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POSITIVE SOLUTIONS TO p-KIRCHHOFF-TYPE ELLIPTIC EQUATION WITH GENERAL SUBCRITICAL GROWTH

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Abstract. In this paper, we study the existence of positive solutions to the p-Kirchhoff elliptic equation involving general subcritical growth

$$(a+\lambda\int_{\mathbb{R}^N}|\nabla u|^p dx+\lambda b\int_{\mathbb{R}^N}|u|^p dx)(-\Delta_p u+b|u|^{p-2}u)=h(u),\ \text{in } \mathbb{R}^N,$$

where $a,\ b>0$, λ is a parameter and the nonlinearity $h(s)$ satisfies the weaker conditions than the ones in our known literature. We also consider the asymptotics of solutions with respect to the parameter λ.

1. Introduction

In this paper, we study the existence of positive solutions to p-Kirchhoff-type problem

(1.1)
$$\begin{cases}(a+\lambda\int_{\mathbb{R}^N}|\nabla u|^p dx+\lambda b\int_{\mathbb{R}^N}|u|^p dx)(-\Delta_p u+b|u|^{p-2}u)=h(u),\ \text{in } \mathbb{R}^N,\\ u(x)>0,\ x\in\mathbb{R}^N,\ u(x)\in W^{1,p}(\mathbb{R}^N),\end{cases}$$

where a and b are positive constants, $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1<p<N$, $\lambda>0$ is a parameter and the general nonlinearity $h(s)$ satisfies the following conditions:

(h_1) $\lim\limits_{s\to+\infty}\frac{h(s)}{s^{p^*-1}}=0$, with $p^*=\frac{pN}{N-p}$;

(h_2) $h\in C(\mathbb{R}_+,\mathbb{R}_+)$ with $\mathbb{R}_+=[0,+\infty)$ and $\lim\limits_{s\to 0}\frac{h(s)}{s^{p-1}}=0$;

(h_3) there exists $\xi>0$ such that $G(\xi)=\int_0^\xi g(s)ds>0$, where $g(s)=h(s)-ab|s|^{p-2}s$.

When $p=2$ in the problem (1.1), the equation reduces to

$$(a+\lambda\textstyle\int_{\mathbb{R}^N}|\nabla u|^2dx+\lambda b\int_{\mathbb{R}^N}|u|^2dx)(-\Delta u+bu)=h(u),\ x\in\mathbb{R}^N. \tag{1.2}$$

The problem (1.2) on a bounded domain $\Omega\subset\mathbb{R}^N$ is viewed as the Kirchhoff-type problem which was proposed by Kirchhoff [15]. Kirchhoff-type problem

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is linked with a generalization of well-known D'Alembert's wave equations for free vibration of elastic strings, especially, considering the changes in length of string produced by transverse oscillations. In addition, the problem (1.2) also models several biological systems, where u describes a process which depends on the average of itself (see [1]).

In recent years, Kirchhoff-type problems in $\mathbb{R}^N$ have been studied by many authors, for example, see [13, 16, 17, 18] and references therein. In [16], Li et al. utilized a cut-off functional to obtain the bounded Palais-Smale sequences and proved the existence of a positive solution to the Kirchhoff-type problem (1.2). Subsequently, in [17], Liu, Liao, and Tang also considered problem (1.2) under the following conditions:

(g_1) $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\mathbb{R}_+ = [0, +\infty)$ and $\lim_{s\to 0} \frac{h(s)}{s} = 0$;

(g_2) $\lim_{s\to+\infty} \frac{h(s)}{s^{2^*-1}} = 0$, with $2^* = \frac{2N}{N-2}$;

(g_3) there exists $\eta > 0$, such that $H(\eta) = \int_0^\eta h(t)dt \geq \frac{ab}{2}\eta^2$.

The conditions (g_1)-(g_3) are weaker than the ones in [16]. The result in [17] covered the asymptotically linear case and superlinear case at infinity.

For the p-Kirchhoff-type problem (1.1), there have been some results. In [9], Cheng and Dai proved the existence of positive solutions for p-Kirchhoff type problem under the following assumptions:

(f_1) there exists a $C > 0$ such that $|h(t)| \leq C(|t|^{p-1} + |t|^{q-1})$ for all $t \geq 0$ and some $q \in (p, p^*)$, here $p^* = \frac{pN}{N-p}$;

(f_2) $\lim_{t\to 0_+} \frac{h(t)}{t^{p-1}} = 0$;

(f_3) $\lim_{t\to+\infty} \frac{h(t)}{t^{p-1}} = +\infty$. For more results, we refer the reader to [2, 7, 6] and the references therein.

In this paper, we are motivated by [9, 16, 17] and study the existence of positive solutions to the problem (1.1). We will adopt the totally different approaches with the ones (namely, cut-off functional techniques and monotonicity methods) in [9, 16, 17] to obtain bounded (PS) sequence. We believe that the conditions (h_1)-(h_3) on the general nonlinearity h are almost optimal.

