

## ON HOMOGENEOUS SQUARE EINSTEIN METRICS

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ABSTRACT. We prove that a homogeneous square Einstein Finsler metric is either Riemannian or flat.

### 1. Introduction

Finsler geometry has proven to be very useful in many scientific fields, including general relativity, biology and medical imaging. However, due to the complexity of the related topics, the study of general Finsler metrics is rather complicated. Recently, the study of some special classes of Finsler metrics is considerably active, including Randers metrics,  $(\alpha, \beta)$ -metrics, and square metrics, etc. On the other hand, the study of Einstein metrics has always been one of the central topics in Riemann-Finsler geometry. Meanwhile, the related topics in this direction are also generally rather involved. For example, up to now, the problem whether there exists an Einstein Finsler metric on an arbitrary manifold, first openly asked by S. S. Chern, is still unsolved.

In this short note we prove the following:

**Theorem 1.1.** *Let  $F = \frac{(\alpha+\beta)^2}{\alpha}$  be a homogeneous square Finsler metric on a reductive coset space  $G/K$  with  $\dim G/K \geq 2$ . If  $F$  is Einstein, then it is either Riemannian or flat.*

### 2. Preliminaries

#### 2.1. Square metrics

A square metric on a smooth manifold  $M$  is a Finsler metric of the form  $F = \frac{(\alpha+\beta)^2}{\alpha}$ , where  $\alpha$  is a Riemannian metric, and  $\beta$  is a 1-form on  $M$ . It is known that  $F$  is positive definite if and only if  $\|\beta_x\|_\alpha < 1, \forall x \in M$  (see [7]).

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Let  $G/K$  be a coset space of a Lie group  $G$ , where  $K$  is a closed subgroup of  $G$ . Recall that  $G/K$  is called a reductive homogeneous space if the Lie algebra  $\mathfrak{g}$  of  $G$  has a decomposition

$$(2.1) \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad (\text{direct sum of subspaces})$$

where  $\mathfrak{k}$  is the Lie algebra of  $K$ , and  $\mathfrak{m}$  is a subspace of  $\mathfrak{g}$  satisfying the condition  $\text{Ad}(k)\mathfrak{m} \subseteq \mathfrak{m}$  for any  $k \in K$ . In the literature, (2.1) is called a reductive decomposition. Note that any homogeneous Finsler metric can be viewed as an invariant Finsler metric on certain coset space.

Now assume that  $G/K$  is equipped with a  $G$ -invariant square metric  $F = \frac{(\alpha+\beta)^2}{\alpha}$ . Note that in this case, given any  $x \in M$ , the inner product on  $T_x(M)$  induced by  $\alpha$  defines a linear isomorphism between the tangent space  $T_x(M)$  and the cotangent space  $T_x^*(M)$ . In particular, if  $\beta$  is a smooth 1-form on  $M$ , then there exists a unique smooth vector field  $\tilde{X}$  such that  $\beta(Y) = \langle \tilde{X}|_x, Y \rangle$  for any  $Y \in T_x(M)$  and  $x \in M$ . We first prove the following:

**Lemma 2.1.** *Let  $F = \frac{(\alpha+\beta)^2}{\alpha}$  be a  $G$ -invariant square metric on a reductive coset space  $G/K$ , with the reductive decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ . Then  $\alpha$  is a  $G$ -invariant Riemannian metric and the vector field  $\tilde{X}$  corresponding to the 1-form  $\beta$  is a  $G$ -invariant vector field.*

*Proof.* Since the restriction of the Riemannian metric  $\alpha$  to  $\mathfrak{m}$  is the inner product  $\langle \cdot, \cdot \rangle$ , there exists a vector  $X$  in  $\mathfrak{m}$  which is dual to  $\langle \cdot, \cdot \rangle$ , such that  $F(y) = \frac{(\sqrt{\langle y, y \rangle} + \langle X, y \rangle)^2}{\sqrt{\langle y, y \rangle}}$ ,  $\forall y \in \mathfrak{m}$ . Since  $F$  is  $G$ -invariant, for any  $k \in K$ , and  $y \in \mathfrak{m}$ , we have  $F(\text{Ad}(k)y) = F(y)$ . Then we have

$$(2.2) \quad \frac{(\sqrt{\langle \text{Ad}(k)y, \text{Ad}(k)y \rangle} + \langle X, \text{Ad}(k)y \rangle)^2}{\sqrt{\langle \text{Ad}(k)y, \text{Ad}(k)y \rangle}} = \frac{(\sqrt{\langle y, y \rangle} + \langle X, y \rangle)^2}{\sqrt{\langle y, y \rangle}}.$$

