique of H .

For an r -graph $H = (V, E)$ and $i \in V$, let $E_i := \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. Let \overline{H} or H^c denote the complement hypergraph of H . For a pair of vertices $i, j \in V$, let $E_{ij} := \{B \in V^{(r-2)} : B \cup \{i, j\} \in E\}$. Let $E_i^c := \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \setminus E\}$, $E_{ij}^c := \{B \in V^{(r-2)} : B \cup \{i, j\} \in V^{(r)} \setminus E\}$, and $E_{i \setminus j} := E_i \cap E_j^c$.

Definition. For an r -uniform graph H with the vertex set $[n]$, edge set $E(H)$, and a vector $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we associate a homogeneous polynomial in n variables, denoted by $\lambda(H, \vec{x})$ as follows:

$$\lambda(H, \vec{x}) := \sum_{i_1 i_2 \dots i_r \in E(H)} x_{i_1} x_{i_2} \dots x_{i_r}.$$

Let $S := \{\vec{x} = (x_1, x_2, \dots, x_n) : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \dots, n\}$. Let $\lambda(H)$ represent the maximum of the above homogeneous multilinear polynomial of degree r over the standard simplex S . Precisely,

$$\lambda(H) := \max\{\lambda(H, \vec{x}) : \vec{x} \in S\}.$$

The value x_i is called the *weight* of the vertex i . A vector $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ is called a feasible weighting for H if $\vec{x} \in S$. A vector $\vec{y} \in S$ is called an *optimal weighting* for H if $\lambda(H, \vec{y}) = \lambda(H)$.

Remark 2.1. Since $\lambda(H)$ is the maximum of a polynomial function in n variables x_1, x_2, \dots, x_n under the constraint $\sum_{i=1}^n x_i = 1$ and the theory of Lagrange multipliers is often used in evaluating $\lambda(H)$, $\lambda(H)$ was called the Lagrangian of H in several papers [7, 8, 11, 13]. In order to emphasize $\lambda(H)$ is a concept related to graph theory, we call it the graph-Lagrangian of H following the suggestion of Franco Giannessi throughout this paper.

In [12], Motzkin and Straus provided the following simple expression for the graph-Lagrangian of a 2-graph.

Theorem 2.2 ([12]). *If H is a 2-graph with n vertices in which a maximum clique has order t then $\lambda(H) = \lambda(K_t^2) = \frac{1}{2}(1 - \frac{1}{t})$. Furthermore, the vector $\vec{x} = (x_1, x_2, \dots, x_n)$ given by $x_i := \frac{1}{t}$ if i is a vertex in a fixed maximum clique and $x_i = 0$ otherwise is an optimal weighting.*

Theorem 2.2 provides solutions to the optimization problem of these types of homogeneous quadratic functions over the standard simplex of an Euclidean plane.

In this paper, we consider a more general question.

Problem 1. Let $\beta \geq 0$ be a constant. For an r -graph H with and a vector $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, define

$$L_\beta(H, \vec{x}) := \lambda(H, \vec{x}) - \beta \lambda(\overline{H}, \vec{x}).$$

Let $S = \{\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \dots, n\}$. Let $L_\beta(H)$ represent the maximum of the above homogeneous multilinear polynomial of degree r over the standard simplex S . Precisely,

$$(1) \quad L_\beta(H) := \max\{L_\beta(H, \vec{x}) : \vec{x} \in S\}.$$

We call $L_\beta(H)$ the generalized graph-Lagrangian of H_n^T . The value x_i is called the weight of the vertex i . A vector $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is called a feasible weighting for H if and only if $\vec{x} \in S$. A vector $\vec{y} \in S$ is called an optimal weighting for H to optimization problem (1) if and only if $L_\beta(H, \vec{y}) = L_\beta(H)$.

Fact 2.3. Let H_1, H_2 be hypergraphs and $H_1 \subseteq H_2$. Then $L_\beta(H_1) \leq L_\beta(H_2)$ for $\beta \geq 0$.

Fact 2.4. Let $0 \leq \beta_2 < \beta_1$. Let H be a hypergraph, then $L_{\beta_1}(H) \leq L_{\beta_2}(H) \leq \lambda'(H)$ for $\beta \geq 0$.

Note that, for an r -uniform hypergraph H if $\beta = 0$, then $L_\beta(H) = \lambda(H)$.

For distinct $A, B \in \mathbb{N}^{(r)}$ we say that A is less than B in the *colex ordering* if $\max(A \triangle B) \in B$, where $A \triangle B := (A \setminus B) \cup (B \setminus A)$. For example we have $246 < 156$ in $\mathbb{N}^{(3)}$ since $\max(\{2, 4, 6\} \triangle \{1, 5, 6\}) \in \{1, 5, 6\}$. In colex ordering, $123 < 124 < 134 < 234 < 125 < 135 < 235 < 145 < 245 < 345 < 126 < 136 < 236 < 146 < 246 < 346 < 156 < 256 < 356 < 456 < 127 < \dots$. Note that the first $\binom{t}{r}$ r -tuples in the colex ordering of $\mathbb{N}^{(r)}$ are the edges of $[t]^{(r)}$.

Let $C_{r,m}$ denote the r -graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$.

Conjecture 2.5. For any r -graph H with m edges, we have $L_\beta(H) \leq L_\beta(C_{r,m})$ for $\beta \geq 0$.

Remark 2.6. Note that, if we set $\beta = 0$, Conjecture 2.5 is the conjecture of Frankl and Füredi given in [7] and [11] as follows.

Conjecture 2.7 ([7]). The r -graph H with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest graph-Lagrangian of all r -graphs with m edges. That is, $\lambda(H) \leq \lambda(C_{r,m})$.

Conjecture 2.7 is true when $r = 2$ by Theorem 2.2. It is obvious that both Theorem 2.2 holds in terms of $L_\beta(H)$ when H is a 2-graph. For general r -graph Talbot in [11] proved the following lemma.

Lemma 2.8 ([11]). For any integers m, t , and r satisfying $\binom{t}{r} \leq m \leq \binom{t}{r} + \binom{t-1}{r-1}$, we have $\lambda(C_{r,m}) = \lambda([t]^{(r)}) = \frac{(t-1) \cdots (t-r+1)}{t^{r-1}}$.

We may also compute $L_\beta(C_{r,m})$ for various values $\beta \geq 0$. As an example, let us consider $L_\beta(C_{r,m})$ for any integers m, t , and r satisfying $\binom{t}{r} \leq m \leq \binom{t}{r} + \binom{t-1}{r-1}$.

Claim 2.9. Let m, t , and r be positive integers satisfying $\binom{t}{r} \leq m \leq \binom{t}{r} + \binom{t-1}{r-1}$. Then $L_\beta(C_{r,m}) = L_\beta([t]^{(r)}) = \lambda([t]^{(r)})$ for $\beta \geq 0$.

Proof. By Lemma 2.8 and Fact 2.4, we have $L_\beta(C_{r,m}) \leq \lambda([t]^{(r)})$. On the other side, since $[t]^{(r)} \subseteq C_{r,m}$. Clearly, $L_\beta([t]^{(r)}) = \lambda([t]^{(r)})$. Hence $L_\beta(C_{r,m}) \geq L_\beta([t]^{(r)}) = \lambda([t]^{(r)})$ by Fact 2.3. Thus $L_\beta(C_{r,m}) = \lambda([t]^{(r)})$ for $\binom{t}{r} \leq m \leq \binom{t}{r} + \binom{t-1}{r-1}$ and $\beta \geq 0$. \square

In order to explore the relationship between the generalized graph-Lagrangian of a hypergraph and the order of its maximum cliques for hypergraphs when the number of edges is in certain ranges, we propose the following conjectures.

Conjecture 2.10. *Let m and t be positive integers satisfying $\binom{t}{r} \leq m \leq \binom{t}{r} + \binom{t-1}{r-1}$. Let H be an r -graph with m edges and contain a clique of order t . Then $L_\beta(H) = L_\beta([t]^{(r)}) = \lambda([t]^{(r)})$ for $\beta \geq 0$.*

Conjecture 2.11. *Let m and t be positive integers satisfying $\binom{t}{r} \leq m \leq \binom{t}{r} + \binom{t-1}{r-1}$. Let H be an r -graph with m edges and contain no clique of order t . Then $L_\beta(H) < L_\beta([t]^{(r)}) = \lambda([t]^{(r)})$ for $\beta \geq 0$.*

Conjecture 2.12. *Let H be an r -graph not containing a clique of order t . Then there exists $\beta_0 \geq 0$ such that $L_\beta(H) < L_\beta([t]^{(r)}) = \lambda([t]^{(r)})$ for $\beta \geq \beta_0$.*

Conjecture 2.13. *Let H be an r -graph containing a maximum clique of order t . Then there exists $\beta_0 \geq 0$ such that $L_\beta(H) = L_\beta([t]^{(r)}) = \lambda([t]^{(r)})$ for $\beta \geq \beta_0$.*

Remark 2.14. Let us give an intuitive explanation for Conjectures 2.12 and 2.13. If the maximum clique of H is less than t .

Note that, if Conjectures 2.11, 2.12, and 2.13 hold, then they can be used to provide heuristics for the maximum clique problem.

Remark 2.15. In [18], both Conjectures 2.10 and 2.11 were posed in the case when $\beta = 0$. For an r -graph H containing a clique of order t , since $\lambda([t]^{(r)}) \leq L_\beta(H) \leq \lambda(H)$, it is easy to see that Conjectures 2.10 and 2.11 in the case when $\beta = 0$ implies Conjectures 2.10 and 2.11 for all other cases when $\beta > 0$, respectively. Also, based on Facts 2.3, 2.4, and Claim 2.9, if both Conjectures 2.10 and 2.11 hold in the case when $\beta = 0$, then Conjecture 2.5 is true for any r -graph H with m edges satisfying $\binom{t}{r} \leq m \leq \binom{t}{r} + \binom{t-1}{r-1}$ for all cases when $\beta \geq 0$.

Thus, Conjecture 2.10 holds for $r = 3$ by Remark 2.15 and Theorem 2.16.

Theorem 2.16 ([18]). *Let m and t be positive integers satisfying $\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t-1}{2}$. Let H be a 3-graph with m edges and contain a clique of order t . Then $\lambda(H) = \lambda([t]^{(3)})$.*

The following theorem is proved in [17].

Theorem 2.17 ([17]). *Let m and t be positive integers satisfying $\binom{t}{r} \leq m \leq \binom{t}{r} + \binom{t-1}{r-1} - (2^{r-3} - 1)(\binom{t-1}{r-2} - 1)$. Let H be an r -graph on vertex set $[t+1]$ with m edges and contain a clique of order t . Then $\lambda(H) = \lambda([t]^{(r)})$.*

Hence Remark 2.15 implies that we have $L_\beta(H) = L_\beta([t]^{(r)})$ for all $\beta \geq 0$ under the condition of Theorem 2.17.

Note that the upper bound $\binom{t}{r} + \binom{t-1}{r-1}$ in Conjecture 2.10 is the best possible, since $L_\beta(C_{r,m}) = \lambda([t]^{(r)})$ does not always hold for $\binom{t}{r} + \binom{t-1}{r-1} < m \leq \binom{t+1}{r} - 1$. For example, if $m = \binom{t}{r} + \binom{t-1}{r-1} + 1$, then $L_\beta(C_{r,m}) > \lambda([t]^{(r)})$ for $\beta < \frac{\binom{t-1}{r-1} - 4\binom{t-2}{r-2} - 4\binom{t-2}{r-2} + 1}{\binom{t-2}{r-2} - 1}$. To see this, take $\vec{x} = (x_1, \dots, x_{t+1}) \in S$, where $x_1 = x_2 = \dots = x_{t-1} = \frac{1}{t}$ and $x_t = x_{t+1} = \frac{1}{2t}$, then $L_\beta(C_{r,m}) \geq L_\beta(C_{r,m}, \vec{x}) > \lambda([t]^{(r)})$ for $\beta < \frac{\binom{t-1}{r-1} - 4\binom{t-2}{r-2} - 4\binom{t-2}{r-2} + 1}{\binom{t-2}{r-2} - 1}$.

The main goal of this paper is to explore Conjecture 2.5 for different r , β and certain ranges of m . We also explore the connection between the clique number and the generalized graph-Lagrangian. Substantial evidence is obtained for Conjectures 2.10-2.13 as well.

3. Some preliminaries

We will impose an additional condition on any optimal weighting $\vec{x} = (x_1, x_2, \dots, x_n)$ for a hypergraph H to Problem 1:

$|\{i : x_i > 0\}|$ is minimal, i.e., if \vec{y} is a feasible weighting for H satisfying $|\{i : y_i > 0\}| < |\{i : x_i > 0\}|$, then $L_\beta(H, \vec{y}) < L_\beta(H, \vec{x})$.