In order to state the results clearly, we introduce some Sobolev spaces. Denote $W^{1,p}(\mathbb{R}^N)$ be the usual Sobolev space equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + b|u|^p)dx\right)^{\frac{1}{p}}$$

and

$$D^{1,p}(\mathbb{R}^N) := \{u \in L^{p^*}(\mathbb{R}^N); \nabla u \in L^p(\mathbb{R}^N)\}$$

endowed with the norm $\|u\|_{D^{1,p}} = (\int_{\mathbb{R}^N} |\nabla u|^p dx)^{\frac{1}{p}}$. Let $W_r^{1,p}(\mathbb{R}^N)$ be the subspace of $W^{1,p}(\mathbb{R}^N)$ of radially symmetric functions. $\|u\|_q = (\int_{\mathbb{R}^N} |u|^q dx)^{\frac{1}{q}}$ for $q \geq 1$ with $u \in L^q(\mathbb{R}^N)$. C_i denote positive constants, $i = 1, 2, \ldots$. S and C_q denote the best constants of Sobolev embeddings $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ and

$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$,

$$S(\textstyle\int_{\mathbb{R}^N} |u|^{p^*}dx)^{p/p^*} \leq \int_{\mathbb{R}^N} |\nabla u|^p dx \ \text{ for all } u \in D^{1,p}(\mathbb{R}^N), \tag{1.3}$$

$$C_q(\textstyle\int_{\mathbb{R}^N} |u|^q dx)^{p/q} \leq \int_{\mathbb{R}^N} (|\nabla u|^p + b|u|^p)dx \ \text{ for all } u \in W^{1,p}(\mathbb{R}^N). \tag{1.4}$$

The following theorem is the first main result in the paper.

Theorem 1.1. *Assume that* (h_1)-(h_3) *hold. There exists a constant* $\lambda_0 > 0$ *such that, for any* $\lambda \in (0, \lambda_0)$, *the problem* (1.1) *admits at least one positive solution* u_λ.

Remark 1.1. Theorem 1.1 covers the result in [17]. Indeed, when $p = 2$, Theorem 1.1 is the main result in [17].

Remark 1.2. We easily prove that the conditions (f_1) and (f_3) in [9] are stronger than the ones (h_1) and (h_3) respectively. In this sense, we improve the main result in [9].

When $\lambda = 0$ in the equation (1.1), the problem reduces to

$$-a\Delta_p u + ab|u|^{p-2}u = h(u), \ x \in \mathbb{R}^N. \tag{1.5}$$

Thw problem (1.5) is viewed as the limit problem of (1.1) when $\lambda \to 0$. We can now state the second main result in this paper.

Theorem 1.2. *If the general nonlinearity h satisfies* (h_1)-(h_3), *then, as* $\lambda \to 0$, u_λ *converges to u in* $W_r^{1,p}(\mathbb{R}^N)$, *where u is a ground state solution to the problem* (1.5).

Remark 1.3. In order to prove the existence of a ground state solution for the problem (1.5), the assumptions (h_1), (h_3) and the additional condition

$(h_2^{'})$ there exists some $q \in (p-1, p^*-1)$ such that

$$\lim_{t\to\infty} \sup \frac{h(t)}{t^q} < \infty$$

were already used by Berestycki and Lions [3], for $p = 2$, and by J. M. do Ó and E. Medeiros [12], for the $1 < p \leq N$ case. Obviously, the condition (h_2) in this paper is weaker than the one $(h_2^{'})$. In this sense, we improve the results in [3, 12].

The rest of the paper is organised as follows. In Section 2, we prove that the limit problem (1.5) has at least a ground state solution. In Section 3, we will find a solution in some neighborhood of the solutions to the limit problem (1.5). Indeed, we view the problem (1.1) as the perturbed problem of (1.5) if λ is sufficiently small. Because of the lack of Ambrosetti-Rabinowitz condition, we use a local deformation approach from Byeon and Jeanjean [4] to obtain a bounded (PS) sequence. In addition, due to the appearance of nonlocal terms $\int_{\mathbb{R}^N} |\nabla u|^p dx$ and $\int_{\mathbb{R}^N} |u|^p dx$, we make a crucial modification on the min-max value which is defined by C_λ, where all paths are requested to be uniformly bounded with respect to λ. Finally, we give the proofs of the main results.

2. Existence of ground state solutions to limit problem

In this section, we prove that the problem (1.5) has at least one ground state solution. Since we consider the positive solutions, we can assume that $h(s) = 0$ for $s \leq 0$. Meanwhile, as the problems (1.1) and (1.5) are autonomous, we can work in $W_r^{1,p}(\mathbb{R}^N)$ (see Theorem 1.28 in [20]). Define the energy functionals of problems (1.1) and (1.5) respectively by

$$I_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{2p}\|u\|^{2p} - \int_{\mathbb{R}^N} H(u)dx$$

and

$$I(u) = \frac{a}{p}\|u\|^p - \int_{\mathbb{R}^N} H(u)dx,$$

where $u \in W_r^{1,p}(\mathbb{R}^N)$ and $H(t) = \int_0^t h(s)ds$.

By the conditions (h_1)-(h_3), we can prove that $I_\lambda,\ I \in C^1(W_r^{1,p}(\mathbb{R}^N), \mathbb{R})$. Indeed, the weak solutions of the problem are the critical points of the corresponding energy functional.

Proposition 2.1. *Suppose that (h_1)-(h_3) hold. Then the limit problem* (1.5) *has at least one ground state solution $u \in W_r^{1,p}(\mathbb{R}^N)$.*

In order to prove the main results, we need the following lemmas.

