TRANSLATION SURFACES OF TYPE 2 IN THE THREE DIMENSIONAL SIMPLY ISOTROPIC SPACE $\mathbb{I}_3^1$

Bahaddin Bukcu, Murat Kemal Karacan, and Dae Won Yoon

Abstract. In this paper, we classify translation surfaces of Type 2 in the three dimensional simply isotropic space $\mathbb{I}_3^1$ satisfying some algebraic equations in terms of the coordinate functions and the Laplacian operators with respect to the first, the second and the third fundamental form of the surface. We also give explicit forms of these surfaces.

1. Introduction

Let $\mathbf{x} : \mathbf{M} \to \mathbb{E}^m$ be an isometric immersion of a connected n-dimensional manifold in the m-dimensional Euclidean space $\mathbb{E}^m$. Denote by $\mathbf{H}$ and Δ the mean curvature and the Laplacian of $\mathbf{M}$ with respect to the Riemannian metric on $\mathbf{M}$ induced from that of $\mathbb{E}^m$, respectively. Takahashi proved that the submanifolds in $\mathbb{E}^m$ satisfying $\Delta \mathbf{x} = \lambda \mathbf{x}$, that is, all coordinate functions are eigenfunctions of the Laplacian with the same eigenvalue $\lambda \in \mathbb{R}$, are either the minimal submanifolds of $\mathbb{E}^m$ or the minimal submanifolds of hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{E}^m$ [15].

As an extension of Takahashi theorem, Garay studied hypersurfaces in $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian, but not necessarily associated to the same eigenvalu in [8]. He considered hypersurfaces in $\mathbb{E}^m$ satisfying the condition

$$(1.1) \qquad \Delta \mathbf{x} = \mathbf{A}\mathbf{x},$$

where $\mathbf{A} \in \mathrm{Mat}(m, \mathbb{R})$ is an $m \times m$-diagonal matrix, and proved that such hypersurfaces are minimal ($\mathbf{H} = 0$) in $\mathbb{E}^m$ and open pieces of either round hyperspheres or generalized right spherical cylinders. Related to this, Dillen, Pas and Verstraelen investigated surfaces in $\mathbb{E}^3$ whose immersions satisfy the

condition

(1.2) $\Delta\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}$,

where $\mathbf{A} \in \text{Mat}(3, \mathbb{R})$ is a 3×3-real matrix and $\mathbf{B} \in \mathbb{R}^3$ [6]. In other words, each coordinate function is of 1-type in the sense of Chen [5]. The notion of an isometric immersion $\mathbf{x}$ is naturally extended to smooth functions on submanifolds of Euclidean space or pseudo-Euclidean space. The most natural one of them is the Gauss map of the submanifold. In particular, if the submanifold is a hypersurface, the Gauss map can be identified with the unit normal vector field to it. Dillen, Pas and Verstraelen studied surfaces of revolution in the three dimensional Euclidean space $\mathbb{E}^3$ such that its Gauss map $\mathbf{G}$ satisfies the condition

(1.3) $\Delta\mathbf{G} = \mathbf{A}\mathbf{G}$,

where $\mathbf{A} \in \text{Mat}(3, \mathbb{R})$ [7]. Yoon studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map $\mathbf{G}$ satisfies the condition (1.3) and also translation surfaces in a 3-dimensional Galilean space $\mathbb{G}_3$ satisfies the condition

(1.4) $\Delta\mathbf{x}_i = \lambda_i\mathbf{x}_i$,

where $\lambda_i \in \mathbb{R}$ and provided some examples of new classes of translation surface [16, 17]. Baba-Hamed, Bekkar and Zoubir classified all translation surfaces in the 3-dimensional Lorentz-Minkowski space $\mathbb{R}_1^3$ under the condition (1.4) [2]. Bekkar and Senoussi studied the translation surfaces in the 3-dimensional Euclidean and Lorentz-Minkowski space under the condition

$$\Delta^{\mathbf{III}}\mathbf{r}_i = \mu_i\mathbf{r}_i,$$

where $\mu_i \in \mathbb{R}$ and $\Delta^{\mathbf{III}}$ denotes the Laplacian of the surface with respect to the third fundamental form $\mathbf{III}$ [3]. They showed that in both spaces a translation surface satisfying the preceding relation is a surface of Scherk. Aydin and Sipus studied constant curvatures of translation surfaces in the three dimensional simply Isotropic space [1, 14]. Karacan, Yoon and Bukcu classified translation surfaces of Type 1 satisfying $\Delta^J\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $j = 1, 2$ and $\Delta^{\mathbf{III}}\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $\lambda_i \in \mathbb{R}$ [4, 10].