Lemma 3.1. *Let $H = (V, E)$ be an r -graph and $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal feasible weighting for H with k ($\leq n$) non-zero weights x_1, x_2, \dots, x_k . Then for every $\{i, j\} \in [k]^{(2)}$, (a) $\frac{\partial L_\beta(H, \vec{x})}{\partial x_i} = \frac{\partial L_\beta(H, \vec{x})}{\partial x_j}$, (b) there is an edge in E containing both i and j .*

Proof. (a) Suppose, for a contradiction, that there exist $\{i, j\} \in [k]^{(2)}$ such that $\frac{\partial L_\beta(H, \vec{x})}{\partial x_i} > \frac{\partial L_\beta(H, \vec{x})}{\partial x_j}$. We define a new feasible weighting \vec{y} as follows. Let $0 < \delta \leq x_j$ and $y_l = x_l$ for $l \neq i, j$, $y_i = x_i + \delta$ and $y_j = x_j - \delta$, then \vec{y} is clearly a feasible weighting for H , and

$$L_\beta(H, \vec{y}) - L_\beta(H, \vec{x}) = \delta \left(\frac{\partial L_\beta(H, \vec{x})}{\partial x_i} - \frac{\partial L_\beta(H, \vec{x})}{\partial x_j} \right) - \delta^2 \frac{\partial^2 L_\beta(H_n^T, \vec{x})}{\partial x_i \partial x_j}.$$

For sufficiently small δ this is strictly positive, contradicting to $\frac{\partial L_\beta(H, \vec{x})}{\partial x_i} > \frac{\partial L_\beta(H, \vec{x})}{\partial x_j}$. Hence Lemma 3.1(a) holds.

(b) Suppose, for a contradiction, that there exist $\{i, j\} \in [k]^{(2)}$ such that for any $e \in E(H)$, $ij \not\subseteq e$. We define a new feasible weighting \vec{y} for H_n^T as follows. Let $y_l = x_l$ for $l \neq i, j$, $y_i = x_i + x_j$ and $y_j = x_j - x_j = 0$, then \vec{y} is clearly a feasible weighting for H_n^T . Note that $\frac{\partial L_\beta(H, \vec{x})}{\partial x_i} = \frac{\partial L_\beta(H, \vec{x})}{\partial x_j}$ by (a). And $\frac{\partial^2 L_\beta(H, \vec{x})}{\partial x_i \partial x_j} = 0$ since $ij \not\subseteq e$ for any $e \in E(H)$. Hence

$$L_\beta(H, \vec{y}) - L_\beta(H, \vec{x}) = x_j \left(\frac{\partial L_\beta(H, \vec{x})}{\partial x_i} - \frac{\partial L_\beta(H_n^T, \vec{x})}{\partial x_j} \right) - x_j^2 \frac{\partial^2 L_\beta(H_n^T, \vec{x})}{\partial x_i \partial x_j} = 0.$$

So \vec{y} is an optimal weighting for H to Problem (1) and $|\{i : y_i > 0\}| = k - 1$, contradicting the minimality of k . Hence Lemma 3.1(b) holds. \square

For an r -graph $H = (V, E)$, when the theory of Lagrange multipliers is applied to find the optimum of $L_\beta(H)$, subject to $\sum_{i=1}^n x_i = 1$, note that $L_\beta(E_i, \vec{x})$ corresponds to the partial derivative of $L_\beta(H, \vec{x})$ with respect to x_i . The following lemma gives some necessary conditions of an optimal weighting of $L_\beta(H)$.

Lemma 3.2. *Let $H = (V, E)$ be an r -graph on the vertex set $[n]$ and $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for H to Problem 1 with k ($\leq n$) non-zero weights x_1, x_2, \dots, x_k . Then for every $\{i, j\} \in [k]^{(2)}$, (a) $L_\beta(E_i, \vec{x}) = L_\beta(E_j, \vec{x}) = rL_\beta(H)$, (b) there is an edge in E containing both i and j .*

Proof. (a) By Lemma 3.1(a) we have $L_\beta(E_i, \vec{x}) = L_\beta(E_j, \vec{x})$ for every $\{i, j\} \in [k]^{(2)}$. Note that

$$x_1 L_\beta(E_1, \vec{x}) + \dots + x_k L_\beta(E_k, \vec{x}) = rL_\beta(H)$$

since each $x_{i_1} x_{i_2} \dots x_{i_r}$ ($i_1 i_2 \dots i_r \in H$) and $-\beta x_{j_1} x_{j_2} \dots x_{j_r}$ ($j_1 j_2 \dots j_r \notin H$) appears r times respectively in $x_1 L_\beta(E_1, \vec{x}) + \dots + x_k L_\beta(E_k, \vec{x})$. Hence $L_\beta(E_i, \vec{x}) = rL_\beta(H)$ for $1 \leq i \leq k$.

(b) The result is directly from Lemma 3.1(b). \square

In [11], Talbot introduced the definition of a left-compressed r -uniform hypergraph. It can be generalize to non-uniform hypergraphs.

Definition. An r -graph $H = ([n], E)$ is *left-compressed* if and only if $j_1 j_2 \dots j_r \in E$ implies $i_1 i_2 \dots i_r \in E$ provided $i_p \leq j_p$ for every p , $1 \leq p \leq r$.

An r -tuple $i_1 i_2 \dots i_r$ is called a *descendant* of an r -tuple $j_1 j_2 \dots j_r$ if $i_s \leq j_s$ for each $1 \leq s \leq r$, and $i_1 + i_2 + \dots + i_r < j_1 + j_2 + \dots + j_r$. In this case, the r -tuple $j_1 j_2 \dots j_r$ is called an *ancestor* of $i_1 i_2 \dots i_r$. The r -tuple $i_1 i_2 \dots i_r$ is called a *direct descendant* of $j_1 j_2 \dots j_r$ if $i_1 i_2 \dots i_r$ is a descendant of $j_1 j_2 \dots j_r$ and $j_1 + j_2 + \dots + j_r = i_1 + i_2 + \dots + i_r + 1$. We say that $j_1 j_2 \dots j_r$ has lower hierarchy than $i_1 i_2 \dots i_r$ if $j_1 j_2 \dots j_r$ is an ancestor of $i_1 i_2 \dots i_r$. This is a partial order on the set of all r -tuples.

Remark 3.3. An r -graph H is left-compressed if all descendants of an edge of H are edges of H . Equivalently, if an r -tuple is not an edge of H , then none of its ancestors will be an edge of H .

Definition (Equivalent Definition of Left-compressed). Let $H = ([n], E)$ be an r -graph. For $e \in E$, and $i, j \in [n]$ with $i < j$, define

$$C_{ij}(e) = \begin{cases} (e \setminus \{j\}) \cup \{i\} & \text{if } i \notin e \text{ and } j \in e, \\ e & \text{otherwise.} \end{cases}$$

and

$$(2) \quad C_{ij}(E) = \{C_{ij}(e) : e \in E\} \cup \{e : e \in E, C_{ij}(e) \in E\}.$$

We say that $E(H)$ is *left-compressed* if $\mathcal{C}_{ij}(E) = E$ for every $1 \leq i < j$.

Note that $|\mathcal{C}_{ij}(E)| = |E|$ from the definition of $\mathcal{C}_{ij}(E)$. We have the following lemma.

Lemma 3.4. *Let $H = ([n], E)$ be an r -graph, $i, j \in [n]$ with $i < j$ and $\vec{x} = (x_1, \dots, x_n)$ be an optimal weighting for H to Problem (1). Write $H_{ij} = ([n], \mathcal{C}_{ij}(E))$. Then,*

$$L_\beta(H, \vec{x}) \leq L_\beta(H_{ij}, \vec{x}).$$

Proof. We can assume that $x_i \geq x_j$ when $i < j$ since otherwise we can just relabel the vertices of H and obtain an optimal weighting $\vec{x} = (x_1, x_2, \dots, x_n)$ satisfying $x_i \geq x_j$ when $i < j$. Note that

$$L_\beta(H_{ij}, \vec{x}) - L_\beta(H, \vec{x}) = \sum_{\substack{e \in E(H), \mathcal{C}_{ij}(e) \notin E(H) \\ i \notin e, j \in e}} (1 + \beta) L_\beta(e \setminus \{j\}, \vec{x}) (x_i - x_j).$$

Hence $L_\beta(H_{ij}, \vec{x}) - L_\beta(H, \vec{x})$ is nonnegative in any case, since $i < j$ implies that $x_i \geq x_j$. So this lemma holds. \square

Remark 3.5. Let $H = (V, E)$ be an r -graph.

(a) In Lemma 3.2, part(a) implies that

$$\lambda(E_i, \vec{x}) - \beta \lambda(E_i^c, \vec{x}) = \lambda(E_j, \vec{x}) - \beta \lambda(E_j^c, \vec{x}) = r L_\beta(H),$$

i.e.,

$$\begin{aligned} & \lambda(E_{i \setminus j}, \vec{x}) + x_j \lambda(E_{ij}, \vec{x}) - \beta \lambda(E_{i \setminus j}^c, \vec{x}) - \beta x_j \lambda(E_{ij}^c, \vec{x}) \\ &= \lambda(E_{j \setminus i}, \vec{x}) + x_i \lambda(E_{ij}, \vec{x}) - \beta \lambda(E_{j \setminus i}^c, \vec{x}) - \beta x_i \lambda(E_{ij}^c, \vec{x}) \\ &= r L_\beta(H). \end{aligned}$$

In particular, if H is left-compressed, then

$$\begin{aligned} & \lambda(E_{i \setminus j}, \vec{x}) + x_j \lambda(E_{ij}, \vec{x}) - \beta x_j \lambda(E_{ij}^c, \vec{x}) \\ &= x_i \lambda(E_{ij}, \vec{x}) - \beta \lambda(E_{i \setminus j}, \vec{x}) - \beta x_i \lambda(E_{ij}^c, \vec{x}) \\ &= r L_\beta(H) \end{aligned}$$

for any i, j satisfying $1 \leq i < j \leq k$ since $E_{j \setminus i} = \emptyset$, $E_{i \setminus j}^c = \emptyset$ and $E_{j \setminus i}^c = E_{i \setminus j}$. Note that $x_i \lambda(E_{ij}, \vec{x}) - \beta \lambda(E_{j \setminus i}^c, \vec{x}) - \beta x_i \lambda(E_{ij}^c, \vec{x}) = r L_\beta(H) > 0$, we have $x_i \lambda(E_{ij}, \vec{x}) - \beta x_i \lambda(E_{ij}^c, \vec{x}) \geq r L_\beta(H) > 0$, i.e., $\lambda(E_{ij}, \vec{x}) > \beta \lambda(E_{ij}^c, \vec{x})$.

(b) If H is left-compressed, then for any i, j satisfying $1 \leq i < j \leq k$,

$$(3) \quad x_i - x_j = \frac{(1 + \beta) \lambda(E_{i \setminus j}, \vec{x})}{\lambda(E_{ij}, \vec{x}) - \beta \lambda(E_{ij}^c, \vec{x})}$$

holds. If H is left-compressed and $E_{i \setminus j} = \emptyset$ for i, j satisfying $1 \leq i < j \leq k$, then $x_i = x_j$.

(c) By (3), if H is left-compressed, then an optimal weighting $\vec{x} = (x_1, x_2, \dots, x_n)$ for H to Problem (1) must satisfy

$$(4) \quad x_1 \geq x_2 \geq \dots \geq x_n \geq 0.$$

In the rest of the paper an optimal weighting for H refers to an optimal weighting for H to Problem (1) unless specifically stated.

4. Results for 3-uniform hypergraphs

In this section, for 3-graphs, we prove several substantial supporting results for Conjectures 2.5 and Conjectures 2.10-2.13.

Theorem 4.1. *Let m and t be integers satisfying $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$. Let $\beta \geq t - 1$ be a constant. Then Conjecture 2.5 is true for $r = 3$ and this value of m .*

Theorem 4.2. *Let m and t be positive integers satisfying $m \leq \binom{t+1}{3} - 1$. Let $\beta \geq 27\binom{t}{2} - 1$ be a constant. Let H be a 3-graph on vertex set $[t + 1]$ with m edges.*

(a) *If H contains a clique of order t , then $L_\beta(H) = L_\beta([t]^{(3)})$. Furthermore, the vector $\vec{x} = (x_1, x_2, \dots, x_n)$, given by $x_i := \frac{1}{t}$ if i is a vertex in a fixed maximum clique and $x_i = 0$ otherwise, is an optimal weighting.*

(b) *If H does not contain a clique of order t , then $L_\beta(H) < L_\beta([t]^{(3)})$.*

Theorem 4.3. *Let m and t be integers satisfying $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$. Let $\beta \geq 27\binom{t}{2} - 1$ be a constant. Let H be a 3-graph with m edges. If H contains a clique order of t , then $L_\beta(H) = L_\beta([t]^{(3)})$. Furthermore, the vector $\vec{x} = (x_1, x_2, \dots, x_n)$, given by $x_i := \frac{1}{t}$ if i is a vertex in a fixed maximum clique and $x_i = 0$ otherwise, is an optimal weighting.*

Theorem 4.4. *Let m and t be integers satisfying $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$. Let $\beta \geq 27\binom{t}{2} - 1$ be a constant. Let H be a 3-graph with m edges. If H does not contain a clique order of t , then $L_\beta(H) < L_\beta([t]^{(3)})$.*

Note that Theorem 4.3 and Theorem 4.4 establish a connection between the generalized graph-Lagrangians and the maximum cliques of H for certain range of β . In [23], we prove the following theorem for 3-uniform hypergraphs.

