

A WEIGHTED FOURIER SERIES WITH SIGNED GOOD KERNELS

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ABSTRACT. It is natural to try to find a kernel such that its convolution of integrable functions converges faster than that of the Fejér kernel. In this paper, we introduce a weighted Fourier partial sums which are written as the convolution of signed good kernels and prove that the weighted Fourier partial sum converges in L^2 much faster than that of the Cesàro means. In addition, we present two numerical experiments.

1. Introduction

Any $f \in L^1([-\pi, \pi]) \equiv L^1$ is associated with its Fourier series

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx},$$

where the Fourier coefficients c_k are given by

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx.$$

The n -th Fourier partial sum of f is defined as

$$(1) \quad s_n(f, x) = \sum_{|k| \leq n} c_k e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x-y) f(y) dy = (D_n * f)(x),$$

where $D_n(x) = \sum_{k=-n}^n e^{ikx}$ is the Dirichlet kernel and $D_n * f$ is the convolution of D_n and f . The kernel $D_n(x)$ also has the closed form of

$$(2) \quad D_n(x) = \frac{\sin((n + \frac{1}{2})x)}{\sin(\frac{x}{2})}.$$

There are many results on $s_n(f, x)$ which has been studied up to now. Even though integral value of the even function $D_n(x)$ is 1 but is not in L^1 , precisely $\|D_n\|_1 = O(\ln n)$. Moreover, $s_n(f, x)$ fails to converge for some continuous functions. In [5], Fejér introduces the Cesàro mean of the Fourier partial sums, by

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which he extends the validity of the series. More precisely, better convergence properties are achieved by means of Cesàro sums which are defined by

$$(3) \quad \sigma_n(f, x) = \frac{1}{n+1} \sum_{k=0}^n s_k(f, x).$$

Setting $K_n(x) = \frac{1}{n+1} \sum_{k=0}^n D_k(x)$, which is called the Fejér kernel, one has $\sigma_n(f, x) = (K_n * f)(x)$. Since the Cesàro mean converges in L^1 , this can be used as alternative for the convergence of the Fourier series. In spite of the advantage, its convergence rate is not better than the Fourier partial sums for smooth functions. So it is of interest to find another weighted Fourier partial sums which have higher convergence rate than Cesàro means.

Before we outline the results of this paper, it will be convenient and necessary to recall some definitions and set down some notational conventions, most of which have been taken from Zygmund [17].

There are many results on Cesàro means for Fourier series. We only make mention of a few recent articles here. For absolute summability, see [2, 6, 12, 16]. There are properties for Cesàro summabilities of positