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REAL HYPERSURFACES WITH $*$-RICCI TENSORS IN COMPLEX TWO-PLANE GRASSMANNIANS

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Abstract. In this article, we consider a real hypersurface of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2}), m \geq 3$, admitting commuting $*$-Ricci and pseudo anti-commuting $*$-Ricci tensor, respectively. As the applications, we prove that there do not exist $*$-Einstein metrics on Hopf hypersurfaces as well as $*$-Ricci solitons whose potential vector field is the Reeb vector field on any real hypersurfaces.

1. Introduction

A complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ consists of all complex two dimensional linear subspaces of $\mathbb{C}^{m+2}$, which is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyper Kähler manifold (see Berndt and Suh [1, 2]). Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$. The Kähler structure J on $G_2(\mathbb{C}^{m+2})$ induces a structure vector field ξ called *Reeb vector field* on M by $\xi := -JN$, where N is the local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. For the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$, its canonical basis $\{J_1, J_2, J_3\}$ induces the almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ on M by $\xi_v := -J_vN$, $v = 1, 2, 3$. It is well known that for the real hypersurface M there exist two natural geometrical conditions that $[\xi] = \mathrm{Span}\{\xi\}$ or $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator A of M. Denote the distribution $\mathfrak{D}$ by the orthogonal complement of the distribution $\mathfrak{D}^\perp$. By using such geometrical conditions, Berndt and Suh in [1] proved that the Reeb vector field ξ either belongs to $\mathfrak{D}$ or $\mathfrak{D}^\perp$ and gave the following classification:

Theorem 1.1. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If $\mathfrak{D}^\perp$ and $[\xi]$ are invariant under the shape operator, then*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ for $\xi \in \mathfrak{D}^\perp$, or*

(B) *M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$ for $\xi \in \mathfrak{D}$, where $m = 2n$.*

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If the Reeb vector field ξ is invariant by the shape operator, M is said to be a *Hopf hypersurface*. Based on the classification of Theorem 1.1 Berndt and Suh later gave a new characterization for the type (B) hypersurfaces of $G_2(\mathbb{C}^{m+2})$.

Theorem 1.2 ([9]). *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2}), m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

As the real hypersurfaces in complex space forms $M_m(c)$ or in quaternionic space forms $Q_m(c)$ with commuting Ricci tensor were considered (cf. [7, 8, 10]), Suh [12] also studied the real hypersurfaces of $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, i.e., $S\phi = \phi S$, where S and ϕ denote the Ricci operator and the structure tensor of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, respectively, and showed that the Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ are of type (A).

Recently Suh [5] introduced a new notion called as *pseudo anti-commuting Ricci tensor*, i.e., it satisfies the following formula:

$$\phi S + S\phi = 2k\phi,$$

where $k = constant$. In this case, it is proved that $k = 4m+2+\frac{\alpha}{2}(h-\alpha)$, where h denotes the mean curvature, or M is the hypersurface of type (B). Since there are no Hopf Einstein real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ (see Corollary in [12]), Suh in [5] further considered a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with a Ricci soliton. The notion of Ricci soliton, introduced firstly by Hamilton in [4], is the generalization of Einstein metric, that is, a Riemannian metric g satisfying

$$\frac{1}{2}\mathcal{L}_W g + Ric - \lambda g = 0,$$

where λ is a constant and Ric is the Ricci tensor of M. The vector field W is called *potential vector field*. Moreover, the Ricci soliton is called shrinking, steady and expanding according as λ is positive, zero and negative respectively. In [5], it is proved that if M is a Hopf hypersurface with potential vector field being the Reeb vector field ξ and Ricci soliton constant $\lambda = k$, then $k = 4(m+1) > 0$, namely the Ricci soliton is shrinking.

As the corresponding of Ricci tensor, Hamada in [3] defined the $*$-Ricci tensor by

$$Ric^*(X,Y) = \frac{1}{2}trace\{\phi \circ R(X,\phi Y)\}, \quad \forall X, Y \in TM, \tag{1}$$

and if the $*$-Ricci tensor is a constant multiple of $g(X,Y)$ for all X, Y orthogonal to ξ, then M is said to be a $*$-*Einstein manifold*. Furthermore, Hamada gave a complete classification of $*$-Einstein Hopf hypersurfaces in non-flat complex space forms. As the generalization of $*$-Einstein metric Kaimakamis and Panagiotidou ([6]) introduced a so-called $*$-Ricci soliton, that is, a Riemannain metric g on M satisfying

$$\frac{1}{2}\mathcal{L}_W g + Ric^* - \lambda g = 0, \tag{2}$$

where λ is constant and Ric^* is the $*$-Ricci tensor of M. They considered the case where W is the Reeb vector field ξ and obtained that a real hypersurface in a complex projective space does not admit a $*$-Ricci soliton as well as that a real hypersurface of complex hyperbolic space admitting a $*$-Ricci soltion is locally congruent to a geodesic hypersphere.

Motivated by the present work, in this paper we first consider the hypersurfaces of $G_2(\mathbb{C}^{m+2})$ with commuting $*$-Ricci tensor, i.e., the $*$-Ricci operator S^* satisfies $\phi S^* = S^*\phi$, where the $*$-Ricci operator S^* is defined by $Ric^*(X,Y) = g(S^*X,Y)$ for any vector fields X, Y, and the following result is proved.

Theorem 1.3. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2}), m \geq 3$, with commuting $*$-Ricci tensor. Then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

In particular, making use of Theorem 1.3 we obtain:

Corollary 1.4. *There do not exist any $*$-Einstein Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.*

For the $*$-Ricci soliton we further get a similar conclusion with the real hypersurfaces in complex projective space $\mathbb{C}P^n$, $n \geq 2$.

Theorem 1.5. *There do not exist real hypersurfaces of $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, admitting a $*$-Ricci soliton, with potential vector field being the Reeb vector field ξ.*

Finally we introduce the notion of *pseudo anti-commuting $*$-Ricci tensor*, i.e. the relation $\phi S^* + S^*\phi = 2k\phi$ holds for constant k, and prove the following:

Theorem 1.6. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2}), m \geq 3$, with pseudo anti-commuting $*$-Ricci tensor. Then $\alpha = 0$ and $k = 4m + 6$.*

This article is organized as follows: In Section 2, some notations and formulas for real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ are presented. In Section 3 we consider Hopf hypersurfaces with commuting $*$-Ricci tensor and give the proofs of Theorem 1.3, Corollary 1.4 and Theorem 1.5. Finally, in Section 4 we study the real Hopf hypersurfaces admitting pseudo anti-commuting $*$-Ricci tensor and prove Theorem 1.6.