Substituting  $y$  with  $-y$  in (2.2), we obtain

$$(2.3) \quad \frac{(\sqrt{\langle -\text{Ad}(k)y, -\text{Ad}(k)y \rangle} + \langle X, -\text{Ad}(k)y \rangle)^2}{\sqrt{\langle -\text{Ad}(k)y, -\text{Ad}(k)y \rangle}} = \frac{(\sqrt{\langle -y, -y \rangle} + \langle X, -y \rangle)^2}{\sqrt{\langle -y, -y \rangle}}.$$

Subtracting (2.3) from (2.2), we get  $\langle X, \text{Ad}(k)y \rangle = \langle X, y \rangle$ . On the other hand, taking the summation of (2.3) and (2.2), we obtain

$$\sqrt{\langle \text{Ad}(k)y, \text{Ad}(k)y \rangle} + \frac{\langle X, \text{Ad}(k)y \rangle^2}{\sqrt{\langle \text{Ad}(k)y, \text{Ad}(k)y \rangle}} = \sqrt{\langle y, y \rangle} + \frac{\langle X, y \rangle^2}{\sqrt{\langle y, y \rangle}}.$$

It follows from the above equation and the equation  $\langle X, \text{Ad}(k)y \rangle = \langle X, y \rangle$  that

$$\left( \sqrt{\langle \text{Ad}(k)y, \text{Ad}(k)y \rangle} - \sqrt{\langle y, y \rangle} \right) \left( \frac{\langle X, y \rangle^2}{\sqrt{\langle \text{Ad}(k)y, \text{Ad}(k)y \rangle} \sqrt{\langle y, y \rangle}} - 1 \right) = 0.$$

If there exists  $y \neq 0$  such that  $\sqrt{\langle \text{Ad}(k)y, \text{Ad}(k)y \rangle} > \sqrt{\langle y, y \rangle}$ , then by the Cauchy-Schwartz inequality and the assumption that  $F$  is positive definite, we have  $\langle X, y \rangle^2 \leq \langle X, X \rangle \langle y, y \rangle \leq \langle y, y \rangle$ . Thus

$$\frac{\langle X, y \rangle^2}{\sqrt{\langle \text{Ad}(k)y, \text{Ad}(k)y \rangle} \sqrt{\langle y, y \rangle}} - 1 < 0.$$

This then implies that  $\sqrt{\langle \text{Ad}(k)y, \text{Ad}(k)y \rangle} = \sqrt{\langle y, y \rangle}$ , which is a contradiction. Therefore we have  $\sqrt{\langle \text{Ad}(k)y, \text{Ad}(k)y \rangle} \leq \sqrt{\langle y, y \rangle}$ . Note that the above inequality is valid for any  $y$  and  $k$ . Thus

$$\begin{aligned} \sqrt{\langle y, y \rangle} &= \sqrt{\langle \text{Ad}(k^{-1})\text{Ad}(k)y, \text{Ad}(k^{-1})\text{Ad}(k)y \rangle} \\ &\leq \sqrt{\langle \text{Ad}(k)y, \text{Ad}(k)y \rangle}. \end{aligned}$$

Therefore we have  $\sqrt{\langle \text{Ad}(k)y, \text{Ad}(k)y \rangle} = \sqrt{\langle y, y \rangle}$  for any  $k \in K$  any  $y \in \mathfrak{m}$ . Then we also have  $\text{Ad}(k)X = X$  for any  $k \in K$ , completing the proof of the lemma.  $\square$

Let  $u_1, u_2, \dots, u_n$  be an orthonormal basis with respect to  $\alpha$  in  $\mathfrak{m}$ . Then we can define a local coordinates on a neighborhood  $U$  of  $o$  via the map:

$$\exp(x^1 u_1) \exp(x^2 u_2) \cdots \exp(x^n u_n) K \mapsto (x_1, \dots, x_n).$$

Set  $gK = (x_1, \dots, x_n) \in U$ . Then we have

$$\begin{aligned} &\frac{\partial}{\partial x^i} \Big|_{gK} \\ &= \frac{d}{dt} \Big|_{t=0} (\exp(x^1 u_1) \cdots \exp(x^{i-1} u_{i-1}) \exp(t + x^i u_i) \exp(x^{i+1} u_{i+1}) \cdots \exp(x^n u_n)) K \\ &= \frac{d}{dt} \Big|_{t=0} \left( \exp(t e^{x^1 \text{ad} u_1} \cdots e^{x^{i-1} \text{ad} u_{i-1}}(u_i)) gK \right). \end{aligned}$$

