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GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD WITH A QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract. The object of study in this paper is generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection. We study the geometry of two types of generic lightlike submanifolds, which are called *recurrent* and *Lie recurrent* generic lightlike submanifolds, of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection.

1. Introduction

Yano-Imai [17] introduced the notion of quarter-symmetric metric connection on a Riemannian manifold. Recently, Jin [7, 10] studied the geometry of lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection. We quote Jin's definition in itself as follow:

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a *quarter-symmetric metric connection* if it is metric, *i.e.*, $\bar{\nabla}\bar{g} = 0$ and its torsion tensor $\bar{T}$, defined by $\bar{T}(\bar{X}, \bar{Y}) = \bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X} - [\bar{X}, \bar{Y}]$, satisfies

$$\bar{T}(\bar{X}, \bar{Y}) = \theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}, \tag{1.1}$$

where J is a $(1, 1)$-type tensor field on $\bar{M}$ and θ is a 1-form associated with a smooth unit vector field ζ on $\bar{M}$ by $\theta(X) = \bar{g}(X, \zeta)$. Throughout this paper, we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

A lightlike submanifold M of an indefinite almost contact manifold $\bar{M}$ is called *generic* if there exists a screen distribution $S(TM)$ of M such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{1.2}$$

where $S(TM)^{\perp}$ is the orthogonal complement of $S(TM)$ in the tangent bundle $T\bar{M}$ of $\bar{M}$, *i.e.*, $T\bar{M} = S(TM) \oplus_{orth} S(TM)^{\perp}$. The notion of generic lightlike submanifolds was introduced by Jin-Lee [12] at 2011 and then, studied by Duggal-Jin [5], Jin [6, 8] and Jin-Lee [14] and several authors. The geometry of

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generic lightlike submanifolds is an extension of that of lightlike hypersurface and half lightlike submanifold of codimension 2, that is, the last two types of lightlike submanifolds are examples of the generic lightlike submanifold. Much of the theory of generic lightlike submanifolds will be immediately generalized in a formal way to general lightlike submanifolds.

The notion of trans-Sasakian manifold, of type (α, β), was introduced by Oubina [16]. If a trans-Sasakian manifold $\bar{M}$ is semi-Riemannian, then $\bar{M}$ is called an *indefinite trans-Sasakian manifold*. Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of trans-Sasakian manifold such that

$$\alpha = 1,\ \beta = 0; \quad \alpha = 0,\ \beta = 1; \quad \alpha = \beta = 0, \quad \text{respectively.}$$

The object of study of this paper is generic lightlike submanifolds of an indefinite trans-Sasakian manifold $\bar{M} \equiv (\bar{M}, J, \zeta, \theta, \bar{g})$ with a quarter-symmetric metric connection subject such that the tensor field J and the 1-form θ, defined by (1.1), are identical with the structure tensor field J and the structure 1-form θ of the indefinite trans-Sasakian structure $(J, \theta, \zeta, \bar{g})$ on $\bar{M}$, respectively.

Remark 1.1. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of $\bar{M}$ with respect to the semi-Riemannian metric $\bar{g}$. It is known [9] that *a linear connection* $\bar{\nabla}$ *on* $\bar{M}$ *is a quarter-symmetric metric connection if and only if* $\bar{\nabla}$ *satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} - \theta(\bar{X})J\bar{Y}. \tag{1.3}$$

2. Preliminaries

An odd-dimensional semi-Riemannian manifold $(\bar{M},\ \bar{g})$ is called an *indefinite trans-Sasakian manifold* if there exist (1) a structure set $\{J,\ \zeta,\ \theta,\ \bar{g}\}$, where J is a $(1,1)$-type tensor field, ζ is a vector field and θ is a 1-form such that

$$\begin{aligned} J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta, \quad \theta(\zeta) = 1, \quad \theta(\bar{X}) = \epsilon\, \bar{g}(\bar{X}, \zeta), \\ \theta \circ J = 0, \qquad \bar{g}(J\bar{X},\ J\bar{Y}) = \bar{g}(\bar{X},\ \bar{Y}) - \epsilon\, \theta(\bar{X})\theta(\bar{Y}), \end{aligned} \tag{2.1}$$

(2) two smooth functions α and β, and a Levi-Civita connection $\widetilde{\nabla}$ such that

$$(\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \epsilon\, \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \epsilon\, \theta(\bar{Y})J\bar{X}\},$$

where ϵ denotes $\epsilon = 1$ or -1 according as ζ is spacelike or timelike respectively. $\{J,\ \zeta,\ \theta,\ \bar{g}\}$ is called an *indefinite trans-Sasakian structure of type* (α, β).

In the entire discussion of this article, we shall assume that the vector field ζ is a spacelike one, *i.e.*, $\epsilon = 1$, without loss of generality.

By directed calculation from (1.3), we see that $(\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = (\bar{\nabla}_{\bar{X}}J)\bar{Y}$. Thus, replacing the Levi-Civita connection $\widetilde{\nabla}$ by the quarter-symmetric metric connection $\bar{\nabla}$ defined by (1.3), the last equation is reformed to

$$(\bar{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}. \tag{2.2}$$

Replacing Y by ζ to (2.2) and using $J\zeta = 0$ and $\theta(\bar{\nabla}_X\zeta) = 0$, we obtain

$$\bar{\nabla}_{\bar{X}}\zeta = -\alpha J\bar{X} + \beta(\bar{X} - \theta(\bar{X})\zeta). \tag{2.3}$$

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite trans-Sasakian manifold $(\bar{M}, \bar{g})$ of dimension $(m+n)$. Then the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of M is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r\,(1 \leq r \leq \min\{m, n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which are called the *screen distribution* and the *co-screen distribution* of M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of (2.1). We use the same notations for any others. Let X, Y, Z and W be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$i, j, k, \ldots, \quad \in \{1, \ldots, r\}, \qquad a, b, c, \ldots, \quad \in \{r+1, \ldots, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

We say that a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is

(1) *r-lightlike submanifold* if $1 \leq r < \min\{m, n\}$;
(2) *co-isotropic submanifold* if $1 \leq r = n < m$;
(3) *isotropic submanifold* if $1 \leq r = m < n$;
(4) *totally lightlike submanifold* if $1 \leq r = m = n$.

The above three classes (2)∼(4) are particular cases of the class (1) as follows:

$$S(TM^{\perp}) = \{0\}, \qquad S(TM) = \{0\}, \qquad S(TM) = S(TM^{\perp}) = \{0\}$$

respectively. The geometry of r-lightlike submanifolds is more general than that of the other three types. For this reason, we consider only r-lightlike submanifolds M, with following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1, \ldots, \xi_r, \; N_1, \ldots, N_r, \; F_{r+1}, \ldots, F_m, \; E_{r+1}, \ldots, E_n\},$$

where $\{F_{r+1}, \ldots, F_m\}$ and $\{E_{r+1}, \ldots, E_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$, respectively. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a \delta_{ab} = \bar{g}(E_a, E_b)$.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss-Weingarten formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X, Y) N_i + \sum_{a=r+1}^{n} h_a^{s}(X, Y) E_a, \tag{2.4}$$

$$\bar{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r}\tau_{ij}(X)N_j + \sum_{a=r+1}^{n}\rho_{ia}(X)E_a, \tag{2.5}$$

$$\bar{\nabla}_X E_a = -A_{E_a}X + \sum_{i=1}^{r}\phi_{ai}(X)N_i + \sum_{b=r+1}^{n}\sigma_{ab}(X)E_b; \tag{2.6}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r}h^*_i(X, PY)\xi_i, \tag{2.7}$$

$$\nabla_X \xi_i = -A^*_{\xi_i}X - \sum_{j=1}^{r}\tau_{ji}(X)\xi_j, \tag{2.8}$$

where ∇ and ∇^* are induced linear connections on M and $S(TM)$ respectively, h^ℓ_i and h^s_a are called the *local second fundamental forms* on M, h^*_i are called the *local screen second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A^*_{\xi_i}$ are linear operators on M, and τ_{ij}, ρ_{ia}, ϕ_{ai} and $\sigma_{\alpha\beta}$ are 1-forms on M.

3. Quarter-symmetric metric connection

Now we assume that ζ is tangent to M. Călin [2] proved that *if ζ is tangent to M, then it belongs to $S(TM)$* which we assume. For a generic M, from (1.2) we show that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^\perp))$ are subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions H_o and H with respect to J, *i.e.*, $J(H_o) = H_o$ and $J(H) = H$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^\perp)) \oplus_{orth} H_o,$$
$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

In this case, the tangent bundle TM of M is decomposed as follow:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp)). \tag{3.1}$$

Consider local null vector fields U_i and V_i for each i, local non-null unit vector fields W_a for each a, and their 1-forms u_i, v_i and w_a defined by

$$U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad W_a = -JE_a, \tag{3.2}$$

$$u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \epsilon_a g(X, W_a). \tag{3.3}$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$JX = FX + \sum_{i=1}^{r}u_i(X)N_i + \sum_{a=r+1}^{n}w_a(X)E_a. \tag{3.4}$$

Applying J to (3.4) and using $(2.1)_1$ and (3.2), we have

$$F^2X = -X + \theta(X)\zeta + \sum_{i=1}^{r}u_i(X)U_i + \sum_{a=r+1}^{n}w_a(X)W_a. \tag{3.5}$$

In the following, we say that F is the *structure tensor field* of M.

Substituting (2.4) and (3.4) into (1.1) and then, comparing the tangent, lightlike transversal and co-screen components of the left-right terms, we get

$$T(X,Y) = \theta(Y)FX - \theta(X)FY, \tag{3.6}$$

$$h_i^\ell(X,Y) - h_i^\ell(Y,X) = \theta(Y)u_i(X) - \theta(X)u_i(Y), \tag{3.7}$$

$$h_a^s(X,Y) - h_a^s(Y,X) = \theta(Y)w_a(X) - \theta(X)w_a(Y), \tag{3.8}$$

where T is the torsion tensor with respect to the connection ∇. Note that, from (3.7) and (3.8), we see that h_i^ℓ and h_a^s are not symmetric.

From the facts that $h_i^\ell(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$ and $\epsilon_a h_a^s(X,Y) = \bar{g}(\bar{\nabla}_X Y, E_a)$, we know that h_i^ℓ and h_a^s are independent of the choice of $S(TM)$. The local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g(A^*_{\xi_i}X,Y) - \sum_{k=1}^{r} h_k^\ell(X,\xi_i)\eta_k(Y), \tag{3.9}$$

$$\epsilon_a h_a^s(X,Y) = g(A_{E_a}X,Y) - \sum_{k=1}^{r} \phi_{ak}(X)\eta_k(Y), \tag{3.10}$$

$$h_i^*(X,PY) = g(A_{N_i}X,PY), \tag{3.11}$$

where $\eta_k s$ are 1-forms such that $\eta_k(X) = \bar{g}(X,N_k)$. Applying $\bar{\nabla}_X$ to $g(\xi_i,\xi_j) = 0$, $\bar{g}(\xi_i,E_a) = 0$, $\bar{g}(N_i,N_j) = 0$, $\bar{g}(N_i,E_a) = 0$ and $\bar{g}(E_a,E_b) = \epsilon\delta_{ab}$, we obtain

$$\begin{aligned} &h_i^\ell(X,\xi_j) + h_j^\ell(X,\xi_i) = 0, \qquad h_a^s(X,\xi_i) = -\epsilon_a\phi_{ai}(X),\\ &\eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0, \quad \bar{g}(A_{E_a}X,N_i) = \epsilon_a\rho_{ia}(X),\\ &\epsilon_b\sigma_{ab} + \epsilon_a\sigma_{ba} = 0 \quad \text{and} \quad h_i^\ell(X,\xi_i) = 0, \quad h_i^\ell(\xi_j,\xi_k) = 0. \end{aligned} \tag{3.12}$$

By directed calculations from (2.3), (2.4), (2.5), (3.4) and (3.11), we have

$$\nabla_X\zeta = -\alpha FX + \beta(X - \theta(X)\zeta), \tag{3.13}$$

$$h_i^\ell(X,\,\zeta) = -\alpha u_i(X), \qquad h_a^s(X,\,\zeta) = -\alpha w_a(X), \tag{3.14}$$

$$h_i^*(X,\,\zeta) = -\alpha v_i(X) + \beta\eta_i(X). \tag{3.15}$$

Applying $\bar{\nabla}_X$ to (3.2), (3.3) and (3.4) by turns and using (2.2), (2.4) $\sim$ (2.8), (3.2) $\sim$ (3.4) and (3.9) $\sim$ (3.11), we have

$$\begin{aligned} &h_j^\ell(X,U_i) = h_i^*(X,V_j), \qquad \epsilon_a h_i^*(X,W_a) = h_a^s(X,U_i),\\ &h_j^\ell(X,V_i) = h_i^\ell(X,V_j), \qquad \epsilon_a h_i^\ell(X,W_a) = h_a^s(X,V_i),\\ &\epsilon_b h_b^s(X,W_a) = \epsilon_a h_a^s(X,W_b), \end{aligned} \tag{3.16}$$

$$\begin{aligned} \nabla_X U_i \;=\; & F(A_{N_i}X) + \sum_{j=1}^{r}\tau_{ij}(X)U_j + \sum_{a=r+1}^{n}\rho_{ia}(X)W_a\\ & - \{\alpha\eta_i(X) + \beta v_i(X)\}\zeta, \end{aligned} \tag{3.17}$$

$$\nabla_X V_i \;=\; F(A^*_{\xi_i}X) - \sum_{j=1}^{r}\tau_{ji}(X)V_j + \sum_{j=1}^{r} h_j^\ell(X,\xi_i)U_j \tag{3.18}$$

$$- \sum_{a=r+1}^{n} \epsilon_a \phi_{ai}(X) W_a - \beta u_i(X)\zeta,$$

$$\nabla_X W_a = F(A_{E_a} X) + \sum_{i=1}^{r} \phi_{ai}(X) U_i + \sum_{b=r+1}^{n} \sigma_{ab}(X) W_b - \epsilon_a \beta w_a(X)\zeta, \tag{3.19}$$

$$\begin{aligned}(\nabla_X F)(Y) = & \sum_{i=1}^{r} u_i(Y) A_{N_i} X + \sum_{a=r+1}^{n} w_a(Y) A_{E_a} X \\ & - \sum_{i=1}^{r} h_i^{\ell}(X,Y) U_i - \sum_{a=r+1}^{n} h_a^{s}(X,Y) W_a \\ & + \alpha\{g(X,Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX,Y)\zeta - \theta(Y)FX\},\end{aligned} \tag{3.20}$$

$$\begin{aligned}(\nabla_X u_i)(Y) = & -\sum_{j=1}^{r} u_j(Y)\tau_{ji}(X) - \sum_{a=r+1}^{n} w_a(Y)\phi_{ai}(X) \\ & - \beta\theta(Y) u_i(X) - h_i^{\ell}(X, FY),\end{aligned} \tag{3.21}$$

$$\begin{aligned}(\nabla_X v_i)(Y) = & \sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) \\ & - \sum_{j=r+1}^{r} u_j(Y)\eta_j(A_{N_i} X) - g(A_{N_i} X, FY) \\ & - \theta(Y)\{\alpha\eta_i(X) + \beta v_i(X)\}.\end{aligned} \tag{3.22}$$

4. Recurrent and Lie recurrent submanifolds

Definition. We say that a lightlike submanifold M of $\bar{M}$ is called

(1) *irrotational* [15] if $\bar{\nabla}_X \xi_i \in \Gamma(TM)$ for all $i \in \{1, \ldots, r\}$,
(2) *solenoidal* [13] if A_{W_a} and A_{N_i} are $S(TM)$-valued,
(3) *statical* [13] if M is both irrotational and solenoidal.

Remark 4.1. From (2.4) and $(3.12)_2$, the item (1) is equivalent to

$$h_j^{\ell}(X, \xi_i) = 0, \qquad h_a^{s}(X, \xi_i) = \phi_{ai}(X) = 0. \tag{4.1}$$

By using $(3.12)_4$, the item (2) is equivalent to

$$\eta_j(A_{N_i} X) = 0, \qquad \rho_{ia}(X) = \eta_i(A_{E_a} X) = 0. \tag{4.2}$$

Denote by λ_{ij}, μ_{ia}, ν_{ia}, κ_{ab} and χ_{ij} the 1-forms on M such that

$$\begin{aligned}&\lambda_{ij}(X) = h_i^{\ell}(X, U_j) = h_j^{*}(X, V_i), && \kappa_{ab}(X) = \epsilon_a h_a^{s}(X, W_b),\\ &\mu_{ia}(X) = h_i^{\ell}(X, W_a) = \epsilon_a h_a^{s}(X, V_i), && \chi_{ij}(X) = h_i^{\ell}(X, V_j),\\ &\nu_{ai}(X) = h_i^{*}(X, W_a) = \epsilon_a h_a^{s}(X, U_i).\end{aligned} \tag{4.3}$$

Definition. The structure tensor field F of M is said to be *recurrent* [11] if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y = \varpi(X)FY.$$

A lightlike submanifold M of an indefinite trans-Sasakian manifold $\bar{M}$ is called *recurrent* if it admits a recurrent structure tensor field F.

Theorem 4.2. *Let M be a recurrent generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. Then the following statements are satisfied*:

(1) *F is parallel with respect to the induced connection ∇ on M,*
(2) *$\bar{M}$ is an indefinite cosymplectic manifold, i.e., $\alpha = \beta = 0$,*
(3) *M is statical,*
(4) *$J(ltr(TM))$, $J(S(TM^\perp))$ and H are parallel distributions on M,*
(5) *M is locally a product manifold $M_r \times M_{n-r} \times M^\sharp$, where M_r, M_{n-r} and $M^\sharp$ are leaves of $J(ltr(TM))$, $J(S(TM^\perp))$ and H, respectively.*

Proof. (1) From the above definition and (3.20), we obtain

$$\begin{aligned}(4.4)\qquad \varpi(X)FY \;=\; &\sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X \\ &- \sum_{i=1}^{r} h_i^\ell(X,Y)U_i - \sum_{a=r+1}^{n} h_a^s(X,Y)W_a \\ &+ \alpha\{g(X,Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX,Y)\zeta - \theta(Y)FX\}.\end{aligned}$$

Replacing Y by ξ_j to this and using the fact that $F\xi_j = -V_j$, we get

$$(4.5)\qquad \varpi(X)V_j = \sum_{k=1}^{r} h_k^\ell(X,\xi_j)U_k + \sum_{b=r+1}^{n} h_b^s(X,\xi_j)W_b - \beta u_j(X)\zeta.$$

Taking the scalar product with U_j, ζ, V_i and W_a by turns, we obtain

$$\varpi = 0, \qquad \beta = 0, \qquad h_i^\ell(X,\xi_j) = 0, \qquad h_a^s(X,\xi_j) = \phi_{aj}(X) = 0,$$

respectively. As $\varpi = 0$, F is parallel with respect to the connection ∇.

(2) Taking the scalar product with U_j to (4.4) with $\varpi = \beta = 0$, we get

$$(4.6)\quad \sum_{i=1}^{r} u_i(Y)g(A_{N_i}X, U_j) + \sum_{a=r+1}^{n} w_a(Y)g(A_{E_a}X, U_j) - \alpha\theta(Y)v_j(X) = 0.$$

Replacing Y by ζ to this equation, we have $\alpha v_j(X) = 0$. It follows that $\alpha = 0$. As $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold.

(3) As $h_i^\ell(X,\xi_j) = 0$ and $h_a^s(X,\xi_j) = 0$, M is irrotational by (4.1). Also, M is solenoidal. In fact, taking the scalar product with N_j to (4.4), we have

$$\sum_{i=1}^{r} u_i(Y)\bar{g}(A_{N_i}X, N_j) + \sum_{a=r+1}^{n} w_a(Y)\bar{g}(A_{E_a}X, N_j) = 0.$$

Taking $Y = U_i$ and $Y = W_a$ by turns, we get (4.2). Thus M is statical.

(4) Taking $Y = U_k$ and $Y = W_b$ to (4.6) by turns, we obtain

$$h_i^*(X, U_j) = \bar{g}(A_{N_i}X, U_j) = 0, \qquad \nu_{ai}(X) = \bar{g}(A_{E_a}X, U_i) = 0. \tag{4.7}$$

Taking the scalar product with V_j and W_b to (4.4) by turns, we have

$$\begin{aligned} h_i^\ell(X, Y) &= \sum_{j=1}^{r} \lambda_{ij}(X)u_j(Y) + \sum_{a=r+1}^{n} \mu_{ia}(X)w_a(Y), \\ \epsilon_a h_a^s(X, Y) &= \sum_{b=r+1}^{n} \kappa_{ba}(X)w_b(Y), \end{aligned} \tag{4.8}$$

due to (3.10), (3.11) and (4.3). Replacing Y by V_j to $(4.8)_{1,2}$, we have

$$\chi_{ij}(X) = h_i^\ell(X, V_j) = 0, \qquad \mu_{ia}(X) = h_a^s(X, V_i) = 0. \tag{4.9}$$

Taking $Y = U_j$ and $Y = W_b$ to (4.4) and using (4.3), $(4.7)_2$ and $(4.9)_2$, we get

$$A_{N_i}X = \sum_{j=1}^{r} \lambda_{ji}(X)U_j, \qquad A_{E_a}X = \sum_{b=r+1}^{n} \epsilon_b \kappa_{ba}(X)W_b. \tag{4.10}$$

Using (3.9), (4.1), $(4.9)_2$ and the non-degenerateness of $S(TM)$, $(4.8)_1$ reduces

$$A^*_{\xi_i}X = \sum_{j=1}^{r} \lambda_{ij}(X)V_j. \tag{4.11}$$

Applying F to $(4.10)_{1,2}$, we have $F(A_{N_i}X) = 0$ and $F(A_{E_a}X) = 0$. Substituting these results into (3.17) and (3.19), we obtain

$$\nabla_X U_i = \sum_{j=1}^{r} \tau_{ij}(X)U_j, \qquad \nabla_X W_a = \sum_{b=r+1}^{n} \sigma_{ab}(X)W_b. \tag{4.12}$$

It follow that $J(ltr(TM))$ and $J(S(TM^\perp))$ are parallel distributions on M with respect to the induced connection ∇ on M, that is,

$$\nabla_X U_i \in \Gamma(J(ltr(TM))), \qquad \nabla_X W_a \in \Gamma(J(S(TM^\perp))).$$

Applying F to (4.11), we get $F(A^*_{\xi_i}X) = \sum_{j=1}^{r} \lambda_{ij}(X)\xi_j$. Thus we have

$$\nabla_X V_i = \sum_{j=1}^{r} \{\lambda_{ij}(X)\xi_j - \tau_{ji}(X)V_j\}. \tag{4.13}$$

Taking $Y \in \Gamma(H)$ to (4.4) and then, taking the scalar product with U_j and W_b to the resulting equation by turns, we obtain

$$h_i^\ell(X, Y) = 0, \quad h_a^s(X, Y) = 0, \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(H). \tag{4.14}$$

By directed calculations from (4.9), $(4.12)_2$, (4.13) and (4.14), we obtain $g(\nabla_X Y, V_i) = 0$ and $g(\nabla_X Y, W_a) = 0$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(H)$. Thus

$$\nabla_X Y \in \Gamma(H), \quad \forall\, X \in \Gamma(TM), \ \forall\, Y \in \Gamma(H).$$

Thus H is also a parallel distribution on M with respect to ∇.

(5) As $J(ltr(TM))$, $J(S(TM^{\perp}))$ and H are parallel distributions and satisfied the decomposition form (3.1), by the decomposition theorem of de Rham [3], M is locally a product manifold $M_r \times M_{n-r} \times M^{\sharp}$, where M_r, M_{n-r} and $M^{\sharp}$ are leaves of $J(ltr(TM)), J(S(TM^{\perp}))$ and H, respectively. □

Definition. The structure tensor field F of M is said to be *Lie recurrent* [11] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X. The structure tensor field F is called *Lie parallel* if $\mathcal{L}_X F = 0$. A lightlike submanifold M is called *Lie recurrent* if it admits a Lie recurrent structure tensor field F.

Theorem 4.3. *Let M be a Lie recurrent generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. Then the following statements are satisfied*:

(1) *F is Lie parallel,*
(2) *$\alpha = 0$ and $d\theta = 0$. Thus $\bar{M}$ is not an indefinite Sasakian manifold,*
(3) *h_i^* is never symmetric on $S(TM)$,*
(4) *τ_{ij} and ρ_{ia} are satisfied $\tau_{ij} \circ F = 0$ and $\rho_{ia} \circ F = 0$. Moreover,*

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X) g(A_{N_k} V_j, N_i) - \beta \delta_{ij} \theta(X).$$

Proof. (1) As $(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]$, using (3.6) and (3.20), we get

$$\begin{aligned}(4.15)\quad \vartheta(X)FY &= -\nabla_{FY}X + F\nabla_Y X - \theta(Y)\{X - \theta(X)\zeta\} \\ &+ \sum_{i=1}^{r} u_i(Y) A_{N_i} X + \sum_{a=r+1}^{n} w_a(Y) A_{E_a} X \\ &- \sum_{i=1}^{r} \{h_i^{\ell}(X, Y) - \theta(Y) u_i(X)\} U_i \\ &- \sum_{a=r+1}^{n} \{h_a^s(X, Y) - \theta(Y) w_a(X)\} W_a \\ &+ \alpha\{g(X, Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \theta(Y)FX\},\end{aligned}$$

by (3.5). Replacing Y by ξ_j and then, Y by V_j to (4.15) by turns, we have

$$\begin{aligned}(4.16)\qquad -\vartheta(X)V_j &= \nabla_{V_j}X + F\nabla_{\xi_j}X + \beta u_j(X)\zeta \\ &- \sum_{i=1}^{r} h_i^{\ell}(X, \xi_j) U_i - \sum_{a=r+1}^{n} h_a^s(X, \xi_j) W_a,\end{aligned}$$

$$(4.17)\qquad \vartheta(X)\xi_j = -\nabla_{\xi_j}X + F\nabla_{V_j}X + \alpha u_j(X)\zeta$$

$$- \sum_{i=1}^{r} h_i^\ell(X, V_j)U_i - \sum_{a=r+1}^{n} h_a^s(X, V_j)W_a,$$

respectively. Taking the scalar product with U_j to (4.16) and then, taking the scalar product with N_j to (4.17), we obtain respectively

$$\begin{aligned} -\vartheta(X) &= g(\nabla_{V_j} X, U_j) - \bar{g}(\nabla_{\xi_j} X, N_j), \\ \vartheta(X) &= g(\nabla_{V_j} X, U_j) - \bar{g}(\nabla_{\xi_j} X, N_j). \end{aligned}$$

Comparing these two equations, we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with ζ to (4.17) satisfying $\vartheta = 0$, we have

$$g(\nabla_{\xi_j} X, \zeta) = \alpha u_j(X).$$

Replacing X by U_j to this equation and using (3.17), we obtain $\alpha = 0$.

Applying $\bar{\nabla}_{\bar{X}}$ to $\theta(\bar{Y}) = \bar{g}(\bar{Y}, \zeta)$ and using (1.1) and (2.3), we obtain

$$d\theta(\bar{X}, \bar{Y}) = \alpha \bar{g}(\bar{X}, J\bar{Y}),$$

due to the fact that $\bar{\nabla}$ is metric. As $\alpha = 0$, we see that $d\theta = 0$.

(3) Replacing X by U_i to (4.15) and using (3.2), (3.3), (3.5), (3.7), (3.8), (3.11), (3.15), $(3.16)_{1,2}$ and (3.17), we obtain

$$\begin{aligned} \sum_{k=1}^{r} u_k(Y) A_{N_k} U_i + \sum_{a=r+1}^{n} w_a(Y) A_{E_a} U_i - \theta(Y) U_i + \beta \eta_i(Y) \zeta \\ - A_{N_i} Y - F(A_{N_i} FY) - \sum_{j=1}^{r} \tau_{ij}(FY) U_j - \sum_{a=r+1}^{n} \rho_{ia}(FY) W_a = 0. \end{aligned} \tag{4.18}$$

Taking $Y = \zeta$ to (4.18) and then, taking the scalar product with PX, we get $h_i^*(\zeta, PX) = -v_i(PX)$. Assume that h_i^* is symmetric on $S(TM)$. Taking $X = PX$ to (3.15), we obtain $h_i^*(\zeta, PX) = 0$. It follows that $v_i(PX) = 0$. It is a contradiction to $v_i(V_i) = 1$. Thus h_i^* is never symmetric on $S(TM)$.

(4) Taking the scalar product with N_i to (4.16) such that $X = W_a$ and using (3.8), (3.10), $(3.12)_4$ and (3.19), we get $h_a^s(U_i, V_j) = \rho_{ia}(\xi_j)$. On the other hand, taking the scalar product with W_a to (4.17) such that $X = U_i$ and using (3.17), we have $h_a^s(U_i, V_j) = -\rho_{ia}(\xi_j)$. Thus $\rho_{ia}(\xi_j) = 0$ and $h_a^s(U_i, V_j) = 0$.

Taking the scalar product with U_i to (4.16) such that $X = W_a$ and using (3.10), $(3.12)_{2,4}$ and (3.19), we get $\epsilon_a \rho_{ia}(V_j) = \phi_{aj}(U_i)$. On the other hand, taking the scalar product with W_a to (4.16) such that $X = U_i$ and using $(3.12)_2$ and (3.17), we get $\epsilon_a \rho_{ia}(V_j) = -\phi_{aj}(U_i)$. Thus $\rho_{ia}(V_j) = 0$ and $\phi_{aj}(U_i) = 0$.

Taking the scalar product with V_i to (4.16) such that $X = W_a$ and using (3.7), (3.8), $(3.12)_2$, $(3.16)_4$ and (3.19), we get $\phi_{ai}(V_j) = -\phi_{aj}(V_i)$. On the other hand, taking the scalar product with W_a to (4.16) such that $X = V_i$ and using $(3.12)_2$ and (3.18), we have $\phi_{ai}(V_j) = \phi_{aj}(V_i)$. Thus $\phi_{ai}(V_j) = 0$.

Taking the scalar product with W_a to (4.16) such that $X = \xi_i$ and using (2.8), (3.9) and $(3.12)_2$, we get $h_i^\ell(V_j, W_a) = \phi_{ai}(\xi_j)$. On the other hand, taking the scalar product with V_i to (4.17) such that $X = W_a$ and using (3.7) and

(3.19), we have $h_i^\ell(V_j, W_a) = -\phi_{ai}(\xi_j)$. Thus $\phi_{ai}(\xi_j) = 0$ and $h_i^\ell(V_j, W_a) = 0$. Summarizing the above results, we obtain

$$\begin{aligned} &\rho_{ia}(\xi_j) = 0,\ \rho_{ia}(V_j) = 0,\ \phi_{ai}(U_j) = 0,\ \phi_{ai}(V_j) = 0,\ \phi_{ai}(\xi_j) = 0, \\ &h_a^s(U_i, V_j) = h_j^\ell(U_i, W_a) = 0, \qquad h_i^\ell(V_j, W_a) = h_a^s(V_j, V_i) = 0. \end{aligned} \tag{4.19}$$

Taking the scalar product with N_i to (4.15) and using $(3.12)_4$, we have

$$\begin{aligned} &-\bar{g}(\nabla_{FY}X, N_i) + \bar{g}(\nabla_Y X, U_i) - \theta(Y)\{\eta_i(X) + \beta v_i(X)\} \\ &+ \sum_{k=1}^{r} u_k(Y)\bar{g}(A_{N_k}X, N_i) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) = 0. \end{aligned} \tag{4.20}$$

Replacing X by V_j to (4.20) and using (3.9), (3.18) and $(4.19)_2$, we have

$$h_j^\ell(FX, U_i) + \tau_{ij}(X) + \beta\delta_{ij}\theta(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}V_j, N_i). \tag{4.21}$$

Replacing X by ξ_j to (4.20) and using (2.8), (3.9) and $(4.19)_1$, we have

$$h_j^\ell(X, U_i) + \delta_{ij}\theta(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}\xi_j, N_i) + \tau_{ij}(FX). \tag{4.22}$$

Taking $X = U_k$ to (4.22), we have

$$h_i^*(U_k, V_j) = h_j^\ell(U_k, U_i) = \bar{g}(A_{N_k}\xi_j, N_i). \tag{4.23}$$

On the other hand, taking the scalar product with V_j to (4.18) and using (3.11), $(3.12)_3$, $(3.16)_1$, $(4.19)_6$ and (4.23), we get

$$h_j^\ell(X, U_i) + \delta_{ij}\theta(X) = -\sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}\xi_j, N_i) - \tau_{ij}(FX).$$

Comparing this equation with (4.22), we obtain

$$\tau_{ij}(FX) + \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}\xi_j, N_i) = 0.$$

Replacing X by U_h to this equation, we have $\bar{g}(A_{N_k}\xi_j, N_i) = 0$. Therefore,

$$\tau_{ij}(FX) = 0, \qquad h_j^\ell(X, U_i) + \delta_{ij}\theta(X) = 0. \tag{4.24}$$

Taking $X = FY$ to $(4.24)_2$, we get $h_j^\ell(FX, U_i) = 0$. Thus (4.21) is reduced to

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}V_j, N_i) - \beta\delta_{ij}\theta(X). \tag{4.25}$$

Replacing Y by W_b to (4.18), we have $A_{E_a}U_i = A_{N_i}W_a$. Taking the scalar product with U_j and using (3.8), (3.10), (3.11) and $(3.16)_2$, we have

$$h_i^*(W_a, U_j) = \epsilon_a h_a^s(U_i, U_j) = \epsilon_a h_a^s(U_j, U_i) = h_i^*(U_j, W_a). \tag{4.26}$$

Taking the scalar product with W_a to (4.18), we have

$$\epsilon_a\rho_{ia}(FY) = -h_i^*(Y, W_a) + \sum_{k=1}^{r} u_k(Y)h_k^*(U_i, W_a) + \sum_{b=r+1}^{n} \epsilon_b w_b(Y)h_b^s(U_i, W_a).$$

Taking the scalar product with U_i to (4.15) and then, taking $X = W_a$ and using (3.8), (3.10), (3.11), $(3.12)_4$, $(3.16)_2$, (3.19) and (4.26), we obtain

$$\epsilon_a\rho_{ia}(FY) = h_i^*(Y, W_a) - \sum_{k=1}^{r} u_k(Y)h_k^*(U_i, W_a) - \sum_{b=r+1}^{n} \epsilon_b w_b(Y)h_b^s(U_i, W_a).$$

Comparing the last two equations, we obtain $\rho_{ia}(FY) = 0$. □

Remark 4.4. Replacing X by ξ_j to (3.9) and using $(3.12)_7$, we have

$$h_i^\ell(\xi_j, X) = g(A^*_{\xi_i}\xi_j, X).$$

Taking $Y = \xi_j$ to (3.7), we obtain $h_i^\ell(X, \xi_j) = h_i^\ell(\xi_j, X)$. From this and $(3.12)_1$, we see that $h_i^\ell(\xi_j, X)$ are skew-symmetric with respect to i and j. It follow that $A^*_{\xi_i}\xi_j = -A^*_{\xi_j}\xi_i$, *i.e.*, $A^*_{\xi_i}\xi_j$ are skew-symmetric with respect to i and j.

In case M is Lie recurrent, taking $Y = U_j$ to (4.18), we have $A_{N_i}U_j = A_{N_j}U_i$. Thus $A_{N_i}U_j$ are symmetric with respect to i and j. Therefore, we get

$$h_i^\ell(\xi_j,\, F(A_{N_j}U_i)) = g(A^*_{\xi_i}\xi_j, F(A_{N_j}U_i)) = 0, \tag{4.27}$$

$$h_i^\ell(\xi_j, W_a) = \epsilon_a h_a^s(\xi_j, V_i) = \epsilon_a h_a^s(V_i, \xi_j) = -\phi_{ji}(V_i) = 0, \tag{4.28}$$

due to $(4.19)_4$. Taking $X = U_i$ (3.7) and using $(4.24)_2$, we obtain

$$h_j^\ell(U_i, X) = 0. \tag{4.29}$$

5. Indefinite generalized Sasakian space forms

Alegre and his collaborators [1] introduced generalized Sasakian space form. Jin [6] extended this notion as follow: An indefinite trans-Sasakian manifold $\bar{M}$ is called *indefinite generalized Sasakian space form* and denoted by $\bar{M}(f_1, f_2, f_3)$ if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned}\bar{R}(\bar{X},\bar{Y})\bar{Z} = {} & f_1\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y}\} \\ & + f_2\{\bar{g}(\bar{X},J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y},J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X},J\bar{Y})J\bar{Z}\} \\ & + f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X} \\ & \qquad + \bar{g}(\bar{X},\bar{Z})\theta(\bar{Y})\zeta - \bar{g}(\bar{Y},\bar{Z})\theta(\bar{X})\zeta\},\end{aligned} \tag{5.1}$$

where the symbol $\bar{R}$ is the curvature tensor of $\bar{M}(f_1, f_2, f_3)$.

Sasakian space form, Kenmotsu space form and cosymplectic space form are important kinds of generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4}, f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4}$$

respectively, where c is a constant J-sectional curvature of each space forms.

Denote by $\bar{R}$, R and R^* the curvature tensors of the quart-symmetric metric connection $\bar{\nabla}$ on $\bar{M}$, and the induced connection ∇ and ∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formulas for M and $S(TM)$, we obtain the Gauss equations for M and $S(TM)$, respectively:

$$
\begin{aligned}
\bar{R}(X,Y)Z = R(X,Y)Z & \\
&+ \sum_{i=1}^{r}\{h_i^\ell(X,Z)A_{N_i}Y - h_i^\ell(Y,Z)A_{N_i}X\} \\
&+ \sum_{a=r+1}^{n}\{h_a^s(X,Z)A_{E_a}Y - h_a^s(Y,Z)A_{E_a}X\} \\
&+ \sum_{i=1}^{r}\{(\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\
&+ \sum_{j=1}^{r}[\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)] \\
&+ \sum_{a=r+1}^{n}[\phi_{ai}(X)h_a^s(Y,Z) - \phi_{ai}(Y)h_a^s(X,Z)] \\
&- \theta(X)h_i^\ell(FY,Z) + \theta(Y)h_i^\ell(FX,Z)\}N_i \\
&+ \sum_{a=r+1}^{n}\{(\nabla_X h_a^s)(Y,Z) - (\nabla_Y h_a^s)(X,Z) \\
&+ \sum_{i=1}^{r}[\rho_{ia}(X)h_i^\ell(Y,Z) - \rho_{ia}(Y)h_a^s(X,Z)] \\
&+ \sum_{b=r+1}^{n}[\sigma_{ba}(X)h_b^s(Y,Z) - \sigma_{ba}(Y)h_b^s(X,Z)] \\
&- \theta(X)h_a^s(FY,Z) + \theta(Y)h_a^s(FX,Z)\}E_a,
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
R(X,\,Y)PZ \;=\; R^*(X,\,Y)PZ & \\
&+ \sum_{i=1}^{r}\{h_i^*(X,\,PZ)A^*_{\xi_i}Y - h_i^*(Y,\,PZ)A^*_{\xi_i}X\} \\
&+ \sum_{i=1}^{r}\{(\nabla_X h_i^*)(Y,\,PZ) - (\nabla_Y h_i^*)(X,\,PZ) \\
&+ \sum_{j=1}^{r}[h_j^*(X,\,PZ)\tau_{ij}(Y) - h_j^*(Y,\,PZ)\tau_{ij}(X)] \\
&- \theta(X)h_i^*(FY,Z) + \theta(Y)h_i^*(FX,Z)\}\xi_i.
\end{aligned}
\tag{5.3}
$$

Comparing the tangential and lightlike transversal components of the two equations (5.1) and (5.2), and using (3.4), we get

$$\begin{aligned}
R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\
& + f_2\{\bar{g}(X,JZ)FY - \bar{g}(Y,JZ)FX + 2\bar{g}(X,JY)FZ\} \\
& + f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\
& + \bar{g}(X,Z)\theta(Y)\zeta - \bar{g}(Y,Z)\theta(X)\zeta\} \\
& + \sum_{i=1}^{r}\{h_i^\ell(Y,Z)A_{N_i}X - h_i^\ell(X,Z)A_{N_i}Y\} \\
& + \sum_{a=r+1}^{n}\{h_a^s(Y,Z)A_{E_a}X - h_a^s(X,Z)A_{E_a}Y\},
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
& (\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\
& + \sum_{j=1}^{r}\{\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)\} \\
& + \sum_{a=r+1}^{n}\{\phi_{ai}(X)h_a^s(Y,Z) - \phi_{ai}(Y)h_a^s(X,Z)\} \\
& - \theta(X)h_i^\ell(FY,Z) + \theta(Y)h_i^\ell(FX,Z) \\
= {} & f_2\{u_i(Y)\bar{g}(X,JZ) - u_i(X)\bar{g}(Y,JZ) + 2u_i(Z)\bar{g}(X,JY)\}.
\end{aligned} \tag{5.5}$$

Taking the scalar product with N_i to (5.3), we have

$$\begin{aligned}
\bar{g}(R(X,Y)PZ,\, N_i) = {} & (\nabla_X h_i^*)(Y,\, PZ) - (\nabla_Y h_i^*)(X,\, PZ) \\
& + \sum_{j=1}^{r}\{\tau_{ij}(Y)h_j^*(X,\, PZ) - \tau_{ij}(X)h_j^*(Y,\, PZ)\} \\
& - \theta(X)h_i^*(FY,Z) + \theta(Y)h_i^*(FX,Z).
\end{aligned}$$

Substituting (5.4) into the last equation and using $(3.12)_4$, we obtain

$$\begin{aligned}
& (\nabla_X h_i^*)(Y,\, PZ) - (\nabla_Y h_i^*)(X,\, PZ) \\
& + \sum_{j=1}^{r}\{\tau_{ij}(Y)h_j^*(X,\, PZ) - \tau_{ij}(X)h_j^*(Y,\, PZ)\} \\
& + \sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(Y)h_a^s(X,\, PZ) - \rho_{ia}(X)h_a^s(Y,\, PZ)\} \\
& + \sum_{j=1}^{r}\{h_j^\ell(X,\, PZ)\eta_i(A_{N_j}Y) - h_j^\ell(Y,\, PZ)\eta_i(A_{N_j}X)\} \\
& - \theta(X)h_i^*(FY,Z) + \theta(Y)h_i^*(FX,Z) \\
= {} & f_1\{g(Y,PZ)\eta_i(X) - g(X,PZ)\eta_i(Y)\}
\end{aligned} \tag{5.6}$$

$$+ f_2\{v_i(Y)\bar{g}(X, JPZ) - v_i(X)\bar{g}(Y, JPZ) + 2v_i(PZ)\bar{g}(X, JY)\}$$
$$+ f_3\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\}\theta(PZ).$$

Theorem 5.1. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a quarter-symmetric metric connection. Then the following properties are satisfied*

(1) *α is a constant,*
(2) *$\alpha\beta = 0$,*
(3) *$f_1 - f_2 = \alpha^2 - \beta^2$ and $f_1 - f_3 = (\alpha^2 - \beta^2) + \alpha - \zeta\beta$.*

Proof. Applying ∇_X to $(3.16)_1$: $h^\ell_j(Y, U_i) = h^*_i(Y, V_j)$ and using (2.1), (3.2), (3.3), (3.4), (3.9), (3.11), $(3.16)_1$, (3.17) and (3.18), we have

$$\begin{aligned}
(\nabla_X h^\ell_j)(Y, U_i) = {} & (\nabla_X h^*_i)(Y, V_j) \\
& - \sum_{k=1}^{r}\{\tau_{kj}(X)h^\ell_k(Y, U_i) + \tau_{ik}(X)h^*_k(Y, V_j)\} \\
& - \sum_{a=r+1}^{n}\{\phi_{aj}(X)h^s_a(Y, U_i) + \epsilon_a\rho_{ia}(X)h^s_a(Y, V_j)\} \\
& + \sum_{k=1}^{r}\{h^*_i(Y, U_k)h^\ell_k(X, \xi_j) + h^*_i(X, U_k)h^\ell_k(Y, \xi_j)\} \\
& - g(A^*_{\xi_j}X, F(A_{N_i}Y)) - g(A^*_{\xi_j}Y, F(A_{N_i}X)) \\
& - \sum_{k=1}^{r} h^\ell_j(X, V_k)\eta_k(A_{N_i}Y) - \alpha^2\, u_j(Y)\eta_i(X) \\
& - \beta^2\, u_j(X)\eta_i(Y) + \alpha\beta\{u_j(X)v_i(Y) - u_j(Y)v_i(X)\}.
\end{aligned}$$

Substituting this into (5.5) such that replace i by j and take $Z = U_i$, we have

$$\begin{aligned}
& (\nabla_X h^*_i)(Y, V_j) - (\nabla_Y h^*_i)(X, V_j) \\
& - \sum_{k=1}^{r}\{\tau_{ik}(X)h^*_k(Y, V_j) - \tau_{ik}(Y)h^*_k(X, V_j)\} \\
& - \sum_{a=r+1}^{n}\epsilon_a\{h^s_a(Y, V_j)\rho_{ia}(X) - h^s_a(X, V_j)\rho_{ia}(Y)\} \\
& - \sum_{k=1}^{r}\{h^\ell_k(Y, V_j)\eta_i(A_{N_k}X) - h^\ell_k(X, V_j)\eta_i(A_{N_k}Y)\} \\
& - \theta(X)h^*_i(FY, V_j) + \theta(Y)h^*_i(FX, V_j)\} \\
& + (\alpha^2 - \beta^2)\{u_j(X)\eta_i(Y) - u_j(Y)\eta_i(X)\} \\
& + 2\alpha\beta\{u_j(X)v_i(Y) - u_j(Y)v_i(X)\} \\
= {} & f_2\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y) + 2\delta_{ij}\bar{g}(X, JY)\}.
\end{aligned}$$

Comparing this with (5.6) such that $PZ = V_j$ and using (3.16), we obtain

$$\begin{aligned} &\{f_1 - f_2 - (\alpha^2 - \beta^2)\}[u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)] \\ &= 2\alpha\beta\{u_j(Y)v_i(X) - u_j(X)v_i(Y)\}. \end{aligned}$$

Taking $X = \xi_i$ and $Y = U_j$, and $X = V_i$ and $Y = U_j$ by turns, we have

$$f_1 - f_2 = \alpha^2 - \beta^2, \qquad \alpha\beta = 0.$$

Applying $\bar{\nabla}_X$ to $\eta_i(Y) = \bar{g}(Y, N_i)$ and using (2.5), we have

$$(\nabla_X \eta_i)Y = -g(A_{N_i}X, Y) + \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y).$$

Applying ∇_Y to $(3.16)_3$ and using (3.13) and (3.22), we have

$$\begin{aligned} (\nabla_X h_i^*)(Y,\zeta) = &-(X\alpha)v_i(Y) + (X\beta)\eta_i(Y) \\ &+ \alpha^2\theta(Y)\eta_i(X) + \beta^2\theta(X)\eta_i(Y) \\ &+ \alpha\{g(A_{N_i}X, FY) + g(A_{N_i}Y, FX) - \sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) \\ &- \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) - \sum_{j=1}^{r} u_j(Y)\eta_i(A_{N_j}X)\} \\ &- \beta\{g(A_{N_i}X, Y) + g(A_{N_i}Y, X) - \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y)\}. \end{aligned}$$

Substituting this and (3.16) into (5.6) such that $PZ = \zeta$, we get

$$\begin{aligned} &\{X\beta + [f_1 - f_3 - (\alpha^2 - \beta^2) - \alpha]\theta(X)\}\eta_i(Y) \\ &- \{Y\beta + [f_1 - f_3 - (\alpha^2 - \beta^2) - \alpha]\theta(Y)\}\eta_i(X) \\ &= (X\alpha)v_i(Y) - (Y\alpha)v_i(X). \end{aligned}$$

Taking $X = \zeta$ and $Y = \xi_i$, and taking $X = U_k$ and $Y = V_i$ by turns, we get

$$f_1 - f_3 = (\alpha^2 - \beta^2) + \alpha - \zeta\beta, \qquad U_i\alpha = 0, \quad \forall\, i.$$

Applying ∇_X to $h_i^\ell(Y,\zeta) = -\alpha u_i(Y)$ and using (3.21) and (3.13), we get

$$\begin{aligned} (\nabla_X h_i^\ell)(Y,\zeta) = -(X\alpha)u_i(Y) + \alpha\{&\sum_{j=1}^{r} u_j(Y)\tau_{ji}(X) + \sum_{a=r+1}^{n} w_a(Y)\phi_{ai}(X) \\ &+ h_i^\ell(X, FY) + h_i^\ell(Y, FX)\}. \end{aligned}$$

Substituting this and (3.16) into (5.5) such that $Z = \zeta$, we obtain

$$(X\alpha)u_i(Y) = (Y\alpha)u_i(X).$$

Replacing Y by U_i to this equation, we obtain $X\alpha = 0$ for all $X \in \Gamma(TM)$. Thus α is a constant. This completes the proof of the theorem. □

We say that $\bar{M}$ (resp. M) is *flat* if $\bar{R} = 0$ (resp. $R = 0$).

Theorem 5.2. *Let M be a recurrent generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a quarter-symmetric metric connection. Then $\bar{M}(f_1, f_2, f_3)$ is flat.*

Proof. As M is recurrent, by Theorem 4.2, we get (4.10), (4.11), (4.12) and the results: $\alpha = \beta = 0$ and $\rho_{ia} = 0$. As $\alpha = \beta = 0$, $f_1 = f_2 = f_3$ by Theorem 5.1. Taking the scalar product with N_j, U_j and W_a to $(4.10)_1$ by turns, we get

$$\eta_j(A_{N_i} X) = 0, \quad h_i^*(X, U_j) = 0, \quad h_a^s(X, U_i) = h_i^*(X, W_a) = 0.$$

Applying ∇_X to $h_i^*(Y, U_j) = 0$ and using $(4.12)_1$, we obtain

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Taking $PZ = U_j$ to (5.6) and using the last two equations, we have

$$f_1\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)\} + f_2\{v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we have $f_1 = 0$. It follows that $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat. □

Theorem 5.3. *Let M be a generic lightlike submanifold of $\bar{M}(f_1, f_2, f_3)$ with a quarter-symmetric metric connection. If M is Lie recurrent, then $\bar{M}(f_1, f_2, f_3)$ is a space form with an indefinite β-Kenmotsu structure such that*

$$f_1 = -\beta^2, \qquad f_2 = 0, \qquad f_3 = \zeta\beta.$$

Proof. Applying ∇_X to $(4.24)_2$: $h_i^\ell(Y, U_j) = -\delta_{ij}\theta(Y)$, we have

$$(\nabla_X h_i^\ell)(Y, U_j) = -\delta_{ij}\{X(\theta(Y)) - \theta(\nabla_X Y)\} - h_i^\ell(Y, \nabla_X U_j).$$

Using this equation, (3.6), $(3.14)_1$, (3.17), $(4.24)_2$ and the facts that $\alpha = 0$, $d\theta = 0$ and $\theta(FX) = 0$, we have

$$\begin{aligned}
&(\nabla_X h_i^\ell)(Y, U_j) - (\nabla_Y h_i^\ell)(X, U_j) \\
= \;& h_i^\ell(X, F(A_{N_j} Y)) - \tau_{ji}(Y)\theta(X) + \sum_{a=r+1}^{n} \rho_{ja}(Y) h_i^\ell(X, W_a) \\
&- h_i^\ell(Y, F(A_{N_j} X)) + \tau_{ji}(X)\theta(Y) - \sum_{a=r+1}^{n} \rho_{ja}(X) h_i^\ell(Y, W_a).
\end{aligned}$$

Replacing Z by U_j to (5.5) and using $(4.24)_2$ and $\theta(FX) = 0$, we obtain

$$\begin{aligned}
&h_i^\ell(X, F(A_{N_j} Y)) - h_i^\ell(Y, F(A_{N_j} X)) \\
&+ \sum_{a=r+1}^{n} \{\rho_{ja}(Y) h_i^\ell(X, W_a) - \rho_{ja}(X) h_i^\ell(Y, W_a)\} \\
&+ \sum_{a=r+1}^{n} \{\phi_{ai}(X) h_a^s(Y, U_j) - \phi_{ai}(Y) h_a^s(X, U_j)\} \\
= \;& f_2\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y) + 2\delta_{ij}\bar{g}(X, JY)\}.
\end{aligned}$$

Taking $Y = U_i$ and $X = \xi_j$ to this equation and using (4.19), (4.27), (4.28) and (4.29), we have $f_2 = 0$. As $f_2 = 0$, we have $f_1 = -\beta^2$ and $f_3 = \zeta\beta$. $\square$

Theorem 5.4. *Let M be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. If U_is are parallel with respect to ∇, then $\tau_{ij} = 0$, $\bar{M}$ is an indefinite cosymplectic manifold and M is solenoidal. Moreover, if $\bar{M} = \bar{M}(f_1, f_2, f_3)$, then it is flat.*

Proof. If U_i is parallel with respect to ∇, then, taking the scalar product with ζ, V_j, W_a, U_j and N_j to (3.17) such that $\nabla_X U_i = 0$ by turns, we get

$$\alpha = \beta = 0, \quad \tau_{ij} = 0, \quad \rho_{ia} = 0, \quad \eta_j(A_{N_i} X) = 0, \quad h_i^*(X, U_j) = 0, \tag{5.7}$$

respectively. As $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold. As $\rho_{ia} = 0$ and $\eta_j(A_{N_i} X) = 0$, M is solenoidal.