2. Preliminaries

In this section we will summarize some basic notations and formulas about the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$. For more detail please refer to [1, 2, 11, 12, 13]. Let $G_2(\mathbb{C}^{m+2})$ be the complex Grassmannian manifold of all complex 2-dimensional linear spaces of $\mathbb{C}^{m+2}$. In fact $G_2(\mathbb{C}^{m+2})$ can be identified with a homogeneous space $SU(m+2)/(S(U(2) \times U(m))$. Up to scaling there exists the unique $S(U(2) \times U(m))$-invariant Riemannian metric $\widetilde{g}$ on $G_2(\mathbb{C}^{m+2})$. The Grassmannian manifold $G_2(\mathbb{C}^{m+2})$ equipped such a metric becomes a symmetric space of rank two, which is both Kähler and quaternionic Kähler. From now on we always assume $m \geq 3$ because it is well known that $G_2(\mathbb{C}^3)$ is isometric to $\mathbb{C}P^2$ and $G_2(\mathbb{C}^4)$ is isometric to the real Grassmannian manifold $G_2^+(\mathbb{R}^6)$ of oriented 2-dimensional linear subspaces of $\mathbb{R}^6$.

Denote J and $\mathfrak{J}$ be the Kähler structure and quaternionic Kähler structure on $G_2(\mathbb{C}^{m+2})$, respectively. A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of almost Hermitian structures J_v such that $J_v J_{v+1} = J_{v+2} = -J_{v+1} J_v$, where the index is taken modulo three. As is well known the Kähler structure J and quaternionic Kähler structure $\mathfrak{J}$ satisfy the following relations:

$$JJ_v = J_v J, \quad trace(JJ_v) = 0, \quad v = 1, 2, 3.$$

We denote $\widetilde{\nabla}$ by the Livi-Civita connection with respect to $\widetilde{g}$, and there exist 1-forms q_1, q_2, q_3 such that

$$\widetilde{\nabla}_X J_v = q_{v+2}(X) J_{v+1} - q_{v+1}(X) J_{v+2}$$

for any vector field X on $G_2(\mathbb{C}^{m+2})$.

Let M be an immersed real hypersurface of $G_2(\mathbb{C}^{m+2})$ with induced metric g. There exists a local defined unit normal vector field N on M and we write

$$\xi := -JN$$

by the structure vector field of M. An induced one-form η is defined by $\eta(\cdot) = \widetilde{g}(J\cdot, N)$, which is dual to ξ. For any vector field X on M the tangent part of JX is denoted by $\phi X = JX - \eta(X)N$. Moreover, the following identities hold:

$$\phi^2 = -Id + \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \phi \circ \xi = 0, \quad \eta(\xi) = 1, \tag{3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{4}$$

where $X, Y \in \mathfrak{X}(M)$. By these formulas, we know that (ϕ, η, ξ, g) is an almost contact metric structure on M. Similarly, for every almost Hermitian structure J_v, it induces an almost contact structure $(\phi_v, \eta_v, \xi_v, g)$ on M by

$$\xi_v = -J_v N, \quad \eta_v(X) = g(\xi_v, X), \quad \phi_v X = J_v X - \eta_v(X)N,$$

for any vector field X. Thus the relations (3) and (4) hold for $(\phi_v, \eta_v, \xi_v, g)$.

Denote ∇, A by the induced Riemannian connection and the shape operator on M, respectively. Then the Gauss and Weigarten formulas are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX, \tag{5}$$

where $\widetilde{\nabla}$ is the connection on $G_2(\mathbb{C}^{m+2})$ with respect to $\widetilde{g}$. Also, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX. \tag{6}$$

Moreover, the following equations are proved (see [5]):

$$\phi_{v+1}\xi_v = -\xi_{v+2}, \quad \phi_v \xi_{v+1} = \xi_{v+2}, \tag{7}$$

$$\phi\xi_v = \phi_v \xi, \quad \eta(\xi_v) = \eta_v(\xi), \tag{8}$$

$$\phi\phi_v X = \phi_v \phi X + \eta_v(X)\xi - \eta(X)\xi_v, \tag{9}$$

$$\nabla_X \xi_v = q_{v+2}(X)\xi_{v+1} - q_{v+1}(X)\xi_{v+2} + \phi_v AX, \tag{10}$$

$$\begin{aligned}\nabla_X(\phi_v \xi) =& q_{v+2}(X)\phi_{v+1}\xi - q_{v+1}(X)\phi_{v+2}\xi \\ &+ \phi_v\phi AX - g(AX,\xi)\xi_v + \eta(\xi_v)AX.\end{aligned} \tag{11}$$

The curvature tensor R and Codazzi equation of M are respectively given as follows:

$$\begin{aligned} & R(X,Y)Z \\ &= g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y + 2g(X,\phi Y)\phi Z \\ &\quad + \sum_{v=1}^{3}\Big\{g(\phi_v Y,Z)\phi_v X - g(\phi_v X,Z)\phi_v Y - 2g(\phi_v X,Y)\phi_v Z\Big\} \\ &\quad + \sum_{v=1}^{3}\Big\{g(\phi_v\phi Y,Z)\phi_v\phi X - g(\phi_v\phi X,Z)\phi_v\phi Y\Big\} \\ &\quad - \sum_{v=1}^{3}\{\eta(Y)\eta_v(Z)\phi_v\phi X - \eta(X)\eta_v(Z)\phi_v\phi Y\} \\ &\quad - \sum_{v=1}^{3}\Big\{\eta(X)g(\phi_v\phi Y,Z) - \eta(Y)g(\phi_v\phi X,Z)\Big\}\xi_v \\ &\quad + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{12}$$

$$\begin{aligned} &(\nabla_X A)Y - (\nabla_Y A)X \\ &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\ &\quad + \sum_{v=1}^{3}\{\eta_v(X)\phi_v Y - \eta_v(Y)\phi_v X - 2g(\phi_v X,Y)\xi_v\} \\ &\quad + \sum_{v=1}^{3}\{\eta_v(\phi X)\phi_v\phi Y - \eta_v(\phi Y)\phi_v\phi X\} \\ &\quad + \sum_{v=1}^{3}\{\eta(X)\eta_v(\phi Y) - \eta(Y)\eta_v(\phi X)\}\xi_v \end{aligned} \tag{13}$$

for any vector fields X, Y, Z on M.