Denote  $e^{x^1 \text{ad} u_1} \cdots e^{x^{i-1} \text{ad} u_{i-1}}(u_i) = f_i^a u_a$ . Then we have  $\frac{\partial}{\partial x^i} \Big|_{gK} = f_i^a \hat{u}_a \Big|_{gK}$ , where  $\hat{u}_a$  denotes the fundamental vector field defined by

$$\hat{u}_a \Big|_{gK} = \frac{d}{dt} \exp(t u_a) gK \Big|_{t=0}.$$

Let  $\Gamma_{ij}^l$  be the Christoffel symbols of the Levi-Civita connection of  $\alpha$  under the coordinate system. Then by [3], we have

$$\Gamma_{ij}^l(o) = \frac{1}{2} (-\langle [u_i, u_j]_{\mathfrak{m}}, u_l \rangle + \langle [u_l, u_i]_{\mathfrak{m}}, u_j \rangle + \langle [u_l, u_j]_{\mathfrak{m}}, u_i \rangle) \quad \text{for } i \geq j.$$

The  $G$ -invariant vector field  $\tilde{X}$  dual to the 1-form  $\beta$  is generated by  $X$  in  $\mathfrak{m}$ . Denote  $X = cu_n$ , ( $|c| < 1$ ) and  $\tilde{X}|_{gH} = c \frac{\partial}{\partial x^n} \Big|_{gH}$ . Then we have the following:

$$\begin{aligned} b_i &= \beta\left(\frac{\partial}{\partial x^i}\right) = \langle \tilde{X}, \frac{\partial}{\partial x^i} \rangle = c \langle \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^i} \rangle = ca_{ni}, \\ b_{i|j} &= \frac{\partial b_i}{\partial x^j} - b_l \Gamma_{ij}^l = c \Gamma_{nj}^k a_{ki}, \end{aligned}$$

$$\begin{aligned}
r_{ij} &= \frac{1}{2}(b_{i|j} + b_{j|i}) = \frac{c}{2}(\Gamma_{nj}^k a_{ki} + \Gamma_{ni}^k a_{kj}), \\
s_{ij} &= \frac{1}{2}(b_{i|j} - b_{j|i}) = \frac{c}{2}(\Gamma_{nj}^k a_{ki} - \Gamma_{ni}^k a_{kj}), \\
s_j &= b^i s_{ij} = a^{il} b_l s_{ij} = cs_{nj}.
\end{aligned}$$

Let  $C_{ij}^k$  be the structure constants defined by the equations  $[u_i, u_j] = C_{ij}^k u_k$ . Then it is shown in [3] that:

$$\begin{aligned}
r_{ij}(o) &= -\frac{c}{2}(C_{nj}^i + C_{ni}^j), \\
s_{ij}(o) &= \frac{c}{2}C_{ij}^n, \\
s_j(o) &= cs_{nj} = \frac{c^2}{2}C_{nj}^n.
\end{aligned}$$

By the above relations, we have the following:

**Lemma 2.2.** *Let  $F = \frac{(\alpha+\beta)^2}{\alpha}$  be a  $G$ -invariant square metric on a coset space  $G/K$  generated by a Riemannian metric  $\alpha$  and a vector  $X \in \mathfrak{m}$  with  $\text{Ad}K(X) = X$  and  $\langle X, X \rangle < 1$ . Then the following assertions are equivalent:*

- (1)  $F$  is a Berwald metric.
- (2) For any  $Z_1, Z_2 \in \mathfrak{m}$ ,

$$\langle [Z_1, X]_{\mathfrak{m}}, Z_2 \rangle + \langle [Z_2, X]_{\mathfrak{m}}, Z_1 \rangle = 0, \quad \langle [Z_1, Z_2]_{\mathfrak{m}}, X \rangle = 0.$$

- (3)  $\Gamma_{nj}^i = 0, \forall i, j$ .
- (4)  $C_{ij}^n = 0 = C_{nj}^i + C_{ni}^j$ .