Theorem 4.5 ([23]). *Let m and t be integers satisfying $\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t-1}{2} - \frac{1}{2}t$. Let H be a 3-graph with m edges and H does not contain a clique order of t . Then $\lambda(H) < \lambda([t]^{(3)})$.*

Next, we generalize Theorem 4.5 as follows.

Theorem 4.6. *Let m and t be positive integers satisfying $\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t-1}{2} - \frac{1}{2}\lceil \frac{t-1}{1+\beta} \rceil - 1$. Let $\beta \geq 0$ be a constant. Let H be a 3-graph with m edges. If H does not contain a clique of order t , then $L_\beta(H) < L_\beta([t]^{(3)})$.*

Theorem 4.7. *Let m and t be positive integers satisfying $\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t-1}{2}$. Let $\beta \geq t - 4$ be a constant. Let H be a 3-graph with m edges. If H does not contain a clique of order t , then $L_\beta(H) < L_\beta([t]^{(3)})$.*

Combing Remark 2.15 and Theorem 4.6, we have:

Theorem 4.8. *Let $\beta \geq 0$ be a constant. Let m and t be integers satisfying $\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t-1}{2} - \frac{1}{2} \lceil \frac{t-1}{1+\beta} \rceil - 1$. Then Conjecture 2.5 is true for $r = 3$ and this value of m .*

4.1. Proof of Theorem 4.1

Proof of Theorem 4.1. Let m and t be integers satisfying $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$. Let $\beta \geq t - 1$ be a constant. Denote $L_{(\beta, m)}^{(3)} := \max\{L_\beta(H) : H \text{ is a 3-graph with } m \text{ edges}\}$. Let H be a 3-graph with m edges such that $L_\beta(H) = L_{(\beta, m)}^{(3)}$, i.e., H is an extremal graph. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for H and k be the number of non-zero weights in \vec{x} . We can assume that H is left-compressed. Otherwise if H is not left-compressed, performing a sequence of left-compressing operations (i.e., if an edge in H has descendants that are not in H , then replace this edge by a descendant not in H with the lowest hierarchy. Repeat this until all descendants of an edge of H are edges of H), we will get a left-compressed 3-graph G . By Lemma 3.4 G is also an extremal graph. Hence we can assume that H is left-compressed and $x_1 \geq x_2 \geq \dots \geq x_k > x_{k+1} = \dots = x_n = 0$. There is an edge e containing both $k-1$ and k by Lemma 3.2(b). Recalling that H is left-compressed, so we have $1(k-1)k \in E$. Let $b := \max\{i : i(k-1)k \in E\}$. We now give three lemmas to be proved later.

Lemma 4.9. $|[k-1]^{(3)} \setminus E| \leq \lceil \frac{k-2}{1+\beta} \rceil$ for $\beta \geq 0$.

Lemma 4.10. $|[k-1]^{(2)} \setminus E_{k-1}| \leq \lceil \frac{b}{1+\beta} \rceil$ for $\beta \geq 0$.

Lemma 4.11. $|[k-2]^{(2)} \setminus E_k| \leq \lceil \frac{b}{1+\beta} \rceil$ for $\beta \geq 0$.

Assume the above lemmas are true, then we may show that $k \leq t + 1$ as follows.

Claim 4.12. Let m and t be positive integers satisfying $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$. Let $\beta \geq 0$ be a constant. Let H be a 3-graph with m edges such that $L_\beta(H) = L_{(\beta, m)}^{(3)}$. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for H and k be the number of non-zero weights in \vec{x} . Then $k \leq t + 1$.

Proof of Claim 4.12. If $k \geq t + 2$, using Lemma 4.9 and Lemma 4.11, we have

$$\begin{aligned} m = |E| &= |E \cap [k-1]^{(3)}| + |[k-2]^{(2)} \cap E_k| + |E_{(k-1)k}| \\ &\geq \binom{k-1}{3} - \lceil \frac{k-2}{1+\beta} \rceil + \binom{k-2}{2} - \lceil \frac{b}{1+\beta} \rceil + b \end{aligned}$$

$$\begin{aligned}
&\geq \binom{t+1}{3} + \binom{t}{2} - \lceil \frac{t}{1+\beta} \rceil \\
&> \binom{t+1}{3} + \binom{t}{2} - \lceil \frac{t}{1+\beta} \rceil > \binom{t+1}{3}.
\end{aligned}$$

It contradicts to $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$. Hence we have $k \leq t + 1$. \square

Now we need the following theorem.

Theorem 4.13. *Let m and t be integers satisfying $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$. Let $\beta \geq 0$ be a constant. Let H be a left-compressed 3-graph on vertex set $[t+1]$ with m edges satisfying $|E(H) \Delta E(C_{3,m})| \leq 4$. Then $L_\beta(H) \leq L_\beta(C_{3,m})$.*

Now we need the following lemmas.

Lemma 4.14 ([22]). *Let m and t be positive integers satisfying $\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t-1}{2}$. Let $H = (V, E)$ be a left-compressed 3-graph on the vertex set $[t+1]$ with m edges and not containing a clique of order t . If $b = |E_{t(t+1)}| \leq 3$, then $\lambda(H) < \lambda([t]^{(3)})$.*

Lemma 4.15. *Let m , t , a and i be positive integers satisfying $m = \binom{t}{3} - a$, $3 \leq a \leq t - 2$ and $i \geq 1$. Let $H = ([t], E)$ be a left-compressed 3-graph with m edges. If the triple with minimum colex ordering in H^c is $(t - 2 - i)(t - 2)t$. Then $L_\beta(H) \leq L_\beta(C_{3,m})$ for $\beta \geq 0$.*

Lemma 4.16. *Let H and H' be left-compressed 3-graphs on vertex set $[t]$ with $m = \binom{t}{3} - a$ edges, where $5 \leq a \leq t - 2$, satisfying $|E(H) \Delta E(C_{3,m})| = |E(H') \Delta E(C_{3,m})| = 4$ and the triples with the minimum colex ordering in H^c and H'^c are $(t - 3)(t - 2)(t - 1)$ and $(t - 4)(t - 2)t$ respectively. Then $L_\beta(H) \leq L_\beta(H') \leq \lambda(C_{3,m})$ for $\beta \geq 0$.*

The proof of Lemma 4.15 and Lemma 4.16 are similar to the proof of Theorem 1.12 in [21], we give the sketch here for completeness.

Proof of Lemma 4.15. Since H is left-compressed, then we have $a \geq 2i + 1$. To show that $\lambda(H) \leq \lambda(C_{3,m})$, we will take an optimal weighting \vec{x} for H , then we take a feasible weighting, say \vec{z} for $C_{3,m}$ by replacing a few coordinators of \vec{x} and show that $\lambda(H, \vec{x}) \leq \lambda(C_{3,m}, \vec{z})$. This would imply that

$$\lambda(H) = \lambda(H, \vec{x}) \leq \lambda(C_{3,m}, \vec{z}) \leq \lambda(C_{3,m}).$$

Let us go into the details. Let $\vec{x} = (x_1, x_2, \dots, x_t)$ be an optimal weighting for H satisfying $x_1 \geq x_2 \geq \dots \geq x_t \geq 0$. First we point out that

$$\begin{aligned}
(5) \quad &\lambda(E_{1(t-2-i)}, \vec{x}) - \lambda(E_{(t-2)(t-1)}, \vec{x}) \\
&= x_{t-2} + x_{t-1} + x_t - x_1 - x_{t-2-i} \geq 0.
\end{aligned}$$

To verify (5), by Remark 3.5(b), we have

$$x_1 = x_{t-1} + \frac{\lambda(E_{1 \setminus (t-1)}, \vec{x})}{\lambda(E_{1(t-1)}, \vec{x})}$$

$$(6) \quad \leq x_{t-1} + \frac{(x_2 + \cdots + x_{t-2})x_t}{x_2 + \cdots + x_{t-2} + x_t} \leq x_{t-1} + x_t;$$

$$\begin{aligned} x_1 &= x_{t-2} + \frac{\lambda(E_{1 \setminus (t-2)}, \vec{x})}{\lambda(E_{1(t-2)}, \vec{x})} \\ &= x_{t-2} + \frac{x_{t-2-i} + \cdots + x_{t-3} + x_{t-1}}{1 - x_1 - x_{t-2}} x_t \\ &\leq x_{t-2} + \frac{x_{t-2-i} + \cdots + x_{t-3} + x_{t-1}}{1 - x_{t-2} - x_{t-1} - x_t} x_t \quad (\text{By (6)}) \\ (7) \quad &\leq x_{t-2} + \frac{x_1 + x_2 + \cdots + x_i + x_{t-2}}{1 - x_{t-2-i} - x_{t-1} - x_t} x_t; \end{aligned}$$

and

$$\begin{aligned} x_{t-2-i} &= x_{t-1} + \frac{\lambda(E_{(t-2-i) \setminus (t-1)}, \vec{x})}{\lambda(E_{(t-2-i)(t-1)}, \vec{x})} \\ &= x_{t-1} + \frac{(x_{t-3-(a-i-2)} + x_{t-3-(a-i-1)} + \cdots + x_{t-3}) - x_{t-2-i}}{1 - x_{t-2-i} - x_{t-1} - x_t} x_t \\ (8) \quad &\leq x_{t-1} + \frac{(x_{i+1} + x_{i+2} + \cdots + x_{a-1}) - x_{t-2-i}}{1 - x_{t-2-i} - x_{t-1} - x_t} x_t \end{aligned}$$

since $a \leq t-2$. Adding (7) and (8), we obtain that

$$\begin{aligned} x_1 + x_{t-2-i} &\leq x_{t-2} + x_{t-1} + \frac{(x_1 + \cdots + x_{a+1}) - x_{t-2-i}}{1 - x_{t-2-i} - x_{t-1} - x_t} x_t \\ &\leq x_{t-2} + x_{t-1} + \frac{(x_1 + \cdots + x_{t-2}) - x_{t-2-i}}{1 - x_{t-2-i} - x_{t-1} - x_t} x_t \\ &= x_{t-2} + x_{t-1} + \frac{1 - x_{t-1} - x_t - x_{t-2-i}}{1 - x_{t-2-i} - x_{t-1} - x_t} x_t \\ &= x_{t-2} + x_{t-1} + x_t. \end{aligned}$$

So, (5) is true. This implies that $\lambda(E_{(t-2)(t-1)}, \vec{x}) \leq \lambda(E_{1(t-2-i)}, \vec{x})$. In what follows, we divide the rest of the proof into three cases: $a = 2i + 1$, $a = 2i + 2$, and $a \geq 2i + 3$.

We first consider the case that $a \geq 2i + 3$. By Remark 3.5(b), we have $x_1 = x_2 = \cdots = x_{t-a-2+i}$ and $x_{t-2-i} = \cdots = x_{t-3}$. Hence $\lambda(C_{3,m}, \vec{x}) - \lambda(G, \vec{x}) = i(x_{t-2-i}x_{t-2}x_t - x_1x_{t-1}x_t)$. Also by Remark 3.5(b), we have

$$\begin{aligned} x_1 &= x_{t-2-i} + \frac{\lambda(E_{1 \setminus (t-2-i)}, \vec{x})}{\lambda(E_{1(t-2-i)}, \vec{x})} \\ &= x_{t-2-i} + \frac{(x_{t-1} + x_{t-2})x_t}{\lambda(E_{1(t-2-i)}, \vec{x})}, \end{aligned}$$

and

$$x_{t-2} = x_{t-1} + \frac{\lambda(E_{(t-2) \setminus (t-1)}, \vec{x})}{\lambda(E_{(t-2)(t-1)}, \vec{x})}$$

$$= x_{t-1} + \frac{(x_{t-3-i} + \cdots + x_{t-a-1+i})x_t}{\lambda(E_{(t-2)(t-1)}, \vec{x})}.$$

Recall that $a \geq 2i + 3$ and $\lambda(E_{(t-2)(t-1)}, \vec{x}) \leq \lambda(E_{1(t-2-i)}, \vec{x})$. We have $x_{t-2} - x_{t-1} \geq x_1 - x_{t-2-i}$. Hence

$$\begin{aligned} \lambda(C_{3,m}, \vec{x}) - \lambda(G, \vec{x}) &= i(x_{t-2-i}x_{t-2}x_t - x_1x_{t-1}x_t) \\ &= i[x_{t-2-i}(x_{t-2} + x_{t-1} - x_{t-1})x_t - x_1x_{t-1}x_t] \\ &\geq i[x_{t-2-i}(x_{t-1} + x_1 - x_{t-2-i})x_t - x_1x_{t-1}x_t] \\ &= i(x_{t-2-i} - x_{t-1})(x_1 - x_{t-2-i})x_t \\ (9) \quad &\geq 0. \end{aligned}$$

Therefore $\lambda(C_{3,m}) \geq \lambda(C_{3,m}, \vec{x}) \geq \lambda(G, \vec{x}) = \lambda(G)$ in this case.

Next, we consider the case that $a = 2i + 2$. Let

$$G' = G \bigcup \{(t-2-i)(t-2)t\} \setminus \{(t-4-i)(t-1)t\}.$$

Then $\lambda(G') \leq \lambda(C_{3,m})$ by the case $a = 2(i-1) + 4$. (Note that $G' = C_{3,m}$ when $i-1 = 0$.) So it is sufficient to prove that $\lambda(G) \leq \lambda(G')$. Clearly,

$$\begin{aligned} \lambda(G', \vec{x}) - \lambda(G, \vec{x}) &= x_{t-2-i}x_{t-2}x_t - x_{t-4-i}x_{t-1}x_t \\ (10) \quad &= x_{t-2-i}x_{t-2}x_t - x_1x_{t-1}x_t. \end{aligned}$$

Consider a new weighting $\vec{y} = (y_1, y_2, \dots, y_t)$ given by $y_j = x_j$ for $j \neq t-4-i$, $j \neq t-2-i$ and $y_{t-4-i} = x_{t-4-i} - \delta$, $y_{t-2-i} = x_{t-2-i} + \delta$. Then

$$\begin{aligned} \lambda(G', \vec{y}) - \lambda(G', \vec{x}) &= \delta[\lambda(E'_{t-2-i}, \vec{x}) - \lambda(E'_{t-4-i}, \vec{x})] - \delta^2\lambda(E'_{(t-4-i)(t-2-i)}, \vec{x}) \\ &= \delta(x_{t-4-i} - x_{t-2-i})\lambda(E'_{(t-4-i)(t-2-i)}, \vec{x}) - \delta^2\lambda(E'_{(t-4-i)(t-2-i)}, \vec{x}). \end{aligned}$$

Let $\delta = \frac{x_{t-4-i} - x_{t-2-i}}{2}$. Clearly, $\vec{y} = (y_1, y_2, \dots, y_t)$ is also a feasible weighting for G and

$$\begin{aligned} \lambda(G', \vec{y}) - \lambda(G', \vec{x}) &= \frac{(x_{t-4-i} - x_{t-2-i})^2}{4}\lambda(E'_{(t-4-i)(t-2-i)}, \vec{x}) \\ (11) \quad &= \frac{(x_1 - x_{t-2-i})^2}{4}\lambda(E_{1(t-2-i)}, \vec{x}). \end{aligned}$$

Let $\vec{z} = (z_1, z_2, \dots, z_t)$ given by $z_j = y_j$ for $j \neq t-2$, $j \neq t-1$ and $z_{t-2} = y_{t-2} + \eta$, $z_{t-1} = y_{t-1} - \eta$. Then

$$\begin{aligned} \lambda(G', \vec{z}) - \lambda(G', \vec{y}) &= \eta[\lambda(E'_{t-2}, \vec{y}) - \lambda(E'_{t-1}, \vec{y})] - \eta^2\lambda(E'_{(t-2)(t-1)}, \vec{y}) \\ &= \eta[(y_{t-2-i}y_t + y_{t-3-i}y_t + y_{t-4-i}y_t) - (y_{t-2} - y_{t-1})\lambda(E'_{(t-2)(t-1)}, \vec{y})] \\ (12) \quad &- \eta^2\lambda(E'_{(t-2)(t-1)}, \vec{y}). \end{aligned}$$

Let

$$\begin{aligned}\eta &= \frac{(y_{t-2-i} + y_{t-3-i} + y_{t-4-i})y_t - (y_{t-2} - y_{t-1})\lambda(E'_{(t-2)(t-1)}, \vec{y})}{2\lambda(E'_{(t-2)(t-1)}, \vec{y})} \\ &= \frac{(x_{t-2-i} + x_{t-3-i} + x_{t-4-i})x_t - (x_{t-2} - x_{t-1})\lambda(E_{(t-2)(t-1)}, \vec{x})}{2\lambda(E_{(t-2)(t-1)}, \vec{x})}.\end{aligned}$$

By Remark 3.5(b), we have

$$(13) \quad x_{t-2} = x_{t-1} + \frac{x_{t-3-i}x_t}{\lambda(E_{(t-2)(t-1)}, \vec{x})}.$$

Hence, $\eta = \frac{(x_{t-2-i} + x_{t-4-i})x_t}{2\lambda(E_{(t-2)(t-1)}, \vec{x})}$ and $\vec{z} = (z_1, z_2, \dots, z_t)$ is also a feasible weighting for G and

$$(14) \quad \lambda(G', \vec{z}) - \lambda(G', \vec{y}) = \frac{(x_{t-2-i} + x_{t-4-i})^2 x_t^2}{4\lambda(E_{(t-2)(t-1)}, \vec{x})}.$$

By Remark 3.5(b), we have

$$(15) \quad x_1 = x_{t-i-2} + \frac{x_{t-2}x_t + x_{t-1}x_t}{\lambda(E_{1(t-i-2)}, \vec{x})}.$$

Combing (10), (11), (14), and (15), we have

$$\begin{aligned}& \lambda(G', \vec{z}) - \lambda(G, \vec{x}) \\ &= x_{t-2-i}x_{t-2}x_t - x_1x_{t-1}x_t + \frac{(x_{t-2} + x_{t-1})^2 x_t^2}{4\lambda(E_{1(t-i-2)}, \vec{x})} + \frac{(x_{t-2-i} + x_{t-4-i})^2 x_t^2}{4\lambda(E_{(t-2)(t-1)}, \vec{x})} \\ &= [x_1 - \frac{x_{t-2}x_t + x_{t-1}x_t}{\lambda(E_{1(t-i-2)}, \vec{x})}]x_{t-2}x_t - x_1x_{t-1}x_t \\ &\quad + \frac{(x_{t-2} + x_{t-1})^2 x_t^2}{4\lambda(E_{1(t-i-2)}, \vec{x})} + \frac{(x_{t-2-i} + x_{t-4-i})^2 x_t^2}{4\lambda(E_{(t-2)(t-1)}, \vec{x})} \\ &\geq \frac{x_t^2}{4\lambda(E_{1(t-i-2)}, \vec{x})} [-4(x_{t-2} + x_{t-1})x_{t-2} + (x_{t-2} + x_{t-1})^2 + 4x_{t-2}^2] \\ &= \frac{x_t^2}{4\lambda(E_{1(t-i-2)}, \vec{x})} (x_{t-2} - x_{t-1})^2 \\ &\geq 0.\end{aligned}$$

Hence $\lambda(G') \geq \lambda(G', \vec{z}) \geq \lambda(G, \vec{x}) = \lambda(G)$ in this case.

What remains is the case that $a = 2i + 1$. Let

$$G'' = G \setminus \{(t-2-i)(t-2)t\} \setminus \{(t-3-i)(t-1)t\}.$$

Then $\lambda(G'') \leq \lambda(C_{3,m})$ by the case $a = 2(i-1) + 3$. (Note that $G' = C_{3,m}$ when $i-1 = 0$.) So it is sufficient to prove that $\lambda(G) \leq \lambda(G'')$. Clearly,

$$(16) \quad \begin{aligned}\lambda(G'', \vec{x}) - \lambda(G, \vec{x}) &= x_{t-2-i}x_{t-2}x_t - x_{t-3-i}x_{t-1}x_t \\ &= x_{t-2-i}x_{t-2}x_t - x_1x_{t-1}x_t.\end{aligned}$$

Consider a new weighting $\vec{u} = (u_1, u_2, \dots, u_t)$ given by $u_j = x_j$ for $j \neq t-2-i$, $j \neq t-3-i$ and $u_{t-2-i} = x_{t-2-i} + \alpha$, $u_{t-3-i} = x_{t-3-i} - \alpha$. Then

$$\begin{aligned} \lambda(G'', \vec{u}) - \lambda(G'', \vec{x}) &= \alpha(x_{t-3-i} - x_{t-2-i})\lambda(E''_{(t-3-i)(t-2-i)}, \vec{x}) \\ &\quad - \alpha^2\lambda(E''_{(t-3-i)(t-2-i)}, \vec{x}). \end{aligned}$$

Let $\alpha = \frac{x_{t-3-i} - x_{t-2-i}}{2}$. Clearly, $\vec{u} = (u_1, u_2, \dots, u_t)$ is also a feasible weighting and

$$\begin{aligned} &\lambda(G'', \vec{u}) - \lambda(G'', \vec{x}) \\ &= \frac{(x_{t-3-i} - x_{t-2-i})^2}{4}\lambda(E''_{(t-3-i)(t-2-i)}, \vec{x}) \\ (17) \quad &= \frac{(x_1 - x_{t-2-i})^2}{4}\lambda(E_{1(t-2-i)}, \vec{x}). \end{aligned}$$

Let $\vec{v} = (v_1, v_2, \dots, v_t)$ given by $v_j = u_j$ for $j \neq t-2$, $j \neq t-1$ and $v_{t-2} = u_{t-2} + \beta$, $v_{t-1} = u_{t-1} - \beta$. Then

$$\begin{aligned} \lambda(G'', \vec{v}) - \lambda(G'', \vec{u}) &= \beta[\lambda(E''_{t-2}, \vec{u}) - \lambda(E''_{t-1}, \vec{u})] - \beta^2\lambda(E''_{(t-2)(t-1)}, \vec{u}) \\ (18) \quad &= \beta(u_{t-2-i}u_t + u_{t-3-i}u_t) - \beta^2\lambda(E''_{(t-2)(t-1)}, \vec{u}). \end{aligned}$$

Let $\beta = \frac{u_{t-2-i}u_t + u_{t-3-i}u_t}{2\lambda(E''_{(t-2)(t-1)}, \vec{u})}$. Clearly, $\beta < u_t$. Hence, $\vec{v} = (v_1, v_2, \dots, v_t)$ is also a feasible weighting for G and

$$(19) \quad \lambda(G'', \vec{v}) - \lambda(G'', \vec{u}) = \frac{(u_{t-2-i} + u_{t-3-i})^2 u_t^2}{4\lambda(E''_{(t-2)(t-1)}, \vec{u})} = \frac{(x_{t-2-i} + x_{t-3-i})^2 x_t^2}{4\lambda(E_{(t-2)(t-1)}, \vec{x})}.$$

By Remark 3.5(b), we have $x_{t-2} = x_{t-1}$ and

$$(20) \quad x_1 = x_{t-2-i} + \frac{2x_{t-1}x_t}{\lambda(E_{1(t-2-i)}, \vec{x})}.$$

Combing (16), (17), (19), and (20), we have

$$\begin{aligned} &\lambda(G'', \vec{v}) - \lambda(G, \vec{x}) \\ &= x_{t-2-i}x_{t-2}x_t - x_1x_{t-1}x_t + \frac{(x_{t-2-i} + x_{t-3-i})^2 x_t^2}{4\lambda(E_{(t-2)(t-1)}, \vec{x})} + \frac{x_{t-1}^2 x_t^2}{\lambda(E_{1(t-2-i)}, \vec{x})} \\ &= -\frac{2x_{t-1}^2 x_t^2}{\lambda(E_{1(t-2-i)}, \vec{x})} + \frac{(x_{t-2-i} + x_{t-3-i})^2 x_t^2}{4\lambda(E_{(t-2)(t-1)}, \vec{x})} + \frac{x_{t-1}^2 x_t^2}{\lambda(E_{1(t-2-i)}, \vec{x})} \\ &\geq 0 \end{aligned}$$

since $\lambda(E_{(t-2)(t-1)}, \vec{x}) \leq \lambda(E_{1(t-2-i)}, \vec{x})$. Hence $\lambda(G'') \geq \lambda(G'', \vec{z}) \geq \lambda(G, \vec{x}) = \lambda(G)$. This completes the proof of Lemma 4.15. \square

Proof of Lemma 4.16. By Lemma 4.15, $\lambda(G') \leq \lambda(C_{3,m})$. So it is sufficient to show $\lambda(G) \leq \lambda(G')$. Let $\vec{x} = (x_1, x_2, \dots, x_t)$ be an optimal weighting for

G satisfying $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. By Remark 3.5(b), $x_{t-2} = x_{t-3}$ and $x_{t-1} = x_t$. Hence

$$(21) \quad \lambda(G', \vec{x}) - \lambda(G, \vec{x}) = (x_{t-3} - x_{t-4})x_{t-2}x_{t-1}.$$

Consider a new weighting $\vec{y} = (y_1, y_2, \dots, y_t)$ given by $y_j = x_j$ for $j \neq t-4$, $j \neq t-3$ and $y_{t-4} = x_{t-4} - \delta$, $y_{t-3} = x_{t-3} + \delta$. Then

$$(22) \quad \begin{aligned} \lambda(G', \vec{y}) - \lambda(G', \vec{x}) &= \delta[\lambda(E'_{t-3}, \vec{x}) - \lambda(E'_{t-4}, \vec{x})] - \delta^2 \lambda(E'_{(t-4)(t-3)}, \vec{y}) \\ &= \delta(x_{t-4} - x_{t-3})\lambda(E'_{(t-4)(t-3)}, \vec{x}) - \delta^2 \lambda(E'_{(t-4)(t-3)}, \vec{x}). \end{aligned}$$

Let $\delta = \frac{x_{t-4} - x_{t-3}}{2}$. Clearly, $\vec{y} = (y_1, y_2, \dots, y_t)$ is also a feasible weighting for G . Also note that $\lambda(E'_{(t-4)(t-3)}, \vec{x}) = \lambda(E_{(t-4)(t-3)}, \vec{x})$. Hence

$$(23) \quad \begin{aligned} \lambda(G', \vec{y}) - \lambda(G', \vec{x}) &= \frac{(x_{t-4} - x_{t-3})^2}{4} \lambda(E'_{(t-4)(t-3)}, \vec{x}) \\ &= \frac{(x_{t-4} - x_{t-3})^2}{4} \lambda(E_{(t-4)(t-3)}, \vec{x}). \end{aligned}$$

Let $\vec{z} = (z_1, z_2, \dots, z_t)$ be given by $z_i = y_i$ for $i \neq t-1$, $i \neq t$ and $z_{t-1} = y_{t-1} + \eta$, $z_t = y_t - \eta$. Then

$$(24) \quad \begin{aligned} \lambda(G', \vec{z}) - \lambda(G', \vec{y}) &= \eta[\lambda(E'_{t-1}, \vec{y}) - \lambda(E'_t, \vec{y})] - \eta^2 \lambda(E'_{(t-1)t}, \vec{y}) \\ &= \eta(y_{t-3}y_{t-2} + y_{t-4}y_{t-2}) - \eta^2 \lambda(E'_{(t-1)t}, \vec{y}) \\ &= \eta(x_{t-3}x_{t-2} + x_{t-4}x_{t-2}) - \eta^2 \lambda(E_{(t-1)t}, \vec{x}) \end{aligned}$$

in view of $y_{t-4} + y_{t-3} = x_{t-4} + x_{t-3}$, $y_{t-2} = x_{t-2}$ and $\lambda(E'_{(t-1)t}, \vec{y}) = \lambda(E_{(t-1)t}, \vec{x})$.

Let $\eta = \frac{x_{t-3}x_{t-2} + x_{t-4}x_{t-2}}{2\lambda(E_{(t-1)t}, \vec{x})}$. By the condition of $|E(G) \triangle E(C_{3,m})| = 4$ we have $\{1, 2\} \subseteq E_{(t-1)t}$, so

$$\eta \leq \frac{x_{t-2}}{2}.$$

Applying Remark 3.5(b), we have

$$(25) \quad \begin{aligned} x_{t-2} &= x_t + \frac{\lambda(E_{(t-2)\setminus t}, \vec{x})}{\lambda(E_{(t-2)t}, \vec{x})} \\ &\leq x_t + \frac{(x_{t-4} + \cdots + x_3)x_{t-1}}{1 - x_t - x_{t-1} - x_{t-2} - x_{t-3}} \\ &\leq x_t + x_{t-1} \\ &= 2x_t. \end{aligned}$$

So $\frac{x_{t-2}}{2} \leq x_t$. Recall that $\eta \leq \frac{x_{t-2}}{2}$. Therefore, $\eta \leq x_t$. Hence, $\vec{z} = (z_1, z_2, \dots, z_t)$ is also a feasible weighting for G' , and

$$(26) \quad \lambda(G', \vec{z}) - \lambda(G', \vec{y}) = \frac{(x_{t-4} + x_{t-3})^2 x_{t-2}^2}{4\lambda(E_{(t-1)t}, \vec{x})}.$$

By Remark 3.5(b), we have

$$(27) \quad x_{t-4} = x_{t-3} + \frac{2x_{t-2}x_t}{\lambda(E_{(t-4)(t-3)}, \vec{x})}.$$

In addition,

$$(28) \quad \begin{aligned} & \lambda(E_{(t-4)(t-3)}, \vec{x}) - \lambda(E_{(t-1)t}, \vec{x}) \\ & \geq (1 - x_{t-4} - x_{t-3}) - (1 - x_{t-4} - x_{t-3} - x_{t-2} - x_{t-1} - x_t) \\ & > 0. \end{aligned}$$

Combining (21), (23), (26), (27), and (28), we have

$$\begin{aligned} & \lambda(G', \vec{z}) - \lambda(G, \vec{x}) \\ &= \frac{(x_{t-4} - x_{t-3})^2 \lambda(E_{(t-4)(t-3)}, \vec{x})}{4} + \frac{(x_{t-4} + x_{t-3})^2 x_{t-2}^2}{4\lambda(E_{(t-1)t}, \vec{x})} - \frac{2x_{t-2}^2 x_t^2}{\lambda(E_{(t-4)(t-3)}, \vec{x})} \\ &= \frac{x_{t-2}^2 x_t^2}{\lambda(E_{(t-4)(t-3)}, \vec{x})} + \frac{(x_{t-4} + x_{t-3})^2 x_{t-2}^2}{4\lambda(E_{(t-1)t}, \vec{x})} - \frac{2x_{t-2}^2 x_t^2}{\lambda(E_{(t-4)(t-3)}, \vec{x})} \\ &\geq 0. \end{aligned}$$

Hence $\lambda(G) = \lambda(G, \vec{x}) \leq \lambda(G', \vec{z}) \leq \lambda(G')$. \square

Now we continue the proof of Theorem 4.1. By Claim 4.12, we have $k \leq t+1$. If $k \leq t$, then $L_\beta(H) \leq L_\beta([t]^{(3)}) \leq L_\beta(C_{3,m})$ since $[t]^{(3)} \subseteq C_{3,m}$. Assume that $k = t+1$. Using Lemma 4.9 and Lemma 4.11 and recalling that $\beta \geq t-1$ we have

$$(29) \quad \begin{aligned} m = |E| &= |E \cap [k-1]^{(3)}| + |[k-2]^{(2)} \cap E_k| + |E_{(k-1)k}| \\ &\geq \binom{k-1}{3} - \lceil \frac{k-2}{1+\beta} \rceil + \binom{k-2}{2} - \lceil \frac{b}{1+\beta} \rceil + b \\ &= \binom{t}{3} + \binom{t-1}{2} - \lceil \frac{t-1}{1+\beta} \rceil - \lceil \frac{b}{1+\beta} \rceil + b \\ &= \binom{t}{3} + \binom{t-1}{2} + b - 2. \end{aligned}$$

Since H is left-compressed, this implies that $|E(H) \Delta E(C_{3,m})| \leq 4$. Hence $L_\beta(H) \leq L_\beta(C_{3,m})$ by Theorem 4.13. We complete the proof of Theorem 4.1. \square

We now prove Lemmas 4.9-4.11. The basic idea in the proof of these lemmas follows from a result given by Talbot in [11]. We remove a vertex and related edges from H and give the weight of this vertex to another vertex. We then insert some other edges to H and check two conditions: (i) the total weight of the 3-graph has not decreased; (ii) the number of edges we have added does not exceed the number previously removed.

Proof of Lemma 4.9. Since E is left-compressed, then $E_i := \{1, \dots, i-1, i+1, \dots, k\}^{(2)}$ for $1 \leq i \leq b$, and $E_{i \setminus j} = \emptyset$ for $1 \leq i < j \leq b$. Hence, by Remark 3.5(b), we have $x_1 = x_2 = \dots = x_b$. We define a new feasible weighting \vec{y} for H as follows. Let $y_i = x_i$ for $i \neq k-1, k$, $y_{k-1} = x_{k-1} + x_k$ and $y_k = 0$.

By Lemma 3.2(a), $L_\beta(E_{k-1}, \vec{x}) = L_\beta(E_k, \vec{x})$, so

$$\begin{aligned}
 L_\beta(H, \vec{y}) - L_\beta(H, \vec{x}) &= x_k(L_\beta(E_{k-1}, \vec{x}) - L_\beta(E_k, \vec{x})) - x_k^2 L_\beta(E_{(k-1)k}, \vec{x}) \\
 &= -x_k^2 L_\beta(E_{(k-1)k}, \vec{x}) \\
 &= -x_k^2 [\lambda(E_{(k-1)k}, \vec{x}) - \beta \lambda(E_{(k-1)k}^c, \vec{x})] \\
 (30) \qquad &= -bx_1 x_k^2 + \beta x_k^2 \sum_{i=b+1}^{k-2} x_i.
 \end{aligned}$$

Since $y_k = 0$ we may remove all edges containing k from E to form a new 3-graph $H^\circ := ([k], E^\circ)$ with $|E^\circ| := |E| - |E_k|$ and $L_\beta(H^\circ, \vec{y}) = L_\beta(H, \vec{y})$. We will show that if Lemma 4.9 fails to hold then there exists a set of edges $F \subset [k-1]^{(3)} \setminus E$ satisfying

$$(31) \qquad (1 + \beta)\lambda(F, \vec{y}) > -bx_1 x_k^2 + \beta x_k^2 \sum_{i=b+1}^{k-2} x_i$$

and

$$(32) \qquad |F| \leq |E_k|.$$

Then, using (30), (31), and (32), the 3-graph $H' := ([k], E')$, where $E' := E^\circ \cup F$, satisfies $|E'| \leq |E|$ and

$$\begin{aligned}
 L_\beta(H', \vec{y}) &= L_\beta(H^\circ, \vec{y}) + (1 + \beta)\lambda(F, \vec{y}) \\
 &> L_\beta(H, \vec{y}) + bx_1 x_k^2 - \beta x_k^2 \sum_{i=b+1}^{k-2} x_i \\
 &= L_\beta(H, \vec{x}).
 \end{aligned}$$

Hence $L_\beta(H') > L_\beta(H)$. This contradicts to $L_\beta(H) = L_{(\beta, m)}^{(3)}$.

We must now construct the set of edges F satisfying (31) and (32). Applying Remark 3.5(b) by taking $i = 1, j = k-1$, we have

$$x_1 = x_{k-1} + \frac{(1 + \beta)\lambda(E_{1 \setminus (k-1)}, \vec{x})}{\lambda(E_{1(k-1)}, \vec{x}) - \beta \lambda(E_{1(k-1)}^c, \vec{x})} = x_{k-1} + \frac{(1 + \beta)\lambda(E_{1 \setminus (k-1)}, \vec{x})}{\lambda(E_{1(k-1)}, \vec{x})},$$

since $E_{1(k-1)}^c = \varphi$. Let $C := [k-2]^{(2)} \setminus E_{k-1}$. Then $\lambda(E_{1 \setminus (k-1)}, \vec{x}) = x_k \sum_{i=b+1}^{k-2} x_i + \lambda(C, \vec{x})$. Applying this and multiplying bx_k^2 to the above equation (note that $\lambda(E_{1(k-1)}, \vec{x}) = \sum_{i=2, i \neq k-1}^k x_i$), we have

$$bx_1 x_k^2 = bx_{k-1} x_k^2 + \frac{b(1 + \beta)x_k^3 \sum_{i=b+1}^{k-2} x_i}{\sum_{i=2, i \neq k-1}^k x_i} + \frac{b(1 + \beta)x_k^2 \lambda(C, \vec{x})}{\sum_{i=2, i \neq k-1}^k x_i}.$$

Since $x_1 \geq x_2 \geq \dots \geq x_k$, then

$$(33) \quad \begin{aligned} bx_1x_k^2 - \beta x_k^2 \sum_{i=b+1}^{k-2} x_i &\leq (b(1 + \frac{(1+\beta)(k-2-b)}{k-3}) \\ &\quad - \beta(k-2-b))x_{k-1}x_k^2 + \frac{b(1+\beta)x_k\lambda(C, \vec{x})}{k-2}. \end{aligned}$$

Define $p := \lceil \frac{b|C|}{k-2} \rceil$ and $q := \lceil b(1 + \frac{k-2-b}{k-3}) - \beta(k-2-b) \rceil$. Note that $q = \lceil b(1 + \frac{k-2-b}{k-3}) - \beta(k-2-b) \rceil \leq k-2$ since $b \leq k-2$. Let the set $F_1 \subset [k-1]^{(3)} \setminus E$ consist of the p heaviest edges in $[k-1]^{(3)} \setminus E$ containing the vertex $k-1$ (note that $|[k-2]^{(2)} \setminus E_{k-1}| = |C| \geq p$). Recalling that $y_{k-1} = x_{k-1} + x_k$, we have

$$(1+\beta)\lambda(F_1, \vec{y}) \geq \frac{b(1+\beta)x_k\lambda(C, \vec{x})}{k-2} + p(1+\beta)x_{k-1}x_k^2.$$

So using (33)

$$(34) \quad (1+\beta)\lambda(F_1, \vec{y}) + \beta x_k^2 \sum_{i=b+1}^{k-2} x_i - bx_1x_k^2 \geq x_{k-1}x_k^2(p(1+\beta) - q).$$

We now distinguish two cases.

Case 1: $p(1+\beta) > q$.

In this case $(1+\beta)\lambda(F_1, \vec{y}) + \beta x_k^2 \sum_{i=b+1}^{k-2} x_i - bx_1x_k^2 > 0$ so defining $F := F_1$ satisfies (31). We need to check that $|F| \leq |E_k|$. Since E is left-compressed, then $[b]^{(2)} \cup \{1, \dots, b\} \times \{b+1, \dots, k-1\} \subset E_k$. Hence

$$(35) \quad |E_k| \geq \frac{b[b-1+2(k-1-b)]}{2} \geq \frac{b(k-1)}{2}$$

since $b \leq k-2$. Since $C \subset [k-2]^{(2)}$, we have $|C| \leq \binom{k-2}{2}$. Note that $|F| = p = \lceil \frac{b|C|}{k-2} \rceil$. Using (35) we obtain

$$|F| \leq \lceil \frac{b(k-3)}{2} \rceil \leq \frac{b(k-1)}{2} \leq |E_k|.$$

So both (31) and (32) are satisfied.