Lemma 2.2 (Pohozǎev equality)**.** *If u is a nontrivial solution of the equation*

$$a(-\Delta_p u + b|u|^{p-2}u) = h(u),\ x \in \mathbb{R}^{\mathrm{N}},$$

then u satisfies the following Pohozǎev equality

$$\frac{a(N-p)}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx = N\int_{\mathbb{R}^N} G(u)dx,\ \text{where } G(u) = H(u) - \frac{ab}{p}|u|^p.$$

Proof. The proof is similar to the one of Lemma 2.6 in [16]. We omit the details. □

For convenience, we give the following notations.

$$\mathcal{L} := \{u \in W_r^{1,p}(\mathbb{R}^N)\backslash\{0\} : \int_{\mathbb{R}^N} G(u)dx = 1\}$$

and

$$\mathcal{P} := \{u \in W_r^{1,p}(\mathbb{R}^N)\backslash\{0\} : \frac{a(N-p)}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx = N\int_{\mathbb{R}^N} G(u)dx\}.$$

From (h_3), we have $\mathcal{L} \neq \emptyset$ and $\mathcal{P} \neq \emptyset$. Set $L = \frac{1}{p}\inf\limits_{u\in\mathcal{L}} \|\nabla u\|_p^p$, $\beta_0 = \inf\limits_{u\in\mathcal{P}} I(u)$ and the mountain pass value

$$k = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], W_r^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, I(\gamma(1)) < 0\}$.

Lemma 2.3. *Assume that* (h_1)-(h_3) *hold. Then* $\beta_0 \le k$ *and*

$$\beta_0 = \frac{p}{N-p}(\frac{a(N-p)}{N})^{\frac{N}{p}} L^{\frac{N}{p}}.$$

Proof. In order to prove $\beta_0 \le k$, it suffices to prove that $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$ for all $\gamma \in \Gamma$.
Set

$$P(u) = \frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{N}{a}\int_{\mathbb{R}^N} G(u)dx.$$

By (h_1) and (h_2), we easily obtain that there exists $\rho > 0$ such that $P(u) > 0$, $0 < \|u\| \le \rho$. For any $\gamma \in \Gamma$, we get $P(\gamma(0)) = 0$ and $P(\gamma(1)) \le \max\{\frac{N-p}{p}, \frac{N}{a}\} I(\gamma(1)) < 0$. Thus, there exists a $t_0 \in (0,1)$ such that $P(\gamma(t_0)) = 0$ with $\|\gamma(t_0)\| > \rho$. This implies that $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$ for all $\gamma \in \Gamma$.

In the following, we prove that $\beta_0 = \frac{p}{N-p}(\frac{a(N-p)}{N})^{\frac{N}{p}} L^{\frac{N}{p}}$. Firstly, we claim that $L > 0$. In fact, if $L = 0$, there is $\{u_n\} \subset \mathcal{L}$ with $\int_{\mathbb{R}^N} G(u_n)dx = 1$ such that $\|\nabla u_n\|_p \to 0$ as $n \to \infty$. From the Sobolev's embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, we have $\|u_n\|_{p^*} \to 0$ as $n \to \infty$. Together with the assumptions (h_1) and (h_2), we get

$$\lim_{n\to\infty} \sup \int_{\mathbb{R}^N} G(u_n)dx \le \lim_{n\to\infty} \sup C_1 \int_{\mathbb{R}^N} |u_n|^{p^*} dx = 0.$$

This is a contradiction with $\int_{\mathbb{R}^N} G(u_n)dx = 1$. Thus, $L > 0$. For any $u \in \mathcal{L}$, define $(\Phi_t(u))(x) = u(\frac{x}{t})$, $T(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx$ and $V(u) = \int_{\mathbb{R}^N} G(u)dx$. We have

$$T(u(\frac{x}{t})) = t^{N-p} T(u)$$

and

$$V(u(\frac{x}{t})) = t^N \int_{\mathbb{R}^3} G(u)dx.$$

Thus, choosing $t_u = (\frac{a(N-p)}{Np})^{\frac{1}{p}} \|\nabla u\|_p$, we get that Φ_{t_u} is a bijection from $\mathcal{L}$ to $\mathcal{P}$. For any $u \in \mathcal{L}$,

$$\begin{aligned} I(\Phi_{t_u}(u)) &= a t_u^{N-p} T(u) - t_u^N V(u) \\ &= \frac{p}{N-p}(\frac{a(N-p)}{Np})^{\frac{N}{p}} \|\nabla u\|_p^N. \end{aligned}$$

Furthermore,

$$\inf_{u\in\mathcal{P}} I(u) = \inf_{u\in\mathcal{L}} I(\Phi_{t_u}(u)),$$

which implies that

$$\beta_0 = \frac{p}{N-p}(\frac{a(N-p)}{N})^{\frac{N}{p}} L^{\frac{N}{p}}.$$

□

Lemma 2.4. *If* $h \in C(\mathbb{R}^N \times \mathbb{R})$ *and assume that*

$$\lim_{t\to 0} \frac{h(x,t)}{t^{p-1}} = 0$$

and

$$\lim_{t\to\infty}\sup\frac{|h(x,t)|}{|t|^{p^*-1}}<\infty$$

hold uniformly in $x\in\mathbb{R}^N$. *For any* $\{u_n\}$ *with* $u_n\to u_0$ *weakly in* $W^{1,p}(\mathbb{R}^N)$ *and* $u_n\to u_0$ *a.e. in* $\mathbb{R}^N$, *we have*

$$\int_{\mathbb{R}^N}H(x,u_n)dx=\int_{\mathbb{R}^N}(H(x,u_n-u_0)+H(x,u_0))dx+o(1),$$

where $H(x,t)=\int_0^t h(x,s)ds$.

Proof. For the subcritical case, we refer to the reference [11]. We omit the details. $\square$

Proof of Proposition 2.1. Assume that there exists $\{u_n\}\subset W_r^{1,p}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N}G(u_n)dx=1$ and $\int_{\mathbb{R}^N}|\nabla u_n|^p dx\to pL$ as $n\to\infty$. By the conditions (h_1) and (h_2), we get that $\|u_n\|_p$ is bounded. So, $\{u_n\}$ is bounded in $W_r^{1,p}(\mathbb{R}^N)$. We may assume that $u_n\to u^*$ weakly in $W_r^{1,p}(\mathbb{R}^N)$. By Lemma 2.4, we have

$$\int_{\mathbb{R}^N}H(u_n)dx=\int_{\mathbb{R}^N}H(u_n-u^*)dx+\int_{\mathbb{R}^N}H(u^*)dx+o(1). \tag{2.1}$$

From the conditions $(h_1)-(h_2)$, for any $\xi>0$, there exists $C_\xi>0$ such that

$$H(s)\le\xi|s|^p+\xi|s|^{p^*}+C_\xi|s|^{k_0},\ k_0\in(p,p^*).$$

Thus

$$\begin{aligned}
&|\int_{\mathbb{R}^N}H(u_n-u^*)dx|\\
&\le\xi\int_{\mathbb{R}^N}|u_n-u^*|^p dx+\xi\int_{\mathbb{R}^N}|u_n-u^*|^{p^*}dx+C_\xi\int_{\mathbb{R}^N}|u_n-u^*|^{k_0}dx\\
&=\xi J_1+\xi J_2+C_\xi J_3,
\end{aligned}$$

where

$$J_1=\int_{\mathbb{R}^N}|u_n-u^*|^p dx,$$
$$J_2=\int_{\mathbb{R}^N}|u_n-u^*|^{p^*}dx$$

and

$$J_3=\int_{\mathbb{R}^N}|u_n-u^*|^{k_0}dx.$$

From the Sobolev's imbedding $W_r^{1,p}(\mathbb{R}^N)\hookrightarrow L^{k_0}(\mathbb{R}^N)$, $k_0\in[p,p^*]$, we obtain $\|u_n\|_{k_0}$ is bounded. In connection with Minkowski inequality, one has

$$|J_1|,\ |J_2|\le C_1,\ \text{where } C_1>0.$$

In addition, since the imbedding $W_r^{1,p}(\mathbb{R}^N)\hookrightarrow L^{k_0}(\mathbb{R}^N)$, $k_0\in(p,p^*)$ is compact, we have $J_3\to0$ as $n\to\infty$. So, $\int_{\mathbb{R}^N}H(u_n-u^*)dx\to0$ as $n\to\infty$. Furthermore, it follows from (2.1) that

$$\int_{\mathbb{R}^N}H(u_n)dx=\int_{\mathbb{R}^N}H(u^*)dx+o(1). \tag{2.2}$$

Next, since $u_n \to u^*$ weakly in $W_r^{1,p}(\mathbb{R}^N)$, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p dx \geq \int_{\mathbb{R}^N}|u^*|^p dx.$$

Then

$$\begin{aligned}1 = \lim_{n\to\infty}\int_{\mathbb{R}^N} G(u_n)dx &= \lim_{n\to\infty}\int_{\mathbb{R}^N}(H(u_n) - \frac{ab}{p}|u_n|^p)dx \\ &\leq \int_{\mathbb{R}^N}(H(u^*) - \frac{ab}{p}|u^*|^p)dx \\ &= \int_{\mathbb{R}^N} G(u^*)dx.\end{aligned}$$

Case 1. $V(u^*) = \int_{\mathbb{R}^N} G(u^*)dx = 1$. Combining with $\int_{\mathbb{R}^N} G(u_n)dx = 1$ and (2.2), then we have

$$\|u_n\|_p^p \to \|u^*\|_p^p \text{ as } n \to \infty. \tag{2.3}$$

Since $T(u_n) \to L$ as $n \to \infty$ and $T(u^*) = L$, we obtain

$$\|\nabla u_n\|_p^p \to \|\nabla u^*\|_p^p \text{ as } n \to \infty. \tag{2.4}$$

It follows from (2.3) and (2.4) that $\|u_n\| \to \|u^*\|$ as $n \to \infty$. Therefore,

$$u_n \to u^* \text{ strongly in } W_r^{1,p}(\mathbb{R}^N) \text{ as } n \to \infty.$$

Case 2. $V(u^*) = \int_{\mathbb{R}^N} G(u^*)dx > 1$. There exists $t_0 > 0$ such that

$$\int_{\mathbb{R}^N} G(u(\frac{x}{t_0}))dx = 1.$$

Together with $V(u(\frac{x}{t_0})) = t_0^N V(u) = 1$, we get that $t_0 = (V(u))^{-\frac{1}{N}}$. Then we have

$$T(u(\frac{x}{t_0})) = t_0^{N-p}T(u) \geq L.$$

Namely,

$$\begin{aligned}T(u) &\geq t_0^{-(N-p)}L \\ &\geq (V(u))^{\frac{N-p}{N}}L \\ &> L.\end{aligned}$$

This is a contradiction with $T(u) \leq L$. So we obtain that $V(u^*) = 1$ and $T(u^*) = L$. Setting $t_{u^*} = (\frac{a(N-p)}{Np})^{\frac{1}{p}}\|\nabla u^*\|_p$, it follows from Coleman, Glazer and Martin [10] that $w = u^*(\frac{x}{t_{u^*}}) \in \mathcal{P}$ is a ground state solution to the limit problem (1.5). □

Let A_r be the set of the radial ground state solution U of the problem (1.5). From Proposition 2.1, we know that $A_r \neq \emptyset$.

Lemma 2.5. *A_r is compact in $W_r^{1,p}(\mathbb{R}^N)$.*

Proof. For any sequence $\{u_n\} \subset A_r$, it follows from similar arguments [4] that u_n is a minimizer of $T(u)$ on the set

$$\{u \in W_r^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} G(u) = \beta_1\},$$

where $\beta_1 = (\frac{a(N-p)L}{N})^{\frac{N}{p}}$.

Set $v_n(x) = u_n(\beta_1^{\frac{1}{N}} x)$, then v_n is a minimizer of $T(u)$ on $\mathcal{L}$. Namely, $\|\nabla v_n\|_p^p \to pL$ with $\int_{\mathbb{R}^N} G(v_n)dx = 1$. From the conditions (h_1)-(h_2) and the Sobolev's imbedding theorem, we can prove that $\{v_n\}$ is bounded in $W_r^{1,p}(\mathbb{R}^N)$. Similar arguments in Proposition 2.1 show that there exists $v_0 \in \mathcal{L}$ such that $v_n \to v_0$ strongly in $W_r^{1,p}(\mathbb{R}^N)$. Furthermore, we can obtain that $u_n \to u_0$ in A_r, where $u_0 = v_0(\beta_1^{-\frac{1}{N}} x)$. The proof is completed. □

Lemma 2.6. *The mountain pass value corresponds with the least energy level, namely,* $k = \beta_0 = I(u_0)$, *where* $u_0 \in A_r$.

Proof. By the assumptions (h_1)-(h_3), we know that the mountain pass value k is well defined. On the one hand, we get that $\beta_0 \le k$. On the other hand, since u_0 is a ground state solution to the limit problem (1.5), we adopt the similar idea in [5] and can prove that there exists a path $\gamma \in \Gamma$ satisfying $\gamma(0) = 0$, $I(\gamma(1)) < 0$ and $\max_{t\in[0,1]} I(\gamma(t)) = I(u_0)$. This implies that $k \le \beta_0$. The proof is completed. □

3. Proofs of main results

Set $U_t(x) = U(\frac{x}{t})$, $U \in A_r$. By Lemma 2.2, we have

$$\begin{aligned} I(U_t) &= \frac{a}{p}\int_{\mathbb{R}^N} |\nabla U_t|^p dx - \int_{\mathbb{R}^N} G(U_t)dx \\ &= \frac{a}{p} t^{N-p} \int_{\mathbb{R}^N} |\nabla U|^p dx - t^N \int_{\mathbb{R}^N} G(U)dx \\ &= (\frac{a}{p} t^{N-p} - \frac{a(N-p)}{Np} t^N) \int_{\mathbb{R}^N} |\nabla U|^p dx. \end{aligned}$$

This shows that $I(U_t) \to -\infty$ as $t \to \infty$. Thus, there exists $t_1 > 1$ such that $I(U_t) < -3$ for $t \in [t_1, +\infty)$.

Define $D_\lambda = \max_{t\in[0,t_1]} I_\lambda(U_t)$. By Lemma 2.5 and Lemma 2.6, we can get that

$$\lim_{\lambda\to 0} D_\lambda = k.$$

In order to get the uniformly bounded set of the mountain pathes, we give the following result.

Lemma 3.1. *There exist* $\lambda_0 > 0$ *and* $C_2 > 0$, *such that for any* $\lambda \in (0, \lambda_0)$, $I_\lambda(U_{t_1}) < -3$, $\|U_t\| \le C_2$, $\forall t \in (0, t_1]$ *and* $\|U\| \le C_2$, $U \in A_r$.

Proof. By Lemma 2.5, there is a constant $C_3 > 0$ such that $\|U\| \le C_3$ for any $U \in A_r$. Meanwhile,

$$\begin{aligned}\|U_t\|^p &= t^{N-p}\|\nabla U\|_p^p + t^N\|U\|_p^p \\ &\le (t^{N-p} + t^N)\|U\|^p \\ &\le ((t_1)^{N-p} + (t_1)^N)C_3^p.\end{aligned}$$

We choose $C_2 = \max\{C_3, ((t_1)^{N-p} + (t_1)^N)^{\frac{1}{p}} C_3\}$ and obtain that

$$\|U\|,\ \|U_t\| \le C_2 \ \text{ for any } U \in A_r.$$

Furthermore

$$\begin{aligned}I_\lambda(U_{t_1}) &= I(U_{t_1}) + \frac{\lambda}{2p}\|U_{t_1}\|^{2p} \\ &\le I(U_{t_1}) + \frac{\lambda}{2p}C_2^{2p}.\end{aligned}$$

It follows from $I(U_{t_1}) < -3$ that there exists $\lambda_0 > 0$ such that

$$I_\lambda(U_{t_1}) < -3 \ \text{ for any } \lambda \in (0, \lambda_0).$$

The proof is completed. □

By Lemma 3.1, we will define a min-max value

$$C_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{s\in[0,t_1]} I_\lambda(\gamma(s)),$$

where $\Gamma_\lambda = \{\gamma \in C([0,t_1], W_r^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(t_1) = U_{t_1}, \|\gamma(t)\| \le C_2 + 2\}$. Obviously, $\Gamma_\lambda \neq \emptyset$ and $C_\lambda \le D_\lambda$ for $\lambda \in (0, \lambda_0)$.

Lemma 3.2. *One has* $\lim\limits_{\lambda\to 0} C_\lambda = k$.