In this paper, we classify the translation surfaces of Type 2 in the three dimensional simply Isotropic space under the condition $\Delta^J\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $J = \mathbf{I}, \mathbf{II}, \mathbf{III}$, where $\lambda_i \in \mathbb{R}$. Δ^J denotes the Laplace operator with respect to the fundamental forms $\mathbf{I}$, $\mathbf{II}$ and $\mathbf{III}$.

2. Preliminaries

A simply isotropic space $\mathbb{I}_3^1$ is a Cayley–Klein space defined from the three dimensional projective space $\mathcal{P}(\mathbb{R}^3)$ with the absolute figure which is an ordered triple (w, f_1, f_2), where w is a plane in $\mathcal{P}(\mathbb{R}^3)$ and f_1, f_2 are two complex-conjugate straight lines in w. The homogeneous coordinates in $\mathcal{P}(\mathbb{R}^3)$ are introduced in such a way that the absolute plane w is given by $x_0 = 0$ and the

absolute lines f_1, f_2 by $x_0 = x_1 + ix_2 = 0$, $x_0 = x_1 - ix_2 = 0$. The intersection point $\mathbb{F}(0:0:0:1)$ of these two lines is called the absolute point. The group of motions of the simply isotropic space is a six-parameter group given in the affine coordinates $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$, $z = \frac{x_3}{x_0}$ by

$$
\begin{aligned}
\overline{x} &= a + x\cos\theta - y\sin\theta, \\
\overline{y} &= b + x\sin\theta + y\cos\theta, \\
\overline{z} &= c + +c_1 x + c_2 y + z,
\end{aligned}
$$

(2.1)

where $a, b, c, c_1, c_2, \theta \in \mathbb{R}$. Such affine transformations are called isotropic congruence transformations or i-motions.

Isotropic geometry has different types of lines and planes with respect to the absolute figure. A line is called non-isotropic (resp. completely isotropic) if its point at infinity does not coincide (coincides) with the point $\mathbb{F}$. A plane is called non-isotropic (resp. isotropic) if its line at infinity does not contain $\mathbb{F}$, otherwise. Completely isotropic lines and isotropic planes in this affine model appear as vertical, i.e., parallel to the z-axis. Finally, the metric of the simply isotropic space $\mathbb{I}_3^1$ is given by

$$ds^2 = dx^2 + dy^2.$$

A surface $\mathbf{M}$ immersed in $\mathbb{I}_3^1$ is called admissible if it has no isotropic tangent planes. For such a surface, the coefficients E, F, G of its first fundamental form are calculated with respect to the induced metric and the coefficients L, M, N of the second fundamental form, with respect to the normal vector field of a surface which is always completely isotropic. The (isotropic) Gaussian and (isotropic) mean curvature are defined by

(2.2) $$\mathbf{K} = k_1 k_2 = \frac{LN - M^2}{EG - F^2}, \quad 2\mathbf{H} = k_1 + k_2 = \frac{EN - 2FM + GL}{EG - F^2},$$

where k_1, k_2 are principal curvatures, i.e., extrema of the normal curvature determined by the normal section (in completely isotropic direction) of a surface. Since $EG - F^2 > 0$, for the function in the denominator we often put $W^2 = EG - F^2$. The surface $\mathbf{M}$ is said to be isotropic flat (resp. isotropic minimal) if $\mathbf{K}$ (resp. $\mathbf{H}$) vanishes [1, 11, 13, 14].

It is well known in terms of local coordinates $\{u, v\}$ of $\mathbf{M}$ the Laplacian operators $\Delta^{\mathbf{I}}$, $\Delta^{\mathbf{II}}$, $\Delta^{\mathbf{III}}$ of the first, the second and the third fundamental form on $\mathbf{M}$ are defined by

(2.3) $$\Delta^{\mathbf{I}}\mathbf{x} = -\frac{1}{\sqrt{EG - F^2}}\left[\frac{\partial}{\partial u}\left(\frac{G\mathbf{x}_u - F\mathbf{x}_v}{\sqrt{EG - F^2}}\right) - \frac{\partial}{\partial v}\left(\frac{F\mathbf{x}_u - E\mathbf{x}_v}{\sqrt{EG - F^2}}\right)\right],$$

(2.4) $$\Delta^{\mathbf{II}}\mathbf{x} = -\frac{1}{\sqrt{|LN - M^2|}}\left[\frac{\partial}{\partial u}\left(\frac{N\mathbf{x}_u - M\mathbf{x}_v}{\sqrt{|LN - M^2|}}\right) - \frac{\partial}{\partial v}\left(\frac{M\mathbf{x}_u - L\mathbf{x}_v}{\sqrt{|LN - M^2|}}\right)\right]$$

and

$$(2.5) \qquad \Delta^{\mathbf{III}}\mathbf{x} = -\frac{\sqrt{EG-F^2}}{LN-M^2} \left[\begin{array}{c} \frac{\partial}{\partial u}\left(\frac{Z\mathbf{x}_u - Y\mathbf{x}_v}{(LN-M^2)\sqrt{EG-F^2}}\right) - \\ \frac{\partial}{\partial v}\left(\frac{Y\mathbf{x}_u - X\mathbf{x}_v}{(LN-M^2)\sqrt{EG-F^2}}\right) \end{array} \right],$$

where

$$X = EM^2 - 2FLM + GL^2,$$
$$Y = EMN - FLN + GLM - FM^2,$$
$$Z = GM^2 - 2FNM + EN^2$$

[2, 3, 4, 9, 10, 12].

3. Translation surfaces in $\mathbb{I}_3^1$

In order to describe the isotropic analogues of translation surfaces of constant curvatures, we consider translation surfaces obtained by translating two planar curves. The local surface parametrization is given by

$$(3.1) \qquad \mathbf{x}(u,v) = \alpha(u) + \beta(v).$$

Since there are, with respect to the absolute figure, different types of planes in $\mathbb{I}_3^1$, there are in total three different possibilities for planes that contain translated curves: the translated curves can be curves in isotropic planes (which can be chosen, by means of isotropic motions, as $y = 0$, resp. $x = 0$); or one curve is in a non-isotropic plane ($z = 0$) and one curve in an isotropic plane ($y = 0$); or both curves are curves in non-isotropic perpendicular planes ($y - z = \pi$, resp. $y + z = \pi$). Therefore, the obtained translation surfaces allow the following parametrizations:

Type 1: The surface **M** is parametrized by

$$(3.2) \qquad \mathbf{x}(u,v) = (u, v, f(u) + g(v)),$$

and the translated curves are $\alpha(u) = (u, 0, f(u))$, $\beta(v) = (0, v, g(v))$.

Type 2: The surface **M** is parametrized by

$$(3.3) \qquad \mathbf{x}(u,v) = (u, f(u) + g(v), v),$$

and the translated curves are $\alpha(u) = (u, f(u), 0)$, $\beta(v) = (0, g(v), v)$. In order to obtain admissible surfaces, $g'(v) \neq 0$ is assumed (i.e., $g(v) \neq$const.).

Type 3: The surface **M** is parametrized by

$$(3.4) \qquad \mathbf{x}(u,v) = \frac{1}{2}(f(u) + g(v), u - v + \pi, u + v),$$

and the translated curves are

$$\alpha(u) = \frac{1}{2}\left(f(u), u + \frac{\pi}{2}, u - \frac{\pi}{2}\right), \beta(v) = \left(g(v), \frac{\pi}{2} - v, \frac{\pi}{2} + v\right).$$

In order to obtain admissible surfaces, $f'(u) + g'(v) \neq 0$ is assumed (i.e., $f'(u) \neq -g'(v) = a =$constant.) [13].

In this paper, we will investigate the translation surfaces of Type 2.

4. Translation surfaces of Type 2 satisfying $\Delta^{\mathbf{I}}\mathbf{x}_i = \lambda_i\mathbf{x}_i$

In this section, we classify translation surface of Type 2 in $\mathbb{I}_3^1$ satisfying the equation

(4.1) $$\Delta^{\mathbf{I}}\mathbf{x}_i = \lambda_i\mathbf{x}_i,$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$ and

$$\Delta^{\mathbf{I}}\mathbf{x} = \left(\Delta^{\mathbf{I}}\mathbf{x}_1, \, \Delta^{\mathbf{I}}\mathbf{x}_2, \, \Delta^{\mathbf{I}}\mathbf{x}_3\right),$$

where

$$\mathbf{x}_1 = u, \ \mathbf{x}_2 = f(u) + g(v), \ \mathbf{x}_3 = v.$$

For the translation surface given by (3.3), the coefficients of the first and second fundamental form are

(4.2) $$E = 1 + f'^2, \ F = f'g', \ G = g'^2,$$

(4.3) $$L = -\frac{f''}{g'}, \ M = 0, N = -\frac{g''}{g'},$$

respectively. The Gaussian curvature $\mathbf{K}$ and the mean curvature $\mathbf{H}$ are

(4.4) $$\mathbf{K} = \frac{f''(u)g''(v)}{g'^4(v)}, \quad \mathbf{H} = -\frac{g'^2(v)f''(u) + \left(1 + f'^2(u)\right)g''(v)}{2g'^3(v)},$$

respectively.