Case 2: $p(1+\beta) \leq q$.

Suppose that Lemma 4.9 fails to hold. So $|[k-1]^{(3)} \setminus E| \geq \lceil \frac{k-2}{1+\beta} \rceil + 1 \geq \lceil \frac{1+q}{1+\beta} \rceil$. Let F_2 consist of any $\lceil \frac{1+q-p(1+\beta)}{1+\beta} \rceil$ edges in $[k-1]^{(3)} \setminus (E \cup F_1)$ and define $F := F_1 \cup F_2$. Then since $(1+\beta)\lambda(F_2, \vec{y}) \geq (1+\beta)\lceil \frac{1+q-p(1+\beta)}{1+\beta} \rceil x_{k-1}^3$ and using (34),

$$\begin{aligned} &(1+\beta)\lambda(F, \vec{y}) + \beta x_k^2 \sum_{i=b+1}^{k-2} x_i - bx_1x_k^2 \\ &\geq x_{k-1}x_k^2(p(1+\beta) - q) + (1+\beta)\lceil \frac{1+q-p(1+\beta)}{1+\beta} \rceil x_{k-1}^3 > 0. \end{aligned}$$

So (31) is satisfied. What remains is to check that $|F| \leq |E_k|$. In fact,

$$|F| = \lceil \frac{1+q}{1+\beta} \rceil \leq 1+q \leq k-1 \leq \frac{b(k-1)}{2} \leq |E_k|$$

when $b \geq 2$. If $b = 1$, then,

$$|F| = \lceil \frac{1+q}{1+\beta} \rceil \leq 3 \leq k-2 = \frac{b[b-1+2(k-1-b)]}{2} \leq |E_k|$$

since $k \geq 5$ (Lemma 4.9 clearly holds for $k \leq 4$). This completes the proof. \square

Proof of Lemma 4.10. We use the notations from Lemma 4.9. If Lemma 4.10 fails to hold, then $|C| = |[k-2] \setminus E_{k-1}| \geq \lceil \frac{b}{1+\beta} \rceil + 1 \geq \frac{b(1 + \frac{(1+\beta)(k-2-b)}{k-3}) - \beta(k-2-b)}{(2 - \frac{b}{k-2})(1+\beta)}$. We again construct a new set of edges $F \subseteq [k-1] \setminus E$ and need to check that F satisfies (31) and (32). Let F consist of all edges in $[k-1]^{(3)} \setminus E$ containing the vertex $k-1$ (So $F = C \times \{k-1\}$). Then, since $y_{k-1} = x_{k-1} + x_k$,

$$(1+\beta)\lambda(F, \vec{y}) = (1+\beta)(x_{k-1} + x_k)\lambda(C, \vec{x}).$$

Using (33) we have

$$\begin{aligned} (1+\beta)\lambda(F, \vec{y}) - bx_1x_k^2 + \beta x_k^2 \sum_{i=b+1}^{k-2} x_i &\geq -\left(b(1 + \frac{(1+\beta)(k-2-b)}{k-3})\right) \\ &\quad - \beta(k-2-b)x_{k-1}x_k^2 \\ &\quad + (2 - \frac{b}{k-2})(1+\beta)x_k\lambda(C, \vec{x}). \end{aligned}$$

In order to show that (31) holds it is sufficient to show

$$(2 - \frac{b}{k-2})(1+\beta)|C| > b(1 + \frac{(1+\beta)(k-2-b)}{k-3}) - \beta(k-2-b).$$

This follows from

$$|C| = |[k-1]^{(2)} \setminus E_{k-1}| > \frac{b(1 + \frac{(1+\beta)(k-2-b)}{k-3}) - \beta(k-2-b)}{(2 - \frac{b}{k-2})(1+\beta)}.$$

For (32), by Lemma 4.9, we have

$$|F| \leq |[k-1]^{(3)} \setminus E| \leq \lceil \frac{b(1 + \frac{k-2-b}{k-3}) - \beta(k-2-b)}{1+\beta} \rceil$$

and in the proof of Lemma 4.9 we have show that $\lceil \frac{b(1 + \frac{k-2-b}{k-3}) - \beta(k-2-b)}{1+\beta} \rceil \leq |E_k|$. Hence F satisfies (32). So, we may construct a new 3-graph H' with at most m edges but $L_\beta(H) < L_\beta(H')$. This contradicts to $L_\beta(H) = L_{(\beta, m)}^{(3)}$. Hence

$$|[k-1]^{(2)} \setminus E_{k-1}| \leq \lceil \frac{b}{1+\beta} \rceil.$$

This completes the proof. \square

Proof of Lemma 4.11. Since E is left-compressed, then $E_i := \{1, \dots, i-1, i+1, \dots, k\}^{(2)}$ for $1 \leq i \leq b$, and $E_{i \setminus j} = \emptyset$ for $1 \leq i < j \leq b$. Hence, by Remark 3.5(b), we have $x_1 = x_2 = \dots = x_b$.

We define a new feasible weighting \vec{z} for H as follows. Let $z_i = x_i$ for $i \neq k-1, k$, $z_{k-1} = 0$ and $z_k = x_{k-1} + x_k$.

By Lemma 3.2(a), $L_\beta(E_{k-1}, \vec{x}) = L_\beta(E_k, \vec{x})$, so

$$\begin{aligned}
 L_\beta(H, \vec{z}) - L_\beta(H, \vec{x}) &= x_k(L_\beta(E_{k-1}, \vec{x}) - L_\beta(E_k, \vec{x})) - x_{k-1}^2 L_\beta(E_{(k-1)k}, \vec{x}) \\
 &= -x_{k-1}^2 L_\beta(E_{(k-1)k}, \vec{x}) \\
 &= -x_{k-1}^2 [\lambda(E_{(k-1)k}, \vec{x}) - \beta \lambda(E_{(k-1)k}^c, \vec{x})] \\
 (36) \quad &= -bx_1 x_{k-1}^2 + \beta x_{k-1}^2 \sum_{i=b+1}^{k-2} x_i.
 \end{aligned}$$

Since $z_{k-1} = 0$ we may remove all edges containing $k-1$ from E to form a new 3-graph $H^* := ([k], E^*)$ with $|E^*| := |E| - |E_k|$ and $L_\beta(H^*, \vec{y}) = L_\beta(H, \vec{y})$.

By Lemma 4.10, we have

$$|[k-1]^{(2)} \setminus E_k| \leq \lceil \frac{b}{1+\beta} \rceil.$$

Hence

$$|E_k| = |[k-2]^{(2)} \cap E_k| + |E_{(k-1)k}| \geq \binom{k-2}{2} - \lceil \frac{b}{1+\beta} \rceil + b \geq \binom{k-2}{2}.$$

If $b = k-2$, Lemma 4.11 clearly holds. Next we assume $b \leq k-3$. We will show that if Lemma 4.11 fails to hold, then there exists a set of edges $G \subset \{1, 2, \dots, k-2, k\}^{(3)} \setminus E$ satisfying

$$(37) \quad (1+\beta)\lambda(G, \vec{z}) > -bx_1 x_k^2 + \beta x_k^2 \sum_{i=b+1}^{k-2} x_i$$

and

$$(38) \quad |G| \leq \binom{k-2}{2}.$$

Then, using (36), (37), and (38), the 3-graph $H' := ([k], E')$, where $E' := E^* \cup G$, satisfies $|E'| \leq |E|$ and

$$\begin{aligned}
 L_\beta(H', \vec{z}) &= L_\beta(H^*, \vec{z}) + (1+\beta)\lambda(G, \vec{z}) \\
 &> L_\beta(H, \vec{z}) + bx_1 x_k^2 - \beta x_k^2 \sum_{i=b+1}^{k-2} x_i \\
 &= L_\beta(H, \vec{x}).
 \end{aligned}$$

Hence $L_\beta(H') > L_\beta(H)$. This contradicts to $L_\beta(H) = L_{(\beta, m)}^{(3)}$.

We must now construct the set of edges G satisfying (37) and (38). Applying Remark 3.5(b) by taking $i = 1, j = k$, we have

$$x_1 = x_k + \frac{(1 + \beta)\lambda(E_{1 \setminus k}, \vec{x})}{\lambda(E_{1k}, \vec{x}) - \beta\lambda(E_{1k}^c, x)} = x_k + \frac{(1 + \beta)\lambda(E_{1 \setminus k}, \vec{x})}{\lambda(E_{1k}, \vec{x})},$$

since $E_{1k}^c = \varphi$. Let $D := [k - 2]^{(2)} \setminus E_k$. Then $\lambda(E_{1 \setminus k}, \vec{x}) = x_{k-1} \sum_{i=b+1}^{k-2} x_i + \lambda(D, \vec{x})$. Applying this and multiplying bx_{k-1}^2 to the above equation (note that $\lambda(E_{1k}, \vec{x}) = \sum_{i=2}^{k-1} x_i$), we have

$$bx_1x_{k-1}^2 = bx_{k-1}^2x_k + \frac{b(1 + \beta)x_{k-1}^3 \sum_{i=b+1}^{k-2} x_i}{\sum_{i=2}^{k-1} x_i} + \frac{b(1 + \beta)x_{k-1}^2 \lambda(D, \vec{x})}{\sum_{i=2}^{k-1} x_i}.$$

Since $x_1 \geq x_2 \geq \dots \geq x_k$, then

$$\begin{aligned} bx_1x_{k-1}^2 - \beta x_{k-1}^2 \sum_{i=b+1}^{k-2} x_i &\leq bx_{k-1}^2x_k + \frac{(1 + \beta)b(k - 2 - b)}{k - 3}x_{k-1}^3 \\ &\quad - \beta(k - 2 - b)x_{k-1}^3 \\ &\quad + \frac{b(1 + \beta)x_{k-1}\lambda(D, \vec{x})}{k - 2}. \end{aligned}$$

Let G consist of those edges in $\{1, 2, \dots, k - 2, k\}^{(3)} \setminus E$ containing the vertex k . So

$$(1 + \beta)\lambda(G, \vec{x}) = (1 + \beta)(x_{k-1} + x_k)\lambda(D, \vec{x}).$$

Suppose that Lemma 4.11 fails to hold, then $|D| \geq \lceil \frac{b}{1+\beta} \rceil + 1$ and $(1 + \beta)|D|(1 - \frac{b}{k-2}) + \beta(k - 2 - b) \geq \frac{(1+\beta)b(k-2-b)}{k-3}$. Therefore

$$\begin{aligned} &(1 + \beta)\lambda(G, \vec{x}) - bx_1x_{k-1}^2 + \beta x_{k-1}^2 \sum_{i=b+1}^{k-2} x_i \\ &\geq ((1 + \beta)(\lceil \frac{b}{1+\beta} \rceil + 1) - b)x_{k-1}x_k^2 \\ &\quad + (1 + \beta)|D|(1 - \frac{b}{k-2})x_{k-1}^3 + \beta(k - 2 - b)x_{k-1}^3 \\ &\quad - \frac{(1 + \beta)b(k - 2 - b)}{k - 3}x_{k-1}^3 \\ &> (1 + \beta)|D|(1 - \frac{b}{k-2})x_{k-1}^3 + \beta(k - 2 - b)x_{k-1}^3 \\ &\quad - \frac{(1 + \beta)b(k - 2 - b)}{k - 3}x_{k-1}^3 \geq 0. \end{aligned}$$

Hence $(1 + \beta)\lambda(G, \vec{x}) > -bx_1x_k^2 + \beta x_k^2 \sum_{i=b+1}^{k-2} x_i$ and so (37) holds. So, we may construct a new 3-graph H' with at most m edges but $L_\beta(H) < L_\beta(H')$. This

contradicts to $L_\beta(H) = L_{(\beta, m)}^{(3)}$. Hence

$$|[k-2]^{(2)} \setminus E_k| \leq \lceil \frac{b}{1+\beta} \rceil.$$

This completes the proof. \square

4.2. Proof of Theorem 4.2

Proof of Theorem 4.2. (a) Clearly $L_\beta(H) \geq L_\beta([t]^{(3)})$ since H contains K_t^3 . Let $\vec{x} = (x_1, x_2, \dots, x_{t+1})$ be an optimal weighting for H satisfying $x_1 \geq x_2 \geq \dots \geq x_{t+1} \geq 0$. If $x_{t+1} = 0$, then $L_\beta(H) \leq L_\beta([t]^{(3)})$. Assume that $x_{t+1} > 0$, we need the following lemma.