Proof. It is clear that $C_\lambda \le D_\lambda \to k$ as $\lambda \to 0$. On the other hand, for any $\gamma \in \Gamma_\lambda$, we have $\tilde{\gamma}(\cdot) = \gamma(t_1\cdot) \in \Gamma$. Together with $I_\lambda(u) \ge I(u)$, we obtain that $C_\lambda \ge k$. So, $\lim\limits_{\lambda\to 0} C_\lambda = k$. □

For $\alpha, d > 0$, set

$$I_\lambda^\alpha = \{u \in W_r^{1,p}(\mathbb{R}^N) : I_\lambda(U) \le \alpha\}$$

and

$$A^d = \{u \in W_r^{1,p}(\mathbb{R}^N) : \inf_{v\in A_r} \|u - v\| \le d\}.$$

Obviously, for all $d > 0$, $A^d \neq \emptyset$. In the following, we will find a solution to the problem (1.1) in the neighborhood of A_r for $\lambda > 0$ small enough.

Lemma 3.3. *For any* $\{u_{\lambda_i}\} \subset A^d$ *satisfying* $\lim\limits_{i\to\infty} I_\lambda(u_{\lambda_i}) \le k$ *and* $\lim\limits_{i\to\infty} I_\lambda'(u_{\lambda_i}) = 0$, *there exists* $u_0 \in A^d$ *such that* $u_{\lambda_i} \to u_0$ *strongly in* $W_r^{1,p}(\mathbb{R}^N)$ *as* $i \to \infty$, *where* $\lim\limits_{i\to\infty} \lambda_i = 0$, *provided that*

$$0 < d < \min\{1, (\tfrac{Nk}{a})^{\frac{1}{p}}\}. \tag{3.1}$$

Proof. For convenience, we replace λ_i by λ. Since $u_\lambda \in A^d$, we have $u_\lambda = U_\lambda + v_\lambda$, where $U_\lambda \in A_r$ and $v_\lambda \in W_r^{1,p}(\mathbb{R}^N)$ with $\|v_\lambda\| \le d$. Because A_r is compact, there exist $U_0 \in A_r$ and $v_0 \in W_r^{1,p}(\mathbb{R}^N)$ such that $U_\lambda \to U_0$ strongly in $W_r^{1,p}(\mathbb{R}^N)$, $v_\lambda \to v_0$ weakly in $W_r^{1,p}(\mathbb{R}^N)$ and $v_\lambda \to v_0$ a.e. in $\mathbb{R}^N$. Let $u_0 = U_0 + v_0$, then $u_0 \in A^d$ and $u_\lambda \to u_0$ weakly in $W_r^{1,p}(\mathbb{R}^N)$.

Firstly, we claim that $u_0 \not\equiv 0$. It follows from $\lim\limits_{i\to\infty} I_\lambda^{'}(u_{\lambda_i}) = 0$ that $I^{'}(u_0) = 0$. Otherwise, if $u_0 \equiv 0$, then $\|U_0\| = \|v_0\| \le d$. By Lemma 2.2, we obtain that $\|\nabla U_0\|_p = (\frac{Nk}{a})^{\frac{1}{p}}$. On the other hand, by (3.1), we have

$$\|\nabla U_0\|_p \le \|U_0\| \le d < (\frac{Nk}{a})^{\frac{1}{p}}.$$

This is a contradiction. So $u_0 \not\equiv 0$ and $I(u_0) \ge k$.

Secondly, we prove that $u_{\lambda_i} \to u_0$ strongly in $W_r^{1,p}(\mathbb{R}^N)$. Indeed, $\{u_{\lambda_i}\}$ is a (PS) sequence of I_λ, that is, $\{u_{\lambda_i}\}$ and $\{I_\lambda(u_{\lambda_i})\}$ are bounded, $I_\lambda^{'}(u_{\lambda_i}) \to 0$ as $i \to \infty$. We obtain

$$u_{\lambda_i} \to u_0 \text{ in } L^q(\mathbb{R}^N),\ q \in (p, p^*)$$

and

$$u_{\lambda_i} \to u_0 \ \text{ a.e. in } \mathbb{R}^N.$$

For convenience, we write u_i for u_{λ_i}. By (h_1) and (h_2), for any $\xi > 0$, there exists $C_\xi > 0$ such that

$$|h(t)| \le \xi|t|^{p-1} + \xi|t|^{p^*-1} + C_\xi|t|^{q-1},\ t \in \mathbb{R},\ q \in (p, p^*). \tag{3.2}$$

Thus, by Hölder inequality, we have

$$\begin{aligned}
&|\int_{\mathbb{R}^N} h(u_i)(u_i - u)dx| \\
\le& \int_{\mathbb{R}^N} |h(u_i)||u_i - u|dx \\
\le& \int_{\mathbb{R}^N} (\xi|u_i|^{p-1}|u_i - u| + \xi|u_i|^{p^*-1}|u_i - u| + C_\xi|u_i|^{q-1}|u_i - u|)dx \\
\le& \xi\|u_i\|_p^{p-1}\|u_i - u\|_p + \xi\|u_i\|_{p^*}^{p^*-1}\|u_i - u\|_{p^*} + C_\xi\|u_i\|_q^{q-1}\|u_i - u\|_q \\
=& \xi\delta_1 + \xi\delta_2 + C_\xi\delta_3,
\end{aligned}$$