Recall that the $*$-Ricci operator S^* of M is defined by

$$g(S^*X,Y) = Ric^*(X,Y) = \frac{1}{2}trace\{\phi \circ R(X,\phi Y)\}$$

for all $X, Y \in TM$. Taking a local frame $\{e_i\}$ of M such that $e_1 = \xi$ and using (4), we derive from (12) that

$$\sum_{i=1}^{4m-1} g(R(X,\phi Y)e_i, \phi e_i)$$

$$
\begin{aligned}
=&g(\phi^2Y,X)-g(\phi Y,\phi X)+g(\phi X,\phi^3Y)-g(\phi^2Y,\phi^2X)-2(4m-2)g(\phi X,\phi Y)\\
&+\sum_{v=1}^{3}\Big\{-g(\phi_v\phi Y,\phi\phi_v X)-g(\phi\phi_v X,\phi_v\phi Y)+2g(\phi_v X,\phi Y)trace(\phi\phi_v)\Big\}\\
&+\sum_{v=1}^{3}\Big\{g(\phi_v\phi X,\phi\phi_v\phi^2Y)-g(\phi_v\phi^2Y,\phi\phi_v\phi X)\Big\}+\sum_{v=1}^{3}\eta(X)g(\phi_v\phi^2Y,\phi\xi_v)\\
&-\sum_{v=1}^{3}\eta(X)g(\xi_v,\phi\phi_v\phi^2Y)+g(AX,\phi A\phi Y)-g(A\phi Y,\phi AX)\\
=&-8mg(\phi X,\phi Y)+2g(AX,\phi A\phi Y)\\
&-2\sum_{v=1}^{3}\Big\{g(\phi_v\phi Y,\phi\phi_v X)-2g(\phi_v X,\phi Y)\eta_v(\xi)\Big\}\\
&-2\sum_{v=1}^{3}\Big\{g(\phi_v\phi X,\phi\phi_v Y)-g(\phi_v\phi X,\phi\phi_v\xi)\eta(Y)\Big\}\\
&-2\sum_{v=1}^{3}\Big\{g(\phi_v Y,\phi\xi_v)-\eta(Y)g(\phi\xi_v,\phi_v\xi)\Big\}\eta(X).
\end{aligned}
$$

In view of (1), the $*$-Ricci tensor is given by

$$
\begin{aligned}
Ric^*(X,Y)=&\,4mg(\phi X,\phi Y)-g(AX,\phi A\phi Y)\\
&+\sum_{v=1}^{3}\Big\{g(\phi_v\phi Y,\phi\phi_v X)-2g(\phi_v X,\phi Y)\eta_v(\xi)\Big\}\\
&+\sum_{v=1}^{3}\Big\{g(\phi_v\phi X,\phi\phi_v Y)-g(\phi_v\phi X,\phi\phi_v\xi)\eta(Y)\Big\}\\
&+\sum_{v=1}^{3}\Big\{g(\phi_v Y,\phi\xi_v)-\eta(Y)g(\phi\xi_v,\ \phi_v\xi)\Big\}\eta(X)\\
=&\,(4m+6)g(\phi X,\phi Y)-g(AX,\phi A\phi Y)\\
&+2\sum_{v=1}^{3}\Big\{-\eta_v(\phi X)\eta_v(\phi Y)-\eta_v(X)\eta_v(Y)\\
&+\eta(Y)\eta(\xi_v)\eta_v(X)-g(\phi_v X,\phi Y)\eta_v(\xi)\Big\}.
\end{aligned}\tag{14}
$$

Thus the $*$-Ricci operator S^* is expressed as

$$
\begin{aligned}
S^*X=&-(4m+6)\phi^2X-(\phi A)^2X+2\sum_{v=1}^{3}\Big\{\eta_v(\phi X)\phi\xi_v-\eta_v(X)\xi_v\\
&+\eta(\xi_v)\eta_v(X)\xi+\eta_v(\xi)\phi\phi_v X\Big\}
\end{aligned}\tag{15}
$$

for all $X \in TM$. From which a straightforward computation gives:

Proposition 2.1. *For a real hypersurface M of $G_2(\mathbb{C}^{m+2})$ the following formulas hold*:

$$(\phi S^* - S^*\phi)X = \phi[(A\phi)^2 - (\phi A)^2]X - 4\sum_{v=1}^{3}\eta_v(\xi)\eta(X)\phi\xi_v, \tag{16}$$

$$S^*\xi = -(\phi A)^2\xi + 4\sum_{v=1}^{3}\Big\{-\eta_v(\xi)\xi_v + \eta(\xi_v)\eta_v(\xi)\xi\Big\}. \tag{17}$$

If M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, i.e., $A\xi = \alpha\xi$, then taking inner product of the Codazzi equation (13) with ξ implies

$$\begin{aligned}
&-2g(\phi X, Y) + \sum_{v=1}^{3}\{\eta_v(X)\eta(\phi_v Y) - \eta_v(Y)\eta(\phi_v X) - 2g(\phi_v X, Y)\eta(\xi_v)\}\\
&\quad + \sum_{v=1}^{3}\{\eta_v(\phi X)\eta(\phi_v\phi Y) - \eta_v(\phi Y)\eta(\phi_v\phi X)\}\\
&\quad + \sum_{v=1}^{3}\{\eta(X)\eta_v(\phi Y) - \eta(Y)\eta_v(\phi X)\}\eta(\xi_v)\\
&= -2g(\phi X, Y) + 2\sum_{v=1}^{3}\{\eta_v(X)\eta(\phi_v Y) - \eta_v(Y)\eta(\phi_v X) - g(\phi_v X, Y)\eta(\xi_v)\}\\
&= g((\nabla_X A)Y - (\nabla_Y A)X, \xi)\\
&= X(\alpha)\eta(Y) - Y(\alpha)\eta(X) + \alpha g(A\phi X + \phi A X, Y) - 2g(A\phi A X, Y)
\end{aligned} \tag{18}$$

for any vector fields X, Y. From (18), by a straightforward computation we have:

Proposition 2.2 ([2])**.** *If M is a Hopf hypersurface such that α is constant, then*

$$\begin{aligned}
A\phi A X = {}& \frac{1}{2}\alpha(A\phi X + \phi A X) + \phi X\\
& + \sum_{v=1}^{3}\{\eta_v(X)\phi\xi_v + \eta(\phi_v X)\xi_v + \eta(\xi_v)\phi_v X\}.
\end{aligned} \tag{19}$$

3. Real hypersurfaces with commuting $*$-Ricci tensor

In this section we will study the real hypersurface M of complex two-plane Grassmannian $G_2(\mathbb{C}^{m+1})$ admitting commuting $*$-Ricci tensor, namely for every vector field $X \in TM$, the $*$-Ricci operator S^* satisfies

$$\phi S^* X = S^*\phi X.$$

We first prove the following key lemma.

Lemma 3.1. *Let M be a Hopf real hypersurface of $G_2(\mathbb{C}^{m+1})$ with $\phi S^* = S^*\phi$. Then the following statements hold*:

(i) *the principal curvature α is constant*;
(ii) *ξ belongs to $\mathfrak{D}$.*

Proof. By assumption, we take an inner product of (16) with Y and put $X = \xi$, then

$$\sum_{v=1}^{3} \eta_v(\xi)\eta(\phi_v Y) = 0.$$

From this, by replacing Y by ϕY, we conclude

$$\sum_{v=1}^{3} \eta_v^2(\xi)\eta(Y) = \sum_{v=1}^{3} \eta_v(\xi)\eta_v(Y). \tag{20}$$

For any $Y \in \mathfrak{D}$ it follows $\eta(Y)\sum_{v=1}^{3} \eta_v^2(\xi) = 0$. That means that either $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.

Next let us put $X = \xi$ in (18), thus we have

$$Y(\alpha) = \xi(\alpha)\eta(Y) - 4\sum_{v=1}^{3} \eta(\xi_v)\eta(\phi_v Y). \tag{21}$$

Since we have proved that either $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$, then the formula (21) yields

$$\operatorname{grad}(\alpha) = \xi(\alpha)\xi. \tag{22}$$

Differentiating (22) along vector field X gives

$$\nabla_X(\operatorname{grad}\alpha) = \nabla_X(\xi(\alpha))\xi + \xi(\alpha)\phi AX.$$

Since $d^2\alpha = 0$, for any vector Y it follows

$$\begin{aligned} 0 &= g(\nabla_X(\operatorname{grad}\alpha), Y) - g(X, \nabla_Y(\operatorname{grad}\alpha)) \\ &= \nabla_X(\xi(\alpha))\eta(Y) - \nabla_Y(\xi(\alpha))\eta(X) + \xi(\alpha)[g(\phi AX, Y) + g(X, \phi AY)]. \end{aligned}$$

Replacing X and Y by ϕX and ϕY in the above equation, respectively, we find

$$\xi(\alpha)g((A\phi - \phi A)X, Y) = 0$$

for any vector fields X, Y. That means that either $\xi(\alpha) = 0$, which implies $\operatorname{grad}\alpha = 0$ from (22), hence α is constant, or $A\phi = \phi A$. The latter equation yields $(\mathcal{L}_\xi g)(X, Y) = g(X, \phi AY) + g(Y, \phi AX) = 0$ for all vectors X, Y, namely the Reeb flow is isometric. In terms of [2, Proposition 6], α is also constant, thus the statement (i) holds.

If $\xi \in \mathfrak{D}^\perp = \operatorname{Span}\{\xi_1, \xi_2, \xi_3\}$, then in this case there exists an Hermitian structure $J_1 \in \mathfrak{J}$ such that $J_1 N = JN$, that is $\xi = \xi_1$. From (7) we have

$$\phi\xi_2 = \phi_2\xi_1 = -\xi_3, \quad \phi_1\xi_2 = \xi_3, \quad \phi\xi_3 = \phi_3\xi_1 = \xi_2, \tag{23}$$

(Notice that the last equal sign of the formula (5.1) in [5] is wrong, which is easily followed from (7) or see Section 5 in [1].) and from (16) the relation $\phi S^* = S^*\phi$ yields

$$\phi[(A\phi)^2 - (\phi A)^2]X = 4\sum_{v=1}^{3}\eta_v(\xi)\eta(X)\phi\xi_v = 0.$$

Since $A\xi = \alpha\xi$, the previous equation implies

$$(A\phi)^2X = (\phi A)^2X. \tag{24}$$

Because the principal curvature α is constant, the formula (19) holds, and by replacing X by ϕX in this, the relation (24) becomes

$$\begin{aligned}&\frac{1}{2}\alpha A\phi^2X + \sum_{v=1}^{3}\{\eta_v(\phi X)\phi\xi_v + \eta(\phi_v\phi X)\xi_v + \eta(\xi_v)\phi_v\phi X\}\\ &= \frac{1}{2}\alpha\phi^2AX + \sum_{v=1}^{3}\{\eta_v(X)\phi^2\xi_v + \eta(\phi_vX)\phi\xi_v + \eta(\xi_v)\phi\phi_vX\}.\end{aligned}$$

Moreover, by substituting (9) into this and a straightforward calculation, we conclude that

$$\sum_{v=1}^{3}\{\eta_v(\phi X)\xi_v - \eta(\phi_vX)\phi\xi_v + 2\eta(X)\eta(\xi_v)\xi_v - 2\eta_v(X)\eta_v(\xi)\xi\} = 0.$$

Now since $\xi \in \mathfrak{D}^\perp$, we find $\eta_v(\xi) = 0$ for $v = 2, 3$. Hence the above equation yields

$$\sum_{v=1}^{3}\{\eta_v(\phi X)\xi_v - \eta(\phi_vX)\phi\xi_v\} = 0.$$

Moreover, we get $\eta(\phi_vX) = 0$ since ξ_v is orthogonal to $\phi\xi_v$ for all v, thus $\eta_2(X) = \eta_3(X) = 0$ for any vector field X by (23), which is impossible. Therefore ξ can not belong to $\mathfrak{D}^\perp$ and we complete the proof of statement (ii). $\square$

Next we apply Lemma 3.1 to prove Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3. Suppose that M is a Hopf real hypersurface of $G_2(\mathbb{C}^{m+2})$ admitting commuting $*$-Ricci tensor. According to Lemma 3.1, the Reeb vector field ξ belongs to $\mathfrak{D}$. By Theorem 1.2, M is the real hypersurface of type (B), i.e., it is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.