*Proof.* Recall that  $F$  is a Berwald metric if and only if the invariant vector field  $\tilde{X}$  is parallel with respect to  $\alpha$ , if and only if  $\Gamma_{nj}^i = 0, \forall i, j$ , if and only if  $-\langle [u_n, u_j]_{\mathfrak{m}}, u_i \rangle + \langle [u_i, u_j]_{\mathfrak{m}}, u_n \rangle + \langle [u_i, u_n]_{\mathfrak{m}}, u_j \rangle = 0$ , if and only if  $\langle [u_n, u_j]_{\mathfrak{m}}, u_i \rangle + \langle [u_n, u_i]_{\mathfrak{m}}, u_j \rangle = 0$  and  $\langle [u_i, u_j]_{\mathfrak{m}}, u_n \rangle = 0$ , if and only if  $r_{ij}(o) = 0$  and  $s_{ij}(o) = 0$ , if and only if  $C_{ij}^n = 0 = C_{nj}^i + C_{ni}^j$ , if and only if  $\langle [Z_1, X]_{\mathfrak{m}}, Z_2 \rangle + \langle [Z_2, X]_{\mathfrak{m}}, Z_1 \rangle = 0$  and  $\langle [Z_1, Z_2]_{\mathfrak{m}}, X \rangle = 0, \forall Z_1, Z_2 \in \mathfrak{m}$ . This completes the proof of the lemma.  $\square$

### 3. Homogeneous Einstein square metrics

#### 3.1. Shen-Yu's description

Einstein square metrics on smooth manifolds have been described in [7] (see also [2, 10] for related results), and the main results can be summarized as the following

**Theorem 3.1** ([7]). *Let  $F = \frac{(\alpha+\beta)^2}{\alpha}$  be a non-Riemannian square metric on an  $n$ -dimensional manifold  $M$ . Then  $F$  is an Einstein metric if and only if the Riemannian metric  $\bar{\alpha} := (1 - b^2)\alpha$  and the 1-form  $\bar{\beta} := \sqrt{1 - b^2}\beta$  satisfy the condition:  ${}^{\bar{\alpha}}\text{Ric} = -(n-1)k^2\bar{\alpha}$  and  $\bar{b}_{i|j} = k\sqrt{1 + b^2}\bar{a}_{ij}$ , where  $k$  is a constant,*

$\bar{b} = \|\bar{\beta}\|_{\bar{\alpha}}$ , and  $\bar{b}_{i|j}$  is the covariant derivation of  $\bar{\beta}$  with respect to  $\bar{\alpha}$ . Moreover, in this case,  $F$  is given in the following form:

$$F = \frac{(\sqrt{1 + \bar{b}^2} \bar{\alpha} + \bar{\beta})^2}{\bar{\alpha}},$$

with the condition  $(1 + \bar{b}^2)(1 - b^2) = 1$ .

**Remark 3.2.** The second condition of Lemma 2.2 can be replaced with the condition that  $\bar{\beta}$  is a closed form and the vector field  $\bar{X}$ , which is the dual to  $\bar{\beta}$  with respect to  $\bar{\alpha}$ , is a conformal field.

**Remark 3.3.** If  $F$  is a  $G$ -invariant square metric on the coset  $G/H$ , and  $\alpha$  is  $G$ -invariant, then so is  $\bar{\alpha}$ . Moreover, let  $X$  be the dual of  $\beta$  with respect to  $\alpha$ , then one easily gets that  $\bar{X} = \frac{1}{\sqrt{1-b^2}}X$ . From this it follows that, if  $\beta$  is  $G$ -invariant, then so is  $\bar{\beta}$ .

### 3.2. Proof of the main theorem

Combining the above description and the above remarks, we get the following characterization of homogeneous non-Riemannian Einstein square metrics on a reductive coset space.

**Theorem 3.4.** Let  $F = \frac{(\alpha+\beta)^2}{\alpha}$  be a homogeneous non-Riemannian square metric on a reductive coset space  $G/K$ , where  $\dim G/K \geq 2$ , and  $\tilde{X}$  be the vector field dual to the  $G$ -invariant 1-form  $\beta$ . Denote  $X = \tilde{X}|_o$  and assume that  $0 < b^2 = \alpha(X, X) < 1$ . Then  $F$  is an Einstein metric if and only if  $\alpha$  is a homogeneous flat Riemannian metric and we have  $\alpha([Z_1, X]_{\mathfrak{m}}, Z_2) + \alpha([Z_2, X]_{\mathfrak{m}}, Z_1) = 0$  and  $\alpha([Z_1, Z_2]_{\mathfrak{m}}, X) = 0$ ,  $\forall Z_1, Z_2 \in \mathfrak{m}$ .