where

$$\delta_1 = \|u_i\|_p^{p-1}\|u_i - u\|_p,$$

$$\delta_2 = \|u_i\|_{p^*}^{p^*-1}\|u_i - u\|_{p^*}$$

and

$$\delta_3 = \|u_i\|_q^{q-1}\|u_i - u\|_q.$$

By the Sobolev's imbedding $W_r^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ and Minkowski inequality, we get that δ_1 and δ_2 are bounded. It follows from $q \in (p, p^*)$ that $\|u_i - u\|_q \to 0$, namely $\delta_3 \to 0$. Thus, we have

$$\int_{\mathbb{R}^N} h(u_i)(u_i - u)dx \to 0.$$

So

$$(a + \lambda\|u_i\|^p)(u_i, u_i - u) = \langle I_\lambda'(u_i), u_i - u\rangle + \int_{\mathbb{R}^N} h(u_i)(u_i - u)dx \to 0,$$

where $(u_i, u_i - u) = \int_{\mathbb{R}^N}(|\nabla u_i|^{p-2}\nabla u_i \cdot \nabla(u_i - u) + b|u_i|^{p-2}u_i(u_i - u))dx$. Noticing that

$$(a + \lambda\|u_i\|^p) \neq 0,$$

we have

$$(u_i, u_i - u) = 0. \tag{3.3}$$

In addition, together with $u_i \to u$ weakly in $W_r^{1,p}(\mathbb{R}^N)$, we have

$$(u, u_i - u) = 0. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\int_{\mathbb{R}^N}((|\nabla u_i|^{p-2}\nabla u_i - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u_i - u) + b(|u_i|^{p-2}u_i - |u|^{p-2}u)(u_i - u))dx \to 0.$$

Combining with the following standard inequality in $\mathbb{R}^N$ given by

$$\langle|\alpha|^{p-2}\alpha - |\beta|^{p-2}\beta, \alpha - \beta\rangle \geq \begin{cases} C_p|\alpha - \beta|^p, \ p \in [2, +\infty), \\ C_p|\alpha - \beta|^2(|\alpha| + |\beta|)^{p-2}, \ p \in (1, 2), \end{cases}$$

we can prove that $u_n \to u$ strongly in $W_r^{1,p}(\mathbb{R}^N)$. □

By Lemma 3.3, there exists a constant d satisfying (3.1) and $C_4 > 0$, $\lambda_0 > 0$ such that $\|I_\lambda'(u)\| \geq C_4$ for $u \in I_\lambda^{D_\lambda} \cap (A^d \backslash A^{\frac{d}{2}})$ and $\lambda \in (0, \lambda_0)$.

Lemma 3.4. *There exists $C_4 > 0$ such that for small $\lambda > 0$, $I_\lambda(\gamma(s)) \geq C_\lambda - C_4$, this shows that $\gamma(s) \in A^{\frac{d}{2}}$, where $\gamma(s) = U(\frac{\cdot}{s})$, $s \in (0, t_1]$.*

Proof. By Pohozǎev equality,

$$\begin{aligned} I_\lambda(\gamma(s)) &= I(\gamma(s)) + \frac{\lambda}{2p}\|\gamma(s)\|^{2p} \\ &= (\frac{a}{p}t^{N-p} - \frac{(N-p)a}{Np}t^N)\int_{\mathbb{R}^N} |\nabla U|^p dx + \frac{\lambda}{2p}\|U(\frac{\cdot}{s})\|^{2p}. \end{aligned}$$

From Lemma 3.1, we have

$$I_\lambda(\gamma(s)) = (\frac{a}{p}t^{N-p} + \frac{(N-p)a}{Np}t^N)\int_{\mathbb{R}^N} |\nabla U|^p dx + O(\lambda).$$

Noticing that $\max\limits_{s\in(0,t_1]} I(\gamma(s)) = k$ can be achieved at $s = 1$, there exists $C_5 > 0$ so small that $\gamma(s) = U(\frac{\cdot}{s}) \in A^{\frac{d}{2}}$ for $|s - 1| \leq C_5$. Combining with $C_\lambda \to k$ as $\lambda \to 0$, there is $C_4 > 0$ such that

$$I(\gamma(s)) \geq C_\lambda - C_4$$

for $\lambda > 0$ small enough. This implies that $|s-1| \le C_5$ and $\gamma(s) \in A^{\frac{d}{2}}$. □

Lemma 3.5. *For any $\lambda > 0$ small enough, there exists a sequence $\{u_n\} \subset I_\lambda^{D_\lambda} \cap A^d$ such that $I_\lambda'(u_n) \to 0$ as $n \to \infty$.*

Proof. Assume by contradiction, there exists $\beta(\lambda) > 0$ such that $|I_\lambda'(u)| \ge \beta(\lambda)$, $u \in I_\lambda^{D_\lambda} \cap A^d$ for some $\lambda > 0$. Then there exists a pseudo-gradient vector field [19] Φ_λ in $W_r^{1,p}(\mathbb{R}^N)$ on a neighborhood Y_λ of $I_\lambda^{D_\lambda} \cap A^d$ such that

$$\|I_\lambda(u)\| \le 2\min\{1, |I_\lambda'(u)|\}$$

and

$$\langle I_\lambda'(u), \Phi_\lambda(u)\rangle \ge \min\{1, |I_\lambda'(u)|\}|I_\lambda'(u)|.$$

Denote ζ_λ be a Lipschitz continuous function on $W_r^{1,p}(\mathbb{R}^N)$ such that $\zeta_\lambda \in [0,1]$ and

$$\zeta_\lambda(u) = \begin{cases} 1, \ u \in I_\lambda^{D_\lambda} \cap A^d \\ 0, \ u \in W_r^{1,p}(\mathbb{R}^N)\backslash Y_\lambda. \end{cases}$$