In the following we remaind to show that a hypersurface of type (B) in $G_2(\mathbb{C}^{m+2})$ admits actually commuting $*$-Ricci tensor. Notice that for a real hypersurface of type (B) Berndt and Suh [1] proved the following:

Proposition 3.2. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic* *dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \delta = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \frac{\pi}{4})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\delta) = 4m - 4 = m(\mu),$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = \mathfrak{J}J\xi, \quad T_\gamma = \mathfrak{J}\xi,\, T_\delta,\, T_\mu,$$

where

$$T_\delta \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\delta = T_\delta, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\delta = T_\mu.$$

Since $\xi \in \mathfrak{D}$, by (16) the condition $\phi S^* = S^*\phi$ is equivalent to

$$\phi[(A\phi)^2 - (\phi A)^2]X = 0. \tag{25}$$

Now by Proposition 3.2 we check the formula (25) as follows:

Case I. $X = \xi \in \mathfrak{D}$. It is obvious.

Case II. $X = \xi_1 \in T_\beta$, then $A\phi\xi_1 = 0$.

$$\phi[(A\phi)^2 - (\phi A)^2]\xi_1 = -\phi(\phi A)^2\xi_1 = \beta A\phi\xi_1 = 0.$$

It is easy to see that the formula (25) holds for ξ_2, ξ_3.

Case III. $X = \phi\xi_1 \in T_\gamma, \gamma = 0$, i.e., $A\phi\xi_1 = 0$.

$$\phi[(A\phi)^2 - (\phi A)^2]\phi\xi_1 = \phi A\phi A(-\xi_1 + \eta(\xi_1)\xi) = -\beta\phi A\phi\xi_1 = 0.$$

Case IV. $X \in T_\delta, \delta = \cot r$. Then $AX = \delta X$, $A\phi X = \mu\phi X$. We compute

$$\phi[(A\phi)^2 - (\phi A)^2]X = \phi[\mu A\phi^2 X - \delta\phi A\phi X] = \phi[-\mu\delta X - \delta\mu\phi^2 X] = 0.$$

Case V. $X \in T_\mu, \mu = -\tan r$. Then $AX = \mu X$ and $A\phi X = \delta\phi X$. We also have

$$\phi[(A\phi)^2 - (\phi A)^2]X = \phi[\delta A\phi^2 X - \mu\phi A\phi X] = \phi[-\delta\mu X - \mu\delta\phi^2 X] = 0.$$

Therefore we see the formula (25) holds for all $X \in TM$ and the proof of Theorem 1.3 is completed. □

Proof of Corollary 1.4. Suppose that M is a $*$-Einstein Hopf hypersurface, i.e., $S^*X = aX$, $a = const.$ for any vector field $X \in \xi^\perp$, where $\xi^\perp$ denotes the orthogonal complement of ξ in TM. Since $\phi S^*X = S^*\phi X = a\phi X$, by virtue of Lemma 3.1, ξ tangents to $\mathfrak{D}$, then M is the real hypersurface of type (B) by Theorem 1.3, and the equation (15) can be simplified as

$$aX = -(4m+6)\phi^2 X - (\phi A)^2 X + 2\sum_{v=1}^{3}\Big\{\eta_v(\phi X)\phi\xi_v - \eta_v(X)\xi_v\Big\}, \ \forall X \in \xi^\perp. \tag{26}$$

Let us consider $X \in T_\delta$ then $\phi X \in T_\mu$ by Proposition 3.2. In such a case we derive from the formula (26)

$$aX = -(4m + 6 + \delta\mu)\phi^2 X = (4m + 5)X, \quad \text{i.e.,} \quad a = 4m + 5.$$

However, if let $X = \xi_v \in T_\beta$ in (26) then $A\phi\xi_v = 0$. We obtain

$$a\xi_v = (4m + 4)\xi_v - \beta\phi A\phi\xi_v = (4m + 4)\xi_v, \quad \text{i.e.,} \quad a = 4m + 4.$$

It leads to a contradiction, thus we complete the proof of Corollary 1.4 in introduction. □

Proof of Theorem 1.5. In order to prove Theorem 1.5 we first give the following lemma.

Lemma 3.3. *If the real hypersurface M admits a $*$-Ricci soliton, then $\phi S^* = S^*\phi$.*

Proof. Since $\mathcal{L}_V g$ and g are symmetry, the $*$-Ricci soliton equation (2) implies the $*$-Ricci tensor is also symmetry, i.e., $Ric^*(X,Y) = Ric^*(Y,X)$ for any vector fields X, Y on M. It yields from (14)

$$4\sum_{v=1}^{3}\Big\{\eta_v(X)\xi - \eta(X)\xi_v\Big\}\eta_v(\xi) = [(\phi A)^2 - (A\phi)^2]X$$

for all $X \in TM$. Thus we get the assertion from (16). □

Proposition 3.4. *If M is a real hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ admitting a $*$-Ricci soliton with potential vector field ξ, then M must be Hopf.*

Proof. From the $*$-Ricci soliton equation (2) it follows

$$S^*\phi X = \lambda\phi X + \frac{1}{2}(A\phi - \phi A)\phi X$$

and

$$\phi S^* X = \lambda\phi X + \frac{1}{2}(\phi A\phi - \phi^2 A)X.$$

Thus we obtain from Lemma 3.3

$$\eta(AX)\xi + \eta(X)A\xi = 2\phi A\phi X + 2AX. \tag{27}$$

Taking $X = \xi$, we get $A\xi = \alpha\xi$, where $\alpha = g(A\xi, \xi)$. □

We assume that M is a real hypersurface in $G_2(\mathbb{C}^{m+2})$ admitting a $*$-Ricci soliton with potential vector field ξ. Then M is Hopf by Proposition 3.4. Moreover, by Lemma 3.3 and Lemma 3.1, the Reeb vector field $\xi \in \mathfrak{D}$.