Suppose that the surface has non zero Gaussian curvature, so

$$f''(u)g''(v) \neq 0, \ \forall \, u, v \in I.$$

By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (4.2) and (2.3) turns out to be

(4.5) $$\Delta^{\mathbf{I}}\mathbf{x} = \left(0, 0, \frac{g'^2 f'' + \left(1 + f'^2\right)g''}{g'^3}\right).$$

Suppose that $\mathbf{M}$ satisfies (4.1). Then from (4.5), we have

(4.6) $$\left(\frac{g'^2 f'' + \left(1 + f'^2\right)g''}{g'^3}\right) = \lambda v,$$

where $\lambda \in \mathbb{R}$. This means that $\mathbf{M}$ is at most of 1-type. First of all, we assume that $\mathbf{M}$ satisfies the condition $\Delta^{\mathbf{I}}\mathbf{x} = 0$. We call a surface satisfying that condition a harmonic surface or isotropic minimal. In this case, we get from (4.6)

(4.7) $$g'^2 f'' + \left(1 + f'^2\right)g'' = 0.$$

Here u and v are independent variables, so each side of (4.7) is equal to a constant, call it p. Hence, the two equations

$$\frac{f''}{1 + f'^2} = p = -\frac{g''}{g'^2}.$$

Thus we get

(4.8)
$$f(u) = c_1 - \frac{\ln\left(\cos\left(pu - c_2\right)\right)}{p},$$
$$g(v) = c_3 + \frac{\ln\left(pv - c_4\right)}{p},$$

where $p, c_i \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

(4.9) $$\mathbf{x}(u,v) = \left(u, \left(c_1 - \frac{\ln\left(\cos\left(pu - c_2\right)\right)}{p}\right) + \left(c_3 + \frac{\ln\left(pv - c_4\right)}{p}\right), v\right).$$

In particular, if $p = 0$, we have

(4.10)
$$f(u) = c_1 u + c_2,$$
$$g(v) = c_3 v + c_4,$$

where $c_i \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

(4.11) $$\mathbf{x}(u,v) = \left(u, \left(c_1 u + c_2\right) + \left(c_3 v + c_4\right), v\right).$$

Using the solution (4.10) gives rise a contradiction with our assumption saying that the solution must be non-degenerate second fundamental form.

Theorem 4.1. *Let $\mathbf{M}$ be a translation surface given by (3.3) in $\mathbb{I}_3^1$. If $\mathbf{M}$ is harmonic or isotropic minimal, then it is congruent to an open part of the surface (4.9). The obtained surface is the isotropic Schrerk's surfaces of the second type.*

If $\lambda \neq 0$, from (4.6), we have

(4.12) $$g'^2 f'' + \left(1 + f'^2\right) g'' = \lambda v g'^3.$$

This differential equation (4.12) admits the solutions

$$f(u) = c_1, \ g(v) = c_2,$$

$$f(u) = c_1, \ g(v) = \pm\frac{\arctan\left(\frac{v\sqrt{\lambda}}{\sqrt{-\lambda v^2 - 2c_1}}\right)}{\sqrt{\lambda}} + c_2,$$

(4.13) $$f(u) = c_1 u + c_2,$$

$$g(v) = c_3 \pm \frac{\sqrt{-1 - c_1^2}\ln\left(v\lambda + \sqrt{\lambda}\sqrt{\lambda v^2 + 2c_4 + 2c_1^2 c_4}\right)}{\sqrt{\lambda}},$$

$$f(u) = \frac{1}{2}c_3 u^2 v\lambda + c_1 u + c_2, \ g(v) = c_3 v + c_4,$$

where $c_i \in \mathbb{R}$. But, using the solutions (4.13) give rise a contradiction with our assumption saying that the solution must be non-degenerate second fundamental form. Based on the selection of the function $f(u)$, it is possible to obtain other form of the function $g(v)$, vice versa. For example, if we choose $g(v) = v^2$, we have

$$f(u) = c_1 + 2v^2 \log \left[\cos \left[\frac{(u + 2v^2 c_2)\sqrt{1 - 4\lambda v^4}}{2v^2} \right] \right],$$

where $c_1, c_2 \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$(4.14) \quad \mathbf{x}(u,v) = \left(u, \left(c_1 + 2v^2 \log \left[\cos \left[\frac{(u + 2v^2 c_2)\sqrt{1 - 4\lambda v^4}}{2v^2} \right] \right] \right) + v^2, v \right).$$

Theorem 4.2. *Let $\mathbf{M}$ be a non harmonic translation surface given by (3.3) in the three dimensional simply isotropic space $\mathbb{I}_3^1$. If The surface $\mathbf{M}$ satisfies the condition $\Delta^{\mathbf{I}} \mathbf{x}_i = \lambda_i \mathbf{x}_i$, where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$, then it is congruent to an open part of the surface (4.14). The obtained surface is the isotropic Schrerk's surface of the second type.*