Lemma 4.17. *Let H be a 3-graph on the vertex set $[t+1]$. Let $\vec{x} = (x_1, x_2, \dots, x_{t+1})$ be an optimal weighting for the graph-Lagrangian of H satisfying $x_1 \geq x_2 \geq \dots \geq x_{t+1} \geq 0$. Then $x_1 < x_{t-1} + x_t + x_{t+1}$ or $L_\beta(H) \leq \lambda(H) \leq \frac{1}{6} \frac{(t-2)^2}{(t-1)t} < \lambda([t]^{(3)}) = L_\beta([t]^{(3)})$.*

Proof. If $x_1 \geq x_{t-1} + x_t + x_{t+1}$, then $2x_1 + x_2 + \dots + x_{t-3} + x_{t-2} > x_1 + x_2 + \dots + x_{t-2} + x_{t-1} + x_t + x_{t+1} = 1$. Recalling that $x_1 \geq x_2 \geq \dots \geq x_{t-2}$, we have $x_1 \geq \frac{1}{t-1}$. By Lemma 3.2(a) (set $\beta = 0$),

$$\begin{aligned} \lambda(H) &= \frac{1}{3} \lambda(E_1, x) \leq \frac{1}{3} \binom{t}{2} \left(\frac{1 - \frac{1}{t-1}}{t} \right)^2 = \frac{1}{6} \frac{(t-2)^2}{(t-1)t} \\ &< \frac{1}{6} \frac{(t-2)(t-1)}{t^2} = \lambda([t]^{(3)}). \end{aligned}$$

The first inequality follows from Theorem 2.2. Hence $x_1 < x_{t-1} + x_t + x_{t+1}$ or $L_\beta(H) \leq \lambda(H) < \lambda([t]^{(3)}) = L_\beta([t]^{(3)})$ holds. \square

Now we continue the proof of Theorem 4.2. Let $H' = [t+1]^{(3)} \setminus \{(t-1)t(t+1)\}$. Without loss of generality, we can assume that $H \subseteq H'$ since $m \leq \binom{t+1}{3} - 1$. By Fact 2.3 $L_\beta(H) \leq L_\beta(H')$. Let $\vec{y} = (y_1, y_2, \dots, y_{t+1})$ be an optimal weighting for H' satisfying $y_1 \geq y_2 \geq \dots \geq y_{t+1} \geq 0$. By Remark 3.5(b), we have $y_1 = y_2 = \dots = y_{t-2}$ and $y_{t-1} = y_t = y_{t+1}$. If $y_{t+1} = 0$, then $L_\beta(H, \vec{y}) \leq L_\beta([t]^{(3)})$. Assume that $y_{t+1} > 0$. By Lemma 4.17, if $y_1 \geq y_{t-1} + y_t + y_{t+1} = 3y_{t+1}$, then $L_\beta(H') \leq \lambda(H') \leq \frac{1}{6} \frac{(t-2)^2}{(t-1)t} < \lambda([t]^{(3)}) = L_\beta([t]^{(3)})$. Assume that $y_1 < y_{t-1} + y_t + y_{t+1} = 3y_{t+1}$. Hence $y_{t+1} > \frac{1}{3}y_1 \geq \frac{1}{3(t+1)}$. Recalling that $\beta \geq 27\binom{t}{2} - 1$, so

$$\begin{aligned} L_\beta(H') &= L_\beta([t+1]^{(3)} \setminus \{(t-1)t(t+1)\}, \vec{y}) \\ &= L_\beta([t+1]^{(3)}, \vec{y}) - (1+\beta)y_{t-1}y_ty_{t+1} \\ &\leq L_\beta([t+1]^{(3)}) - (1+\beta)y_{t-1}y_ty_{t+1} \\ &< \binom{t+1}{3} \frac{1}{(t+1)^3} - 27\binom{t}{2} \frac{1}{27(t+1)^3} \end{aligned}$$

$$= \binom{t}{3} \frac{1}{t^3} = L_\beta([t]^{(3)}).$$

Clearly, the vector $\vec{x} = (x_1, x_2, \dots, x_n)$, given by $x_i := \frac{1}{t}$ if i is a vertex in a fixed maximum clique and $x_i = 0$ otherwise, is an optimal weighting. This completes the proof of part (a).

(b) Let $\vec{x} = (x_1, x_2, \dots, x_{t+1})$ be an optimal weighting for H satisfying $x_1 \geq x_2 \geq \dots \geq x_{t+1} \geq 0$. If $x_{t+1} = 0$, $L_\beta(H) < L_\beta([t]^{(3)})$ since H does not contain $[t]^{(3)}$. Since H does not contain a clique of order t , $|\{1, 2, \dots, t-1, t+1\}^{(3)} \setminus E| \geq 1$. Let $F = [t]^{(3)} \setminus E$ and $H' = H \cup F$, then H' has at most $\binom{t+1}{3} - 1$ edges and $L_\beta(H) = L_\beta(H, \vec{x}) < L_\beta(H', \vec{x}) \leq L_\beta(H')$. By Fact 2.3, we can assume $H' = [t+1]^{(3)} \setminus \{(t-1)t(t+1)\}$ without loss of generality. We have proved that $L_\beta(H') = L_\beta([t]^{(3)})$ in part (a). Hence $L_\beta(H) \leq L_\beta([t]^{(3)})$. This completes the proof part(b). \square

4.3. Proof of Theorem 4.3

Denote $L_{(\beta, m, t)}^{(3)} := \max\{L_\beta(H) : H \text{ is a 3-graph with } m \text{ edges and } H \text{ containing } K_t^3\}$. We give the following remark.

Remark 4.18. Let m and t be a positive integers satisfying $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$. Let $\beta \geq 0$ be a constant. Let H be a left-compressed 3-graph with m edges containing a clique order of t such that $L_\beta(H) = L_{(\beta, m, t)}^{(3)}$. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for H and k be the number of non-zero weights in \vec{x} . Then the results of Lemmas 4.9, 4.10 and 4.11 also hold. The proofs are similar to the proofs of Lemmas 4.9, 4.10 and 4.11. So we omit the details here.

Proof of Theorem 4.3. Let m and t be integers satisfying $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$. Let $\beta \geq 27\binom{t}{2} - 1$ be a constant. Let H be a 3-graph with m edges containing a clique order of t . We can assume that $L_\beta(H) = L_{(\beta, m, t)}^{(3)}$, i.e., H is an extremal graph. We can also assume H is left-compressed. Otherwise if H is not left-compressed, performing a sequence of left-compressing operations we will get a left-compressed 3-graph G . The order of its maximum complete 3-graph is still t . By Lemma 3.4 G is also an extremal graph. Since H contains $[t]^{(3)}$, we have $L_\beta(H) \geq L_\beta([t]^{(3)})$. Next we prove that $L_\beta(H) \leq L_\beta([t]^{(3)})$.

Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for H and k be the number of non-zero weights in \vec{x} . By Remark 4.18 and Claim 4.12, we have $k \leq t+1$. Hence $L_\beta(H) = L_\beta([t]^{(3)})$ by Theorem 4.2(a). \square

4.4. Proof of Theorem 4.4

Denote $L_{(\beta, m, t)}^{(3)-} := \max\{L_\beta(H) : H \text{ is a 3-graph with } m \text{ edges and } H \text{ not containing } K_t^3\}$. The following lemma implies that we only need to consider left-compressed 3-graphs when Theorem 4.4 is proved.

Lemma 4.19. *Let m and t be positive integers satisfying $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$. Let $\beta \geq 27\binom{t}{2} - 1$ be a constant. Then Theorem 4.4 holds or there exists a left-compressed 3-graph H with m edges not containing K_t^3 such that $L_\beta(H) = L_{(\beta, m, t)}^{(3)-}$.*

Proof. Let H be a 3-graph on the vertex set $[n]$ with m edges not containing K_t^3 such that $L_\beta(H) = L_{(\beta, m, t)}^{(3)-}$, i.e., H is an extremal graph. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting of H . We can also assume that $x_1 \geq x_2 \geq \dots \geq x_k > x_{k+1} = \dots = x_n = 0$ since otherwise we can just relabel the vertices of H and obtain another extremal graph not containing K_t^3 with an optimal weighting $\vec{x} = (x_1, x_2, \dots, x_t)$ satisfying $x_1 \geq x_2 \geq \dots \geq x_k > x_{k+1} = \dots = x_n = 0$. If $k \leq t+1$, then Theorem 4.4 holds by Theorem 4.2(b). Hence we assume $k \geq t+2$. Next we obtain a new 3-graph H' from H by performing the following:

- (1) If $(t-2)(t-1)t \in E(H)$, then there is at least one triple in $[t]^{(3)} \setminus E(H)$ since H does not contain K_t^3 . We replace $(t-2)(t-1)t$ by such a triple. Denote the new graph as H' .
- (2) If an edge in H' has a descendant other than $(t-2)(t-1)t$ that is not in $E(H')$, then replace this edge by a descendant other than $(t-2)(t-1)t$ with the lowest hierarchy. Repeat this until there is no such an edge.

Then H' satisfies the following properties:

- (1) The number of edges in H' is the same as the number of edges in H .
- (2) $L_\beta(H) = L_\beta(H, \vec{x}) \leq L_\beta(H', \vec{x}) \leq L_\beta(H')$.
- (3) $(t-2)(t-1)t \notin E(H')$.
- (4) For any edge in $E(H')$, all its descendants other than $(t-2)(t-1)t$ will be in $E(H')$.

If H' is not left-compressed, then there is an ancestor uvw of $(t-2)(t-1)t$ such that $uvw \in E(H')$. uvw must be $(t-2)(t-1)(t+1)$ or an ancestor of $(t-2)(t-1)(t+1)$. Hence $(t-2)(t-1)(t+1)$ and all descendants of $(t-2)(t-1)(t+1)$ other than $(t-2)(t-1)t$ will be in $E(H')$. Let \vec{y} be an optimal weighting for H' . We can assume that \vec{y} has at least $t+2$ positive weights. Since otherwise $L_\beta(H, \vec{x}) \leq L_\beta(H', \vec{x}) \leq L_\beta(H', \vec{y}) < L_\beta([t]^{(3)})$ by Theorem 4.2(b). This confirms Theorem 4.4. Without loss generality, we can assume the first $t+2$ weights of \vec{y} are positive. By Lemma 3.2(a) and the structure of H' , $1(t+1)(t+2) \in E(H')$. Hence all triples in the form of $1j(t+2)$ (where $2 \leq j \leq t+1$). So

$$m \geq \binom{t+1}{3} - (t-1) - 1 + t = \binom{t+1}{3}$$

which contradicts to $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$.

So we can assume that H' is left-compressed. Clearly H' does not contain $[t]^{(3)}$. Since H' is left-compressed, then H' does not contain a clique of order

t . Therefore we get a left-compressed extremal graph. Hence we can assume that H is left-compressed. This completes the proof of Lemma 4.19. \square

We also need the following remark in the proof of Theorem 4.4.

Remark 4.20. Let m and t be positive integers satisfying $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$. Let $\beta \geq 0$ be a constant. Let H be a left-compressed 3-graph with m edges not containing a clique order of t such that $L_\beta(H) = L_{(\beta, m, t)}^{(3)-}$. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for H and k be the number of non-zero weights in \vec{x} . Then the results of Lemmas 4.9, 4.10 and 4.11 also hold or $L_\beta(H) < L_\beta([t]^{(3)})$ holds. The proofs are similar to the proofs of Lemmas 4.9, 4.10 and 4.11. So we omit the details here.

Proof of Theorem 4.4. Let m and t be integers satisfying $\binom{t}{3} \leq m \leq \binom{t+1}{3} - 1$. Let $\beta \geq 27\binom{t}{2} - 1$ be a constant. Let H be a 3-graph with m edges not containing a clique of order t . We can assume that $L_\beta(H) = L_{(\beta, m, t)}^{(3)-}$, i.e., H is an extremal graph. By Lemma 4.19 we can assume H is left-compressed. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for H satisfying $x_1 \geq x_2 \geq \dots \geq x_n \geq 0$ with k positive weights. If $L_\beta(H) < L_\beta([t]^{(3)})$, then Theorem 4.4 holds. Otherwise by Remark 4.20 and Claim 4.12, we have $k \leq t + 1$. Hence $L_\beta(H) < L_\beta([t]^{(3)})$ by Theorem 4.2(b), i.e., Theorem 4.4 holds. \square

4.5. Proof of Theorems 4.6 and 4.7

The following lemma implies that we only need to consider left-compressed 3-graphs when Theorems 4.6 and 4.7 are proved.

Lemma 4.21. *Let m and t be positive integers satisfying $\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t-1}{2}$. Let $\beta \geq 0$ be a constant. Then Theorem 4.6 and Theorem 4.7 hold or there exists a left-compressed 3-graph H with m edges not containing $[t]^{(3)}$ such that $L_\beta(H) = L_{(\beta, m, t)}^{(3)-}$.*

Proof. Let H be a 3-graph on the vertex set $[n]$ with m edges not containing K_t^3 such that $L_\beta(H) = L_{(\beta, m, t)}^{(3)-}$, i.e., H is an extremal graph. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting of H satisfying $x_1 \geq x_2 \geq \dots \geq x_k > x_{k+1} = \dots = x_n = 0$. If $k \geq t + 2$, then use the same method in the proof of Lemma 4.19 we can obtain a left-compressed extremal graph not containing $[t]^{(3)}$.

If $k \leq t$, since H does not contain K_t^3 , we have $m \leq \binom{t}{3} - 1$. If H is not left-compressed, performing a sequence of left-compressing operations, we will get a left-compressed 3-graph H' with m edges. Since $m \leq \binom{t}{3} - 1$, H' does not contain $[t]^{(3)}$. By Lemma 3.4 H' is also an extremal graph.

Hence we only need to consider $k = t + 1$. Next we obtain a new 3-graph H' from H as we did in the proof of Lemma 4.19.