Define μ_λ be a Lipschitz continuous function on $\mathbb{R}$ such that $\mu_\lambda \in [0,1]$ and

$$\mu_\lambda(t) = \begin{cases} 1, \ |t - C_\lambda| \le \frac{C_4}{2}, \\ 0, \ |t - C_\lambda| \ge C_4, \end{cases}$$

where C_4 is given in Lemma 3.4. Set

$$\eta_\lambda(u) = \begin{cases} -\zeta_\lambda(u)\mu_\lambda(I_\lambda(u))\Phi_\lambda(u), \ u \in Y_\lambda, \\ 0, \ u \in W_r^{1,p}(\mathbb{R}^N)\backslash Y_\lambda. \end{cases}$$

Then, the following initial value problem

$$\begin{cases} \frac{d}{dt} Z_\lambda(u,t) = \eta_\lambda(Z_\lambda(u,t)), \\ Z_\lambda(u,0) = u, \end{cases}$$

admits a unique global solution $Z_\lambda : W_r^{1,p}(\mathbb{R}^N) \times \mathbb{R}_+ \to W_r^{1,p}(\mathbb{R}^N)$ which satisfies

(i) $Z_\lambda(u,t) = u$, if $t = 0$ or $u \notin Y_\lambda$ or $|I_\lambda(u) - C_\lambda| \ge C_4$;
(ii) $\|\frac{d}{dt} Z_\lambda(u,t)\| \le 2$ for $(u,t) \in W_r^{1,p}(\mathbb{R}^N) \times \mathbb{R}_+$;
(iii) $\frac{d}{dt} I_\lambda(Z_\lambda(u,t)) \le 0$.

We adopt similar idea in [8] and obtain that for any $s \in (0, t_1]$, there is $t_s > 0$ such that

$$Z_\lambda(\gamma(s), t_s) \in I_\lambda^{C_\lambda - \frac{C_4}{2}}, \text{ where } \gamma(s) = U(\frac{\cdot}{s}), \ s \in (0, t_1].$$

Let $\gamma_0(s) = Z_\lambda(\gamma(s), t_*(s))$, where $t_*(s) = \inf\{t \ge 0, \ Z_\lambda(\gamma(s), t) \in I_\lambda^{C_\lambda - \frac{C_4}{2}}\}$. By similar ideas in [8, 21], we can prove that $\gamma_0(s)$ is continuous in $[0, t_1]$ and $\|\gamma_0(s)\| \le C_2 + 2$. Therefore, we have $\gamma_0 \in \Gamma_\lambda$ with $\max\limits_{t \in [0,t_1]} I_\lambda(\gamma_0(t)) \le C_\lambda - \frac{C_4}{2}$. This is a contradiction with $C_\lambda = \inf\limits_{\gamma \in \Gamma_\lambda} \max\limits_{s \in [0,t_1]} I_\lambda(\gamma(s))$. The proof is completed. □

Now, we give the proofs of the main results.

Proof of Theorem 1.1. By Lemma 3.5, there exists a bounded (PS) sequence $\{u_n\} \subset I_\lambda^{D_\lambda} \cap A^d$. Without loss of generality, we may assume that $u_n \to u_\lambda$ weakly in $W_r^{1,p}(\mathbb{R}^N)$. In connection with Lemma 3.3 and Lemma 3.4, we can obtain that $I_\lambda^{'}(u_\lambda) = 0$ and $u_\lambda \in I_\lambda^{D_\lambda} \cap A^d$. Furthermore, it follows from similar arguments in Lemma 3.3 that $u_\lambda \not\equiv 0$ under the proper choice of d satisfying (3.1). By the strong maximum principle, we adopt similar idea in [18] and can prove that u_λ is a positive solution of the problem (1.1). □

Proof of Theorem 1.2. For any $\phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$I_\lambda^{'}(u_\lambda)\phi = I^{'}(u_\lambda)\phi + \lambda\|u_\lambda\|^p \int_{\mathbb{R}^N} |u_\lambda|^{p-2}u_\lambda\phi dx = 0.$$

Then

$$I^{'}(u_\lambda)\phi = -\lambda\|u_\lambda\|^p \int_{\mathbb{R}^N} |u_\lambda|^{p-2}u_\lambda\phi dx \to 0 \text{ as } \lambda \to 0.$$

Combining with that

$$I_\lambda(u_\lambda) = I(u_\lambda) + \frac{\lambda}{2p}\|u_\lambda\|^{2p},$$

we have

$$I(u_\lambda) \le C_\lambda \text{ and } I^{'}(u_\lambda) \to 0 \text{ as } \lambda \to 0.$$

Namely, $\{u_\lambda\}$ is a bounded (PS) sequence for the energy functional I. We may assume that $u_\lambda \to u^*$ weakly in $W_r^{1,p}(\mathbb{R}^N)$, then $I^{'}(u^*) = 0$. Similar proof as the one in Lemma 3.3 demonstrates that $u_\lambda \to u^*$ strongly in $W_r^{1,p}(\mathbb{R}^N)$. By the proper choice of $d > 0$, we can prove that $u^* \not\equiv 0$. Hence $I(u^*) \ge k$. Meanwhile, we have $I(u^*) \le k$ since $I(u_\lambda) \le D_\lambda \to k$ as $\lambda \to 0$. So $I(u^*) = k$. By Lemma 2.6 , u^* is a ground state solution to the limit problem (1.5). The proof is completed. □

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