As $\alpha = \beta = 0$, $f_1 = f_2 = f_3$ by Theorem 5.1. Applying ∇_Y to $(5.7)_5$ and using $(5.7)_5$ and the fact that $\nabla_X U_i = 0$, we obtain

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Substituting this equation and (5.7) into (5.6) with $PZ = U_j$, we have

$$f_1\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)\} + f_2\{v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we obtain $f_1 = 0$. Therefore, $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat. $\square$

Theorem 5.5. *Let M be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. If V_is are parallel with respect to ∇, then $\tau_{ij} = 0$, $\alpha = -1$ and $\beta = 0$, i.e., $\bar{M}$ is an indefinite Sasakian manifold, and $\phi_{ai} = h_i^\ell(X, \xi_j) = 0$, i.e., M is irrotational. Moreover, if $\bar{M} = \bar{M}(f_1, f_2, f_3)$, then $\bar{M}(f_1, f_2, f_3)$ is a space form with an indefinite Sasakian structure of the curvature functions*

$$f_1 = f_3 = \tfrac{2}{3}, \qquad f_2 = -\tfrac{1}{3}.$$

Proof. If V_i is parallel with respect to ∇, then, taking the scalar product with ζ, U_j, V_j, W_a and N_j to (3.18) with $\nabla_X V_i = 0$ by turns, we get respectively

$$\beta = 0, \quad \tau_{ji} = 0, \quad h_j^\ell(X, \xi_i) = 0, \quad \phi_{ai} = 0, \quad h_i^\ell(X, U_j) = 0 \tag{5.8}$$

and we have $F(A^*_{\xi_i} X) = 0$. As $h_j^\ell(X, \xi_i) = 0$ and $\phi_{ai} = 0$, M is irrotational. Replacing Y by ξ_j and U_j to (3.7) by turns and using $(5.8)_{3,5}$, we have

$$h_i^\ell(\xi_j, X) = 0, \qquad h_i^\ell(U_j, X) = \delta_{ij}\theta(X). \tag{5.9}$$

Taking $X = U_i$ to $(3.14)_1$ and using $(5.9)_2$, we get

$$-\alpha = -\alpha u_i(U_i) = h_i^\ell(U_i, \zeta) = \theta(\zeta) = 1.$$

As $\alpha = -1$ and $\beta = 0$, $\bar{M}$ is an indefinite Sasakian manifold.

Applying ∇_X to $(5.8)_5$ and using (3.4), $(3.14)_1$, (3.17) and $(5.8)_3$, we have

$$(\nabla_X h_i^\ell)(Y, U_j) = \ h_i^\ell(Y, V_k) g(A_{N_j} X, N_k)$$

$$- \sum_{a=r+1}^{n} \rho_{ja}(X) h_i^{\ell}(Y, W_a) - u_i(Y)\eta_j(X).$$

Substituting the last two equations into (5.5) with $Z = U_j$, we obtain

$$\begin{aligned} & h_i^{\ell}(Y, V_k) g(A_{N_j} X, N_k) - h_i^{\ell}(X, V_k) g(A_{N_j} Y, N_k) \\ & + u_i(X)\eta_j(Y) - u_i(Y)\eta_j(X) \\ & + \sum_{a=r+1}^{n} \{\rho_{ja}(Y) h_i^{\ell}(X, W_a) - \rho_{ja}(X) h_i^{\ell}(Y, W_a)\} \\ = \; & f_2\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y) + 2\delta_{ij}\bar{g}(X, JY)\}. \end{aligned}$$

Taking $X = \xi_j$ and $Y = U_i$ to this and using (5.9), we obtain $3f_2 = -1$. As $f_2 = -\frac{1}{3}$, we have $f_1 = f_3 = \frac{2}{3}$ by Theorem 5.1. $\square$

Definition. A screen distribution $S(TM)$ is called *totally umbilical* [4] in M if there exist smooth functions γ_i such that $A_{N_i} X = \gamma_i PX$, or equivalently,

$$h_i^*(X, PY) = \gamma_i g(X, Y).$$

In case $\gamma_i = 0$ for all i, we say that $S(TM)$ is *totally geodesic* in M.

Theorem 5.6. *Let M be a generic lightlike submanifold of $\bar{M}(f_1, f_2, f_3)$ with a quarter-symmetric metric connection. If $S(TM)$ is totally umbilical in M, then $\bar{M}(f_1, f_2, f_3)$ is flat and $S(TM)$ is totally geodesic.*

Proof. Assume that $S(TM)$ is totally umbilical. Then (3.17) is reduced to $\gamma_i\theta(X) = -\alpha v_i(X) + \beta\eta_i(X)$ for all i. Replacing X by V_i, ξ_i and ζ to this equation by turns, we have $\alpha = \beta = \gamma_i = 0$. As $\gamma_i = 0$, $S(TM)$ is totally geodesic. As $\alpha = 0$, $f_1 = f_2 = f_3$ by Theorem 5.1. Taking $PZ = U_k$ to (5.6) with $h_i^* = 0$ and using the facts that $h_a^s(X, U_k) = h_k^*(X, W_a) = 0$ and $h_j^{\ell}(X, U_k) = h_k^*(X, V_j) = 0$, we get

$$f_1\{v_k(Y)\eta_i(X) - v_k(X)\eta_i(Y)\} + f_2\{v_i(Y)\eta_k(X) - v_i(X)\eta_k(Y)\} = 0.$$

Taking $X = \xi_i$ and $Y = V_k$ to this equation, we get $f_1 = 0$. Thus $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat. $\square$

References

[1] P. Alegre, D. E. Blair, and A. Carriazo, *Generalized Sasakian space form*, Israel J. Math. **141** (2004), 157–183.

[2] C. Călin, *Contributions to geometry of CR-submanifold*, Thesis, University of Iasi (Romania, 1998).

[3] G. de Rham, *Sur la réductibilité d'un espace de Riemannian*, Comment. Math. Helv. **26** (1952), 328–344.

[4] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Acad. Publishers, Dordrecht, 1996.

[5] K. L. Duggal and D. H. Jin, *Generic lightlike submanifolds of an indefinite Sasakian manifold*, Int. Electron. J. Geom. **5** (2012), no. 1, 108–119.

[6] D. H. Jin, *Indefinite generalized Sasakian space form admitting a generic lightlike submanifold*, Bull. Korean Math. Soc. **51** (2014), no. 6, 1711–1726.

[7] ______, *Lightlike hypersurfaces of a trans-Sasakian manifold with a quarter-symmetric metric connection*, Appl. Math. Sci. **9** (2015), no. 28, 1393–1406.

[8] ______, *Generic lightlike submanifolds of an indefinite trans-Sasakian manifold of a quasi-constant curvature*, Appl. Math. Sci. **9** (2015), no. 60, 2985–2997.

[9] ______, *Geometry of lightlike hypersurface of an indefinite Kaehler manifold with a quarter-symmetric metric connection*, Appl. Math. Sci. **10** (2016), no. 6, 289–299.

[10] ______, *Geometry of lightlike hypersurface of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection*, Appl. Math. Sci. **10** (2016), no. 13, 625–636.

[11] ______, *Special lightlike hypersurfaces of indefinite Kaehler manifolds*, Filomat **30** (2016), no. 7, 1919–1930.

[12] D. H. Jin and J. W. Lee, *Generic lightlike submanifolds of an indefinite cosymplectic manifold*, Math. Probl. Eng. **2011** (2011), Art ID 610986, 1–16.

[13] ______, *A semi-Riemannian manifold of quasi-constant curvature admits lightlike submanifolds*, Inter. J. Math. Anal. **9** (2015), no. 25, 1215–1229.

[14] ______, *Generic lightlike submanifolds of an indefinite Kaehler manifold*, Inter. J. Pure Appl. Math. **101** (2015), no. 4, 543–560.

[15] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, Kluwer Academic, 366, 1996.

[16] J. A. Oubina, *New classes of almost contact metric structures*, Publ. Math. Debrecen **32** (1985), no. 3-4, 187–193.

[17] K. Yano and T. Imai, *Quarter-symmetric metric connection and their curvature tensors*, Tensor N.S. **38** (1982), 13–18.

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