5. Translation surfaces of Type 2 satisfying $\Delta^{\mathbf{II}} \mathbf{x}_i = \lambda_i \mathbf{x}_i$

In this section, we classify translation surface of Type 2 with non-degenerate second fundamental form in $\mathbb{I}_3^1$ satisfying the equation

$$(5.1) \qquad\qquad \Delta^{\mathbf{II}} \mathbf{x}_i = \lambda_i \mathbf{x}_i,$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$ and

$$\Delta^{\mathbf{II}} \mathbf{x} = \left(\Delta^{\mathbf{II}} \mathbf{x}_1, \Delta^{\mathbf{II}} \mathbf{x}_2, \Delta^{\mathbf{II}} \mathbf{x}_3 \right),$$

where

$$\mathbf{x}_1 = u, \ \mathbf{x}_2 = f(u) + g(v), \ \mathbf{x}_3 = v.$$

By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (4.3) and (2.4) turns out to be

$$(5.2) \qquad \Delta^{\mathbf{II}} \mathbf{x} = \left(-\frac{g' f'''}{2 f''^2}, \frac{g'}{2} \left(4 - \frac{f' f'''}{f''^2} - \frac{g' g'''}{g''^2} \right), -\frac{g' g'''}{2 g''^2} \right).$$

The equation (5.1) by means of (5.2) gives rise to the following system of ordinary differential equations

$$(5.3) \qquad\qquad -\frac{g' f'''}{2 f''^2} = \lambda_1 u,$$

$$(5.4) \qquad \frac{g'}{2} \left(4 - \frac{f' f'''}{f''^2} - \frac{g' g'''}{g''^2} \right) = \lambda_2 \left(f(u) + g(v) \right),$$

$$(5.5) \qquad\qquad -\frac{g' g'''}{2 g''^2} = \lambda_3 v,$$

where $\lambda_i \in \mathbb{R}$. This means that $\mathbf{M}$ is at most of 3-types. Combining equations (5.3), (5.4) and (5.5), we have

$$(5.6) \qquad g'(v)\left(2 + \lambda_1 u + \lambda_3 v\right) = \lambda_2\left(f(u) + g(v)\right).$$

We discuss six cases according to constants $\lambda_1, \lambda_2, \lambda_3$.

Case 1: Let $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0$, from (5.6), we obtain

$$(5.7) \qquad g'\left(2 + \lambda_3 v\right) = \lambda_2 f + \lambda_2 g.$$

Here u and v are independent variables, so each side of (5.7) is equal to a constant, call it p. Hence, the two equations

$$(5.8) \qquad g'\left(2 + \lambda_3 v\right) - \lambda_2 g = p = \lambda_2 f.$$

Their general solutions are

$$(5.9) \qquad \begin{aligned} f(u) &= \frac{p}{\lambda_2}, \\ g(v) &= -\frac{p}{\lambda_2} + c_1\left(2 + \lambda_3 v\right)^{\frac{\lambda_2}{\lambda_3}}, \end{aligned}$$

where for some constant $c_1 \neq 0$ and $\lambda_2 \neq 0, \lambda_3 \neq 0$. In particular, if $p = 0$, then we have

$$(5.10) \qquad \begin{aligned} f(u) &= 0, \\ g(v) &= c_1\left(2 + \lambda_3 v\right)^{\frac{\lambda_2}{\lambda_3}}, \end{aligned}$$

where $c_1 \in \mathbb{R}$.

Case 2: Let $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$, from (5.6), we obtain

$$g'(v)\left(2 + \lambda_3 v\right) = 0.$$

Thus we get

$$(5.11) \qquad g(v) = c_1,$$

where $c_1 \in \mathbb{R}$.

Case 3: Let $\lambda_1 \neq 0, \lambda_2 = 0$, $\lambda_3 = 0$, from (5.6), we obtain

$$(5.12) \qquad g'\left(2 + \lambda_1 u\right) = 0.$$

Thus we get (5.11).

Case 4: Let $\lambda_1 = 0, \lambda_2 \neq 0$, $\lambda_3 = 0$, from (5.6), we obtain

$$(5.13) \qquad 2g' - \lambda_2 g = \lambda_2 f.$$

Here u and v are independent variables, so each side of (5.13) is equal to a constant, call it p. Hence, the two equations

$$2g' - \lambda_2 g = p = \lambda_2 f.$$

Their general solutions are

$$(5.14) \qquad \begin{aligned} f(u) &= \frac{p}{\lambda_2}, \\ g(v) &= -\frac{p}{\lambda_2} + c_1 e^{\frac{\lambda_2 v}{2}}, \end{aligned}$$

where for some constant $c_1 \neq 0$ and $\lambda_2 \neq 0$.