On the other hand, by replacing X by ϕX in (27), we find $A\phi X = \phi AX$ holds for any vector field X. Thus for any vector fields X, Y, it follows from (6)

$$(\mathcal{L}_\xi g)(X,Y) = g(\phi AX - A\phi X, Y) = 0,$$

which shows the Reeb flow is isometric, namely ξ is Killing. According to the main theorem in [2] we complete the proof of Theorem 1.5. □

4. Real hypersurfaces with pseudo anti-commuting $*$-Ricci tensor

In this section we consider the real hypersurface M admitting pseudo anti-commuting $*$-Ricci tensor, i.e. the $*$-Ricci operator S^* satisfies

$$S^*\phi X + \phi S^* X = 2k\phi X, \quad k = const. \tag{28}$$

for every vector field X on M. From this condition we have $\phi S^*\xi = 0$, which further shows $S^*\xi = 0$ since $\eta(S^*\xi) = 0$ followed from (17). Moreover, using (17) again we also get Eq. (20), thus as the proof of Lemma 3.1 we obtain the following lemma.

Lemma 4.1. *Let M be a Hopf real hypersurface of $G_2(\mathbb{C}^{m+1})$ admitting pseudo anti-commuting $*$-Ricci tensor. Then the principle curvature α is constant and ξ either belongs to $\mathfrak{D}$ or $\mathfrak{D}^\perp$.*

Proof of Theorem 1.6. We first show that the Reeb vector field ξ must belong to $\mathfrak{D}^\perp$. Making use of (15), the formula (28) becomes

$$\begin{aligned}0 = 2(4m+6-k)\phi X - (\phi A)^2\phi X - \phi(\phi A)^2 X - 4\sum_{v=1}^{3}\Big\{\eta_v(\phi X)\xi_v \\ + [\eta_v(X) - \eta(\xi_v)\eta(X)]\phi\xi_v - 2\eta_v(\phi X)\eta(\xi_v)\xi + \eta(\xi_v)\phi_v X\Big\}.\end{aligned} \tag{29}$$

When $\xi \in \mathfrak{D}$, we have

$$\begin{aligned}0 = 2(4m+6-k)\phi X - (\phi A)^2\phi X - \phi(\phi A)^2 X \\ - 4\sum_{v=1}^{3}\Big\{\eta_v(\phi X)\xi_v + \eta_v(X)\phi\xi_v\Big\}.\end{aligned} \tag{30}$$

Now by Proposition 3.2, when $X = \xi_1 \in \mathfrak{D}^\perp$, then $A\phi\xi_1 = 0$. It follows from (30)

$$\begin{aligned}2k\phi\xi_1 &= 2(4m+6)\phi\xi_1 - 4\{\eta_2(\phi\xi_1)\xi_2 + \eta_3(\phi\xi_1)\xi_3\} - 4\phi\xi_1 \\ &= (8m+8)\phi\xi_1.\end{aligned}$$

That means $k = 4m + 4$. However, when $X \in T_\delta, \delta = \cot r$, i.e., $AX = \delta X$, $A\phi X = \mu\phi X$. Using (30), we have

$$2k\phi X = 2(4m+6)\phi X + (\mu\delta + \delta\mu)\phi X = [2(4m+6) + 2\delta\mu]\phi X.$$

This shows $k = 4m+6+\delta\mu = 4m+5$. From the difference of k we conclude that there does not exist pseudo anti-commuting $*$-Ricci tensor in the hypersurfaces of type (B). Therefore $\xi \in \mathfrak{D}^\perp$ by Lemma 4.1.

Since $\xi \in \mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$, without loss general we may put $\xi = \xi_1$. Let us take the covariant derivative of equation (28) along vector X, namely

$$(\nabla_X S^*)\phi Y + S^*(\nabla_X\phi)Y + (\nabla_X\phi)S^*Y + \phi(\nabla_X S^*)Y = 2k(\nabla_X\phi)Y. \tag{31}$$

Since $(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi$ and $S^*\xi = 0$, we have

$$S^*(\nabla_X\phi)Y$$

$$= \eta(Y)S^*AX - g(AX,Y)S^*\xi$$
$$= \eta(Y)\Big\{ -(4m+6)\phi^2 AX - (\phi A)^2 AX$$
$$+ 2\sum_{v=1}^{3}[\eta_v(\phi AX)\phi\xi_v - \eta_v(AX)\xi_v + \eta(\xi_v)\eta_v(AX)\xi + \eta_v(\xi)\phi\phi_v AX]\Big\}.$$

By (15), we directly compute

$$\phi(\nabla_X S^*)Y$$
$$= -(4m+6)[\eta(Y)\phi^2 AX] - \phi\nabla_X(\phi A)\phi AY - \phi(\phi A)\nabla_X(\phi A)Y$$
$$+ 2\sum_{v=1}^{3}\Big\{[q_{v+2}(X)\eta_{v+1}(\phi Y) - q_{v+1}(X)\eta_{v+2}(\phi Y)$$
$$+ g(\phi_v AX, \phi Y) + \eta(Y)\eta_v(AX) - g(AX,Y)\eta(\xi_v)]\phi^2\xi_v$$
$$+ \eta_v(\phi Y)[q_{v+2}(X)(\eta_{v+1}(\xi)\xi - \xi_{v+1}) - q_{v+1}(X)(\eta_{v+2}(\xi)\xi - \xi_{v+2})$$
$$- \phi_v AX + \eta_v(\phi AX)\xi + \eta(\xi_v)\phi AX]\Big\}$$
$$+ 2\sum_{v=1}^{3}\Big\{ -[q_{v+2}(X)\eta_{v+1}(Y) - q_{v+1}(X)\eta_{v+2}(Y) + g(\phi_v AX, Y)]\phi\xi_v$$
$$- \eta_v(Y)[q_{v+2}(X)\phi\xi_{v+1} - q_{v+1}(X)\phi\xi_{v+2} + \phi\phi_v AX]\Big\}$$
$$+ 2\sum_{v=1}^{3}\Big\{\eta_v(Y)\phi^2 AX + \phi\nabla_X(\phi\phi_v)Y\Big\}\eta(\xi_v)$$
$$+ 2\sum_{v=1}^{3}\Big\{-\phi_v Y + \eta(\phi_v Y)\xi\Big\}[q_{v+2}(X)\eta_{v+1}(\xi) - q_{v+1}(X)\eta_{v+2}(\xi) + 2\eta(\phi_v AX)].$$