*Proof.* We first prove the “only if” part. By Lemma 2.1, for any  $k \in K$ , we have  $\text{Ad}(k)(X) = X$ . Now assume that  $F$  is a  $G$ -invariant Einstein metric. Then by Remark 3.3,  $\bar{X}$  is a homothetic vector field on the homogeneous Riemannian manifold  $(G/K, \bar{\alpha})$ . We first prove that  $(G/K, \bar{\alpha})$  is flat. Suppose conversely that it is non-flat. Then  $(G/K, \bar{\alpha})$  is an Einstein Riemannian manifold with negative Ricci scalar. Now there are only two cases:

(1)  $\dim G/K > 2$ . Then by Theorem 10 in [6], one easily deduces that the homothetic vector field  $\bar{X}$  on  $(G/K, \bar{\alpha})$  is just a Killing vector field. Hence by Theorem 3.1, the sectional curvature of  $(G/K, \bar{\alpha})$  must be vanishing, which is a contradiction to our assumption  $\bar{\alpha}Ric < 0$ .

(2)  $\dim G/K = 2$ . Then  $(G/K, \bar{\alpha})$  must be a space of constant curvature. It follows easily from the main results of [8] and [9] that  $(G/K, \bar{\alpha})$  is isometric to a hyperbolic space, and  $(G/K, \bar{\alpha})$  must be an irreducible symmetric space. But in this case there does not exist any nonzero  $G$ -invariant vector field on  $G/K$ , which is also a contradiction.

Therefore  $(G/K, \bar{\alpha})$  must be flat, and  $\bar{X}$  must be a Killing vector field on  $(G/K, \bar{\alpha})$ . Then  $\tilde{X}$  is also a Killing vector field on  $(G/K, \alpha)$  and  $\alpha$  is a homogeneous Ricci-flat Einstein metric. By the main result of [1],  $\alpha$  must be flat. Now a similar argument as in pp. 189–190 in [3] implies that

$$\bar{\alpha}([Z_1, X]_{\mathfrak{m}}, Z_2) + \bar{\alpha}([Z_2, X]_{\mathfrak{m}}, Z_1) = 0, \forall Z_1, Z_2 \in \mathfrak{m}.$$

From this it follows that  $\alpha([Z_1, X]_{\mathfrak{m}}, Z_2) + \alpha([Z_2, X]_{\mathfrak{m}}, Z_1) = 0, \forall Z_1, Z_2 \in \mathfrak{m}$ . On the other hand, since  $\bar{\beta}$  is closed, we have  $\bar{\alpha}([Z_1, Z_2]_{\mathfrak{m}}, \bar{X}|_o) = 0, \forall Z_1, Z_2 \in \mathfrak{m}$ . Then one easily deduces that  $\alpha([Z_1, Z_2]_{\mathfrak{m}}, X) = 0, \forall Z_1, Z_2 \in \mathfrak{m}$ . This completes the proof of the “only if” part.

Conversely, if  $\alpha$  is a homogeneous flat Riemannian metric, and we have

$$\alpha([Z_1, X]_{\mathfrak{m}}, Z_2) + \alpha([Z_2, X]_{\mathfrak{m}}, Z_1) = 0,$$

and

$$\alpha([Z_1, Z_2]_{\mathfrak{m}}, X) = 0, \quad \forall Z_1, Z_2 \in \mathfrak{m}.$$

Then the vector field generated by  $X$  is a Killing field and the corresponding 1-form  $\beta$  is closed. From this it follows that  $F$  is a homogeneous Einstein square metric with zero Ricci scalar.  $\square$

Now by the above theorem and Lemma 2.2, we have:

**Corollary 3.1.** *A homogeneous Einstein square metric must be Berwald.*

*Proof of Theorem 1.1.* If  $F = \frac{(\alpha+\beta)^2}{\alpha}$  is a homogeneous Finsler-Einstein metric on the coset space  $G/K$ , then by Corollary 3.1,  $F$  is Berwald. Now, it is shown in [5] that a Berwald-Einstein Finsler metric is either Riemannian or Ricci flat. Now we suppose that  $F$  is non-Riemannian but Ricci flat. It is proved in [4] that there exists a  $G$ -invariant Riemannian metric  $\tilde{\alpha}$  on  $G/K$  such that the Chern connection of  $F$  and the Levi-Civita connection of  $\tilde{\alpha}$  coincide. Thus  $\tilde{\alpha}$  is Ricci flat. Now by the main result of [1], a Ricci flat homogeneous Riemannian metric must be flat. Thus  $\tilde{\alpha}$  is flat. From this it follows that  $F$  is also flat.  $\square$

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