Case 5: Let $\lambda_1 = \lambda_2 = \lambda_3 = 0$, from (5.6), we obtain

$$2g'(v) = 0.$$

Thus we get (5.11). Here, the function $f(u)$ independent of selection of the function $g(v)$.

Case 6: Let $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$, from (5.6), we obtain

$$(5.15) \qquad g'(2 + \lambda_1 u) = \lambda_2 f + \lambda_2 g.$$

Here u and v are independent variables, so each side of (5.15) is equal to a constant, call it p. Hence, the two equations

$$(5.16) \qquad 2g' - \lambda_1 u g' - \lambda_2 g = p = \lambda_2 f.$$

Their general solutions are

$$(5.17) \qquad f(u) = \frac{p}{\lambda_2}, \quad g(v) = -\frac{p}{\lambda_2} + c_1 e^{-\frac{\lambda_2 v}{-2 + \lambda_1 u}},$$

where for some constant $c_1 \neq 0$ and $\lambda_2 \neq 0$.

Using the solutions (5.9), (5.10), (5.11), (5.14) and (5.17) give rise a contradiction with our assumption saying that the solution must be non-degenerate second fundamental form.

Definition. A surface of in the three dimensional simple isotropic space is said to be **II**-harmonic if it satisfies the condition $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{0}$.

Theorem 5.1. *Let $\mathbf{M}$ be a translation surface given by (3.3) in the three dimensional simply isotropic space $\mathbb{I}_3^1$. Then $\mathbf{M}$ is **II**-harmonic if and only if it is an open part of the following non isotropic plane*

$$\mathbf{x}(u, v) = (u, f(u) + c_1, v).$$

The obtained surfaces is a cylindrical surface.

Theorem 5.2. *Let $\mathbf{M}$ be a non **II**-harmonic translation surface with non-degenerate second fundamental form given by (3.3) in the three dimensional simply isotropic space $\mathbb{I}_3^1$. There are no surfaces satisfy the condition*

$$\Delta^{\mathbf{II}}\mathbf{x}_i = \lambda_i \mathbf{x}_i,$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$.

6. Translation surfaces of Type 2 satisfying $\Delta^{\mathbf{III}}\mathbf{x}_i = \lambda_i \mathbf{x}_i$

In this section, we classify translation surface of Type 2 in $\mathbb{I}_3^1$ satisfying the equation

$$(6.1) \qquad \Delta^{\mathbf{III}}\mathbf{x}_i = \lambda_i \mathbf{x}_i,$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$ and

$$\Delta^{\mathbf{III}}\mathbf{x} = \left(\Delta^{\mathbf{III}}\mathbf{x}_1, \ \Delta^{\mathbf{III}}\mathbf{x}_2, \ \Delta^{\mathbf{III}}\mathbf{x}_3 \right),$$

where

$$\mathbf{x}_1 = u, \ \mathbf{x}_2 = f(u) + g(v), \ \mathbf{x}_3 = v.$$

Suppose that the surface has non zero Gaussian curvature, so $f''(u)g''(v) \neq 0, \forall u \in I$. By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (4.2), (4.3) and (2.5) turns out to be

$$(6.2) \quad \Delta^{\mathbf{III}}\mathbf{x} = \left(-\frac{g'f'''}{2f''^2}, -\frac{-4g'f''^2 g''^2 + f'g'g''^2 f''' + g'^2 f''^2 g'''}{2f''^2 g''^2}, -\frac{g'g'''}{2g''^2} \right).$$

Equation (6.1) by means of (6.2) gives rise to the following system of ordinary differential equations

$$(6.3) \qquad\qquad\qquad \left(-\frac{g'f'''}{2f''^2} \right) = \lambda_1 u,$$

$$(6.4) \qquad -\frac{-4g'f''^2 g''^2 + f'g'g''^2 f''' + g'^2 f''^2 g'''}{2f''^2 g''^2} = \lambda_2 \left(f(u) + g(v) \right),$$

$$(6.5) \qquad\qquad\qquad \left(-\frac{g'g'''}{2g''^2} \right) = \lambda_3 v,$$

where λ_1, λ_2 and $\lambda_3 \in \mathbb{R}$. This means that $\mathbf{M}$ is at most of 3- types. Combining equations (6.3), (6.4) and (6.5), we have

$$(6.6) \qquad\qquad 2g' + \lambda_1 u f' + \lambda_3 v g' = \lambda_2 \left(f(u) + g(v) \right).$$