Since $A\xi = \alpha\xi$, using the above two formulas with $\xi \in \mathfrak{D}^\perp$, it follows from (31) with $Y = \xi$

$$2k[AX - \alpha\eta(X)\xi] = S^*AX + \phi(\nabla_X S^*)\xi \tag{32}$$
$$= -2(4m+5)\phi^2 AX - (\phi A)^2 AX - A(\phi A)^2 X$$
$$+ 4\sum_{v=1}^{3}\eta_v(AX)\phi^2\xi_v - 4\sum_{v=1}^{3}[q_{v+2}(X)\eta_{v+1}(\xi) - q_{v+1}(X)\eta_{v+2}(\xi)]\phi\xi_v$$
$$- 2[q_3(X)\phi\xi_2 - q_2(X)\phi\xi_3] + 2\phi\nabla_X(\phi\phi_1)\xi - 4\sum_{v=1}^{3}\eta(\phi_v AX)\phi\xi_v.$$

By (11) and (23), we compute

$$\phi\nabla_X(\phi\phi_1)\xi = q_3(X)\xi_3 + q_2(X)\xi_2 + \phi^2 AX \tag{33}$$

and

$$\sum_{v=1}^{3}\eta_v(AX)\phi^2\xi_v=\sum_{v=1}^{3}\eta(\phi_vAX)\phi\xi_v. \tag{34}$$

Substituting (33) and (34) into (32), we obtain

$$2(4m+4-k)\phi^2AX+A(\phi A)^2X+(\phi A)^2AX=0. \tag{35}$$

Now making use of (19), we compute

$$\begin{aligned}(\phi A)^2AX&=\frac{1}{2}\alpha(\phi A\phi AX-A^2X+\alpha^2\eta(X)\xi)+\phi^2AX\\&\quad+\sum_{v=1}^{3}\{\eta_v(AX)\phi^2\xi_v+\eta(\phi_vAX)\phi\xi_v\}+\phi\phi_1AX\\&=\frac{1}{2}\alpha(\phi A\phi AX-A^2X+\alpha^2\eta(X)\xi)+\phi^2AX\\&\quad-2\{\eta_2(AX)\xi_2+\eta_3(AX)\xi_3\}+\phi\phi_1AX\end{aligned} \tag{36}$$

and

$$\begin{aligned}A(\phi A)^2X&=\frac{1}{2}\alpha(A\phi A\phi X+A\phi^2AX)+A\phi^2X\\&\quad+\sum_{v=1}^{3}\{\eta_v(X)A\phi^2\xi_v+\eta(\phi_vX)A\phi\xi_v\}+A\phi\phi_1X\\&=\frac{1}{2}\alpha(A\phi A\phi X-A^2X+\alpha^2\eta(X)\xi)+\phi^2AX\\&\quad-2\{\eta_2(X)A\xi_2+\eta_3(X)A\xi_3\}+A\phi\phi_1X.\end{aligned} \tag{37}$$

By substituting (36) and (37) into (35), we get

$$\begin{aligned}&(8m+11-2k)\phi^2AX+\frac{1}{2}\alpha(\phi A\phi AX+A\phi A\phi X)\\&\quad+\alpha(-A^2X+\alpha^2\eta(X)\xi)-2\{\eta_2(X)A\xi_2+\eta_3(X)A\xi_3\}+A\phi\phi_1X=0.\end{aligned} \tag{38}$$

Using (19) again, we compute

$$\phi A\phi AX+A\phi A\phi X=\alpha(A\phi^2X+\phi A\phi X)+2\phi^2X-4(\eta_2(X)\xi_2+\eta_3(X)\xi_3)+2\phi\phi_1X,$$

then the relation (38) becomes

$$\begin{aligned}&(8m+11-2k+\frac{1}{2}\alpha^2)\phi^2AX+\frac{1}{2}\alpha^2\phi A\phi X+\alpha\phi^2X+\alpha\phi\phi_1X\\&-2\alpha(\eta_2(X)\xi_2+\eta_3(X)\xi_3)+\alpha(-A^2X+\alpha^2\eta(X)\xi)\\&-2\{\eta_2(X)A\xi_2+\eta_3(X)A\xi_3\}+A\phi\phi_1X=0.\end{aligned} \tag{39}$$

Now putting $X=\xi_2$ in (39) and using (23), we have

$$(8m+12-2k+\frac{1}{2}\alpha^2)A\xi_2+\frac{1}{2}\alpha^2\phi A\xi_3+2\alpha\xi_2+\alpha A^2\xi_2=0. \tag{40}$$

Moreover, taking inner product of the above formula with $X \in \mathfrak{D}$, we get

$$(8m + 12 - 2k + \frac{1}{2}\alpha^2)\eta_2(AX) - \frac{1}{2}\alpha^2\eta_3(A\phi X) + \alpha\eta_2(A^2X) = 0 \quad \text{for all } X \in \mathfrak{D}. \tag{41}$$

In the following we divide into two cases.

Case I. $\alpha = 0$. Then the relation (40) implies $(4m+6-k)A\xi_2 = 0$. Similarly, taking $X = \xi_3$ in (39) and using (23), we get

$$(4m + 6 - k)A\xi_3 = 0.$$

We claim $k = 4m + 6$. Otherwise, if $k \neq 4m + 6$, $A\xi_2 = A\xi_3 = 0$. In view of the relation $2\beta_2\beta_3 - \alpha(\beta_2 + \beta_3) - 4 = 0$, where $A\xi_\mu = \beta_\mu\xi_\mu$, $\mu = 2, 3$ (see [1, Lemma 9]), we derive a contradiction.