If H' is not left-compressed, then there is an ancestor uvw of $(t-2)(t-1)t$ such that $uvw \in E(H')$. We claim that uvw must be $(t-2)(t-1)(t+1)$.

Otherwise $uvw = (t-2)t(t+1)$ or $uvw = (t-1)t(t+1)$, then $m \geq \binom{t+1}{3} - 2 > \binom{t}{3} + \binom{t-1}{2}$. This contradicts to $m \leq \binom{t}{3} + \binom{t-1}{2}$. Hence uvw must be $(t-2)(t-1)(t+1)$. Since $m \leq \binom{t}{3} + \binom{t-1}{2}$ and all the descendants of uvw other than $(t-2)(t-1)t$ of an edge in H' will be an edge in H' , then there are two possibilities.

Case 1: $H' = ([t]^{(3)} \setminus \{(t-2)(t-1)t\}) \cup \{ij(t+1), ij \in [t-1]^{(2)}\}$.

Case 2: $H' = ([t]^{(3)} \setminus \{(t-2)(t-1)t\}) \cup \{ij(t+1), ij \in [t-1]^{(2)}\} \cup \{1t(t+1)\}$.

We will show if these two cases happen, then Theorems 4.6 and 4.7 hold. Note that $([t]^{(3)} \setminus \{(t-2)(t-1)t\}) \cup \{ij(t+1), ij \in [t-1]^{(2)}\} \subseteq ([t]^{(3)} \setminus \{(t-2)(t-1)t\}) \cup \{ij(t+1), ij \in [t-1]^{(2)}\} \cup \{1t(t+1)\}$. So it is sufficient to assume $H' = ([t]^{(3)} \setminus \{(t-2)(t-1)t\}) \cup \{ij(t+1), ij \in [t-1]^{(2)}\} \cup \{1t(t+1)\}$ and show that $L_\beta(H', \vec{x}) < L_\beta([t]^{(3)})$ since then $L_\beta(H) = L_\beta(H, \vec{x}) \leq L_\beta(H', \vec{x}) < L_\beta([t]^{(3)})$, i.e., Theorems 4.6 and 4.7 hold.

Let $H^* = [t]^{(3)} \cup \{ij(t+1), ij \in [t-1]^{(2)}\} \cup \{1t(t+1)\} \setminus \{(t-2)(t-1)(t+1)\}$. Then $L_\beta(H', \vec{x}) \leq L_\beta(H^*, \vec{x})$ since $x_1 \geq x_2 \geq \dots \geq x_{t+1} > 0$. Note that H^* contains $[t]^{(3)}$ and the number of the edges in H^* is $\binom{t}{3} + \binom{t-1}{2}$. Hence $L_\beta(H^*) = L_\beta([t]^{(3)})$ by Theorem 2.16. We claim \vec{x} is not an optimal weight for H^* . So $L_\beta(H^*, \vec{x}) < L_\beta(H^*) = L_\beta([t]^{(3)})$. To show this we prove that an optimal weighting for H^* must have t positive weights which contradicts to \vec{x} has $t+1$ positive weights. Clearly, the optimal weighting for H^* has at least t positive weights. Let $\vec{y} = (y_1, y_2, \dots, y_{t+1})$ be an optimal weighting for H^* . Note that H^* is left-compressed. Hence $\vec{y} = (y_1, y_2, \dots, y_{t+1})$ satisfies $y_1 \geq y_2 \geq \dots \geq y_{t+1} \geq 0$. Suppose $y_{t+1} > 0$ for a contradiction. Let $G = H^* \setminus \{1t(t+1)\} \cup \{(t-2)(t-1)t\}$. Since G contains $[t]^{(3)}$ and the number of the edges in G is $\binom{t}{3} + \binom{t-1}{2}$, we have $L_\beta(G) = L_\beta([t]^{(3)})$ by Theorem 2.16. Clearly,

$$(39) \quad L_\beta(G, \vec{y}) - L_\beta(H^*, \vec{y}) = y_{t-2}y_{t-1}y_{t+1} - y_1y_t y_{t+1}.$$

Using Remark 3.5(b), we have

$$(40) \quad y_1 = y_t + \frac{(1+\beta)(y_2 + \dots + y_{t-1})y_{t+1}}{y_2 + \dots + y_{t-1} + y_{t+1}} < y_t + (1+\beta)y_{t+1},$$

$$(41) \quad y_1 = y_{t-2} + \frac{(1+\beta)(y_{t-1} + y_t)y_{t+1}}{y_2 + \dots + y_{t-3} + y_{t-1} + y_t + y_{t+1}},$$

and

$$(42) \quad y_{t-1} = y_t + \frac{(1+\beta)(y_2 + \dots + y_{t-3})y_{t+1}}{y_1 + \dots + y_{t-2} - \beta y_{t+1}}.$$

Combing (40), (41) and (42), we have

$$(43) \quad 0 < y_1 - y_{t-2} < y_{t-1} - y_t$$

for $t \geq 6$ (We have $y_{t-2}y_{t-1}y_{t+1} - y_1y_t y_{t+1} > 0$ for $t \leq 5$ by a direction calculation). Applying (43) to (39), we have

$$\begin{aligned} L_\beta(G, \vec{y}) - L_\beta(H^*, \vec{y}) &= y_{t-2}y_{t-1}y_{t+1} - y_1y_t y_{t+1} \\ &= [(y_{t-1} - y_t)y_{t-2} - (y_1 - y_{t-2})y_t]y_{t+1} \\ &> (y_1 - y_{t-2})(y_{t-2} - y_t)y_{t+1} \geq 0 \end{aligned}$$

for $y_{t+1} > 0$. Hence $L_\beta(H^*, \vec{y}) < L_\beta(G, \vec{y}) \leq L_\beta([t]^{(3)}) = L_\beta(H^*)$. This contradicts to \vec{y} is an optimal weighting for H^* . Hence $y_{t+1} = 0$.

So we can assume that H' is left-compressed. Clearly H' does not contain $[t]^{(3)}$. Since H' is left-compressed, then H' does not contain a clique of order t . Therefore we get a left-compressed extremal graph. Hence we can assume that H is left -compressed. This completes the proof of Lemma 4.21. \square

Proof of Theorem 4.6. Let m and t be a positive integers satisfying $\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t-1}{2} - \frac{1}{2}\lceil \frac{t-1}{1+\beta} \rceil$. Let $\beta \geq 0$ be a constant. Let H be a 3-graph with m edges not containing a clique order of t such that $L_\beta(H) = L_{(\beta, m, t)}^{(3)-}$. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for H and k be the number of non-zero weights in \vec{x} . By Lemma 4.21 we can assume that H is left-compressed and $x_1 \geq x_2 \geq \dots \geq x_k > x_{k+1} = \dots = x_n = 0$. So there is an edge e containing both $k-1$ and k by Lemma 3.2(a). Recalling that H is left-compressed, we have $1(k-1)k \in E$. Let $b := \max\{i : i(k-1)k \in E\}$. If $k \leq t$, then $L_\beta(H) < L_\beta([t]^{(3)})$. Hence we assume $k \geq t+1$. By Remark 4.20 and Claim 4.12, we only need to consider the case of $k = t+1$. Now we need the following lemma.

Lemma 4.22 ([22]). *Let H be a 3-graph on the vertex set $[t+1]$. Let $\vec{x} = (x_1, x_2, \dots, x_{t+1})$ be an optimal weighting for the graph-Lagrangian of H satisfying $x_1 \geq x_2 \geq \dots \geq x_{t+1} \geq 0$. Then $x_1 < x_{t-2} + x_{t-1}$ or $\lambda(H) \leq \frac{1}{6} \frac{(t-2)^2}{(t-1)t} < \lambda([t]^{(3)})$.*

Let $D = [t]^{(3)} \setminus E$. Also let $b = |E_{t(t+1)}|$ with a little notation abuse. By Lemma 4.11 and Remark 4.18, we have $|D| \leq \lceil \frac{2b}{1+\beta} \rceil$. So $\lfloor \frac{|D|}{2} \rfloor \leq b$ and the triples $1t(t+1), \dots, \lfloor \frac{|D|}{2} \rfloor t(t+1)$ are in H . Let $H' = H \cup D \setminus \{1t(t+1), \dots, \lfloor \frac{|D|}{2} \rfloor t(t+1)\}$. By Lemma 4.9 and Remark 4.18, we have $|D| \leq \lceil \frac{t-1}{1+\beta} \rceil$. So

$$\begin{aligned} |H'| &= |H| + |D| - \lfloor \frac{|D|}{2} \rfloor \\ &\leq \binom{t}{3} + \binom{t-1}{2} - \frac{1}{2} \lceil \frac{t-1}{1+\beta} \rceil - 1 + \lceil \frac{t-1}{1+\beta} \rceil - \frac{1}{2} \lceil \frac{t-1}{1+\beta} \rceil + 1 \\ &= \binom{t}{3} + \binom{t-1}{2}. \end{aligned}$$

Note that H' contains $[t]^{(3)}$. By Remark 2.15, we have $L_\beta(H', \vec{x}) \leq L_\beta(H')$.

Next we show that $L_\beta(H, \vec{x}) < L_\beta(H', \vec{x})$. By Remark 3.5(b), $x_1 = x_2 = \dots = x_{\lfloor \frac{|D|}{2} \rfloor}$. If $\lambda(H) < \lambda([t]^{(3)})$, then $L_\beta(H) < L_\beta([t]^{(3)})$ and Theorem 4.6 holds. Otherwise we have $x_1 < x_{t-2} + x_{t-1}$ by Lemma 4.22. Hence

$$\begin{aligned} L_\beta(H', \vec{x}) - L_\beta(H, \vec{x}) &= (1 + \beta)(\lambda(D, \vec{x}) - \lfloor \frac{|D|}{2} \rfloor x_1 x_t x_{t+1}) \\ &\geq (1 + \beta)(|D| x_{t-2} x_{t-1} x_t - \lfloor \frac{|D|}{2} \rfloor x_1 x_t x_{t+1}) \\ &\geq (1 + \beta)(|D| x_{t-2} x_{t-1} x_t - \lfloor \frac{|D|}{2} \rfloor (x_{t-2} + x_{t-1}) x_t x_{t+1}). \end{aligned}$$

Recalling that $x_1 \geq x_2 \geq \dots \geq x_{t+1}$, we have

$$\begin{aligned} |D| x_{t-2} x_{t-1} x_t - \lfloor \frac{|D|}{2} \rfloor (x_{t-2} + x_{t-1}) x_t x_{t+1} &> |D| x_{t-2} x_{t-1} x_t - |D| x_{t-2} x_t x_{t+1} \\ &\geq 0. \end{aligned}$$

Hence $L_\beta(H, \vec{x}) < L_\beta(H', \vec{x}) \leq L_\beta([t]^{(3)}) = \lambda([t]^{(3)})$. This completes the proof of Theorem 4.6. \square

Proof of Theorem 4.7. Let m and t be a positive integers satisfying $\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t-1}{2}$. Let $\beta \geq t - 4$ be a constant. Let H be a 3-graph with m edges not containing a clique order of t such that $L_\beta(H) = L_{(\beta, m, t)}^{(3)-}$. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for H and k be the number of non-zero weights in \vec{x} . By Lemma 4.21 we can assume that H is left-compressed and $x_1 \geq x_2 \geq \dots \geq x_k > x_{k+1} = \dots = x_n = 0$. So there is an edge e containing both $k-1$ and k by Lemma 3.2(a). Recalling that H is left-compressed, we have $1(k-1)k \in E$. Let $b := \max\{i : i(k-1)k \in E\}$. If $k \leq t$, then $L_\beta(H) < L_\beta([t]^{(3)})$. Hence we assume $k \geq t+1$. Using Remark 4.18 and Claim 4.12, we have $k = t+1$.

Clearly $b \leq t-3$, otherwise $m \geq \binom{t}{3} + \binom{t-1}{2} + 1$ since H is left-compressed which contradicts to $\binom{t}{3} \leq m \leq \binom{t}{3} + \binom{t-1}{2}$.

By Lemma 4.14, if $b \leq 3$, then $L_\beta(H) \leq \lambda(H) < \lambda([t]^{(3)})$. Hence we can assume $4 \leq b \leq t-3$. By Remark 4.18, Lemma 4.9 and 4.11 and similar to (29), we have

$$\begin{aligned} m = |E| &\geq \binom{t}{3} + \binom{t-1}{2} - \lceil \frac{t-1}{1+\beta} \rceil - \lceil \frac{b}{1+\beta} \rceil + b \\ &\geq \binom{t}{3} + \binom{t-1}{2} + 1 \end{aligned}$$

for $\beta \geq t-4$. This contradicts to $m \leq \binom{t}{3} + \binom{t-1}{2}$. We complete the proof of Theorem 4.7. \square

5. Concluding remarks

At this moment, we are not able to extend the arguments in this paper to verify Conjectures 2.5, 2.10, and 2.11 for $r \geq 4$. When $r \geq 4$, the computation is more complex. If there is some technique to overcome this difficulty, then the idea used in proving Theorems 4.1-4.8. can be used to improve our results much further.

The potential applications of the results in this paper are in the areas of polynomial optimization, extremal graph theory, and providing heuristics for the maximum clique problem which has important applications in different domains.

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