Here u and v are independent variables, so each side of (6.6) is equal to constant, call it p. Hence, we have

$$(6.7) \qquad\qquad \lambda_1 u f' - \lambda_2 f = p = -\lambda_3 v g' - 2g' + \lambda_2 g.$$

Their general solutions are given by

$$(6.8) \qquad\qquad \begin{aligned} f(u) &= -\frac{p}{\lambda_2} + c_1 u^{\frac{\lambda_2}{\lambda_1}}, \\ g(v) &= \frac{p}{\lambda_2} + c_2 \left(2 + \lambda_3 v \right)^{\frac{\lambda_2}{\lambda_3}}, \end{aligned}$$

where $p, c_i \in \mathbb{R}$ with $\lambda_i \neq 0$. In this case, $\mathbf{M}$ is parametrized by

$$(6.9) \qquad \mathbf{x}(u,v) = \frac{1}{2} \left(u, \left(-\frac{p}{\lambda_2} + c_1 u^{\frac{\lambda_2}{\lambda_1}} \right) + \left(\frac{p}{\lambda_2} + c_2 \left(2 + \lambda_3 v \right)^{\frac{\lambda_2}{\lambda_3}} \right), v \right).$$

In particular, if $p = 0$,

$$(6.10) \qquad\qquad \begin{aligned} f(u) &= c_1 u^{\frac{\lambda_2}{\lambda_1}}, \\ g(v) &= c_2 \left(2 + \lambda_3 v \right)^{\frac{\lambda_2}{\lambda_3}}. \end{aligned}$$

In this case, $\mathbf{M}$ is parametrized by

$$(6.11) \qquad \mathbf{x}(u,v) = \left(u, \left(c_1 u^{\frac{\lambda_2}{\lambda_1}} \right) + \left(c_2 \left(2 + \lambda_3 v \right)^{\frac{\lambda_2}{\lambda_3}} \right), v \right).$$

We discuss four cases according to constants $\lambda_1, \lambda_2, \lambda_3$.

Case 1: Let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda \neq 0$, from (6.7), we obtain

$$\lambda u f' - \lambda f = p = -\lambda v g' - 2g' + \lambda g,$$

and their general solutions are

(6.12)
$$f(u) = -\frac{p}{\lambda} + c_1 u,$$
$$g(v) = \frac{p}{\lambda} + c_2 \left(2 + \lambda v\right),$$

where $\lambda, p, c_i \in \mathbb{R}$. Using the solution (6.11) gives rise a contradiction with our assumption saying that the solution must be non-degenerate second fundamental form.

Case 2: Let $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$, from (6.7), we obtain

(6.13)
$$p = -\lambda_3 v g' - 2g'.$$

Thus we have

(6.14)
$$f(u), \ g(v) = c_1 - \frac{p \ln \left(2 + cv\right)}{\lambda_3},$$

where $c_1, p \in \mathbb{R}$. Here, the function $f(u)$ is independent of selection of the function $g(v)$. In this case, $\mathbf{M}$ is parametrized by

(6.15)
$$\mathbf{x}(u, v) = \left(u, f(u) + \left(c_1 - \frac{p \ln \left(2 + cv\right)}{\lambda_3}\right), v\right).$$

Case 3: Let $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$, from (6.7), we obtain

(6.16)
$$\lambda_1 u f' - \lambda_2 f = p = -2g' + \lambda_2 g,$$

and

(6.17)
$$f(u) = -\frac{p}{\lambda_2} + c_1 u^{\frac{\lambda_2}{\lambda_1}},$$
$$g(v) = \frac{p}{\lambda_2} + ce^{\frac{\lambda_2 v}{2}},$$

where $c_i, p \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

(6.18)
$$\mathbf{x}(u, v) = \left(u, \left(-\frac{p}{\lambda_2} + c_1 u^{\frac{\lambda_2}{\lambda_1}}\right) + \left(\frac{p}{\lambda_2} + ce^{\frac{\lambda_2 v}{2}}\right), v\right).$$

Case 4: Let $\lambda_1 = \lambda_2 = \lambda_3 = 0$, from (4.12), we obtain

(6.19)
$$g(v) = c_1 + \frac{p}{2}v,$$

where $c_i \in \mathbb{R}$. Here, the function $f(u)$ is independent of selection of the function $g(v)$. In this case, $\mathbf{M}$ is parametrized by

(6.20)
$$\mathbf{x}(u, v) = \left(u, v, f(u) + \left(c_1 + \frac{p}{2}v\right)\right).$$

Definition. A surface of in the three dimensional simple isotropic space is said to be **III**-harmonic if it satisfies the condition $\Delta^{\mathbf{III}}\mathbf{x} = \mathbf{0}$.