Case II. $\alpha \neq 0$. Since $\xi = \xi_1 \in \mathfrak{D}^\perp$, it yields from (29)

$$0 = 2(4m + 6 - k)\phi X - (\phi A)^2\phi X - \phi(\phi A)^2 X - 4\sum_{v=1}^{3}\Big\{\eta_v(\phi X)\xi_v + \eta_v(X)\phi\xi_v\Big\} - 4\phi_1 X. \tag{42}$$

Making use of (23) and (19), the formula (42) becomes

$$2(4m + 7 - k)\phi X + \alpha(A\phi X + \phi AX) - 4(\eta_3(X)\xi_2 - \eta_2(X)\xi_3) - 2\phi_1 X = 0 \quad \text{for all } X \in TM. \tag{43}$$

For every $X \in \mathfrak{D}$, we take an inner product of the above formula with ξ_3 and get

$$\alpha(\eta_3(A\phi X) + \eta_3(\phi AX)) = 0,$$

which shows

$$\eta_3(A\phi X) = \eta_2(AX).$$

Hence the relation (41) becomes

$$(8m + 12 - 2k)\eta_2(AX) + \alpha\eta_2(A^2X) = 0 \quad \text{for all } X \in \mathfrak{D}.$$

From this we have $(8m + 12 - 2k)A\xi_2 + \alpha A^2\xi_2 \in \mathfrak{D}^\perp$. Write $T := (8m + 12 - 2k)A + \alpha A^2$. Thus $T\xi_2 \in \mathfrak{D}^\perp$ and the equation (40) can be rewritten as

$$T\xi_2 + \frac{1}{2}\alpha^2(\phi A\xi_3 + A\xi_2) + 3\alpha\xi_2 = 0.$$

Taking an inner product of this with ξ_2 gives

$$g(T\xi_2, \xi_2) + \frac{1}{2}\alpha^2(g(A\xi_3, \xi_3) + g(A\xi_2, \xi_2)) + 3\alpha = 0. \tag{44}$$

On the other hand, putting $X = \xi_2$ in (43), we have

$$(8m + 12 - 2k)\xi_3 + \alpha(A\xi_3 - \phi A\xi_2) = 0. \tag{45}$$

Taking an inner product of (45) with ξ_3 and substituting the result into (44), we have

$$g(T\xi_2, \xi_2) = \frac{1}{2}\alpha(8m + 12 - 2k) - 3\alpha.$$

That shows $T\xi_2 = \mu\xi_2$, where $\mu = \frac{1}{2}\alpha(8m + 12 - 2k) - 3\alpha$.

Letting $X = \xi_3$ in (39) and $X = \xi_2$ in (43), respectively, we can also derive that $T\xi_3 = \mu\xi_3$ by the same method as before. Actually this shows that $g(T\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$ since $T\xi_1 = T\xi = [(8m+12-2k)\alpha+\alpha^3]\xi$. Moreover, because of the fact that $AT = TA$, there exists a basis X_1, X_2, X_3 of $\mathfrak{D}^{\perp}$ with $AX_i = \lambda_i X_i$ and $TX_i = \lambda_i X_i,\ i = 1, 2, 3$, which satisfies

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = SO(3) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix},$$

where $SO(3)$ denotes the special orthogonal group. Accordingly, we prove that $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$. That means that the distribution $\mathfrak{D}^{\perp}$ is invariant under the shape operator A.

Summarizing the above discussion, in view of Theorem 1.1 we prove the following result.

Proposition 4.2. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$ with pseudo anti-commuting $*$-Ricci tensor. Suppose $A\xi = \alpha\xi$, then one of following holds*:

(1) *If $\alpha = 0$, $k = 4m + 6$;*
(2) *If $\alpha \neq 0$, M is a real hypersurface of type (A) in $G_2(\mathbb{C}^{m+2})$.*

Moreover, notice that for a real hypersurface of type (A) the follow conclusion was given by Berndt and Suh [1].

Proposition 4.3. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^{\perp}$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \frac{\pi}{2}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \frac{\pi}{\sqrt{8}})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN, \\ T_\beta &= \mathbb{C}^{\perp}\xi = \mathbb{C}^{\perp}N, \\ T_\lambda &= \{X | X \perp \mathbb{H}\xi, JX = J_1 X\}, \\ T_\mu &= \{X | X \perp \mathbb{H}\xi, JX = -J_1 X\}. \end{aligned}$$

Since Berndt and Suh [2] proved that the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. Thus the relation $\phi A = A\phi$ is satisfied on M. In view of (16), for $\xi \in \mathfrak{D}^{\perp}$ the condition $S^*\phi + \phi S^* = 2k\phi$ implies

$$(46)\quad (4m+6)\phi X + A^2\phi X - 2\sum_{v=1}^{3}\Big\{\eta_v(X)\phi\xi_v + \eta_v(\phi X)\xi_v\Big\} - 2\phi_1 X = k\phi X.$$

Now we consider the following cases for the above formula.

Case I. When $X = \xi_2$ in (46), we get

$$\begin{aligned}&(4m+6)\phi\xi_2 + \beta^2\phi\xi_2 - 2\{\phi\xi_2 - \xi_3\} - 2\phi_1\xi_2\\ &= [4m+4+\beta^2]\phi\xi_2 = k\phi\xi_2,\end{aligned}$$

i.e., $k = 4m + 4 + \beta^2$.

Case II. $X \in T_\lambda$, $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$. We have $AX = \lambda X$ and $A\phi X = \lambda\phi X$ since $A\phi = \phi A$. From (46) we derive

$$(4m+6)\phi X + \lambda^2\phi X - 2\phi X = k\phi X.$$

So in this case $k = 4m + 4 + \lambda^2$.

Case III. $X \in T_\mu, \mu = 0$, i.e., $A\phi X = 0$. Thus the relation (46) gives

$$(4m+6)\phi X + 2\phi X = k\phi X.$$

This case gives $k = 4m + 8$.

In view of Case I and Case II, we derive that $\lambda^2 = \beta^2$, i.e., $\tan^2(\sqrt{2}r) = \cot^2(\sqrt{2}r)$. However, together Case II and Case III, we get $\lambda^2 = 4$, that shows $\tan^2(\sqrt{2}r) = 2$. It comes to a contradiction. Therefore there can not exist pseudo anti-commuting $*$-Ricci tensor in the real hypersurfaces of type (A) in $G_2(\mathbb{C}^{m+2})$.

Therefore by virtue of Proposition 4.2 we complete the proof of Theorem 1.6. □

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