Theorem 6.1. *Let* **M** *be a translation surface by* (3.3) *in the three dimensional simply isotropic space* $\mathbb{I}_3^1$. *If* **M** *is* **III**-*harmonic, then it is congruent to an open part of the surface* (6.20). *The obtained surfaces is a cylindrical surface.*

Theorem 6.2 (Classification)**.** *Let* **M** *be a translation surface with non - degenerate second fundamental form given by* (3.3) *in the three dimensional simply isotropic space* $\mathbb{I}_3^1$. *The surfaces* **M** *satisfies the condition* $\Delta^{\mathbf{III}}\mathbf{x}_i=\lambda_i\mathbf{x}_i$, *where* $\lambda_i\in\mathbb{R}$, *then it is congruent to an open part of the surfaces* (6.9), (6.11), (6.15) *and* (6.18). *The obtained surfaces are the isotropic Schrerk's surfaces of the first type.*

References

[1] M. E. Aydin, *A generalization of translation surfaces with constant curvature in the isotropic space*, J. Geom. **107** (2016), no. 3, 603–615.

[2] Ch. Baba-Hamed, M. Bekkar, and H. Zoubir, *Translation surfaces in the three dimensional Lorentz-Minkowski space satisfying* $\Delta r_i = Ar_i$, Int. J. Math. Anal. **4** (2010), no. 17, 797–808.

[3] M. Bekkar and B. Senoussi, *Translation surfaces in the 3-dimensional space satisfying* $\Delta^{III}r_i = \mu_i r_i$, J. Geom. **103** (2012), no. 3, 367–374.

[4] B. Bukcu, D. W. Yoon, and M. K. Karacan, *Translation surfaces in the 3-dimensional simply isotropic space* $\mathbb{I}_3^1$ *satisfying* $\Delta^{III}x_i = \lambda_i x_i$, Konuralp J. Math. **4** (2016), no. 1, 275–281.

[5] B. Y. Chen, *A report on submanifold of finite type*, Soochow J. Math. **22** (1996), no. 2, 117–337.

[6] F. Dillen, J. Pas, and L. Vertraelen, *On surfaces of finite type in Euclidean 3-space*, Kodai Math. J. **13** (1990), no. 1, 10–21.

[7] _____, *On the Gauss map of surfaces of revolution*, Bull. Inst. Math. Acad. Sinica **18** (1990), no. 3, 239–246.

[8] O. J. Garay, *An extension of Takahashi's theorem*, Geom. Dedicata **34** (1990), no. 2, 105–112.

[9] G. Kaimakamis, B. Papantoniou, and K. Petoumenos, *Surfaces of revolution in the 3-dimensional Lorentz-Minkowski space satisfying* $\Delta^{III}r = Ar$, Bull. Greek Math. Soc. **50** (2005), 75–90.

[10] M. K. Karacan, D. W. Yoon, and B. Bukcu, *Translation surfaces in the three dimensional simply isotropic space* $\mathbb{I}_3^1$, Int. J. Geom. Methods Mod. Phys. **13** (2016), no. 7, 1650088, 9 pp.

[11] H. Sachs, *Isotrope geometrie des raumes*, Vieweg Verlag, Braunschweig, 1990.

[12] B. Senoussi and M. Bekkar, *Helicoidal surfaces with* $\Delta^J r = Ar$ *in 3-dimensional Euclidean space*, Stud. Univ. Babes-Bolyai Math. **60** (2015), no. 13, 437–448.

[13] Z. M. Sipus, *Translation surfaces of constant curvatures in a simply Isotropic space*, Period Math. Hungar. **68** (2014), no. 2, 160–175.

[14] K. Strubecker, *Differentialgeometrie des Isotropen raumes III. Flächentheorie*, Math. Z. **48** (1942), 369–427.

[15] T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385.

[16] D. W. Yoon, *On the Gauss map of translation surfaces in Minkowski 3-space*, Taiwanese J. Math. **6** (2002), no. 3, 389–398.

[17] _____, *Some classification of translation surfaces in Galilean 3-Space*, Int. J. Math. Anal. **6** (2012), no. 28, 1355–1361.

BAHADDIN BUKCU
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES AND ARTS
GAZI OSMAN PASA UNIVERSITY
60250, TOKAT, TURKEY
E-mail address: bbukcu@yahoo.com

MURAT KEMAL KARACAN
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES AND ARTS
USAK UNIVERSITY
1 EYLUL CAMPUS 64200, USAK, TURKEY
E-mail address: murat.karacan@usak.edu.tr

DAE WON YOON
DEPARTMENT OF MATHEMATICS
EDUCATION AND RINS
GYEONGSANG NATIONAL UNIVERSITY
JINJU 660-701, KOREA
E-mail address: dwyoon@gnu.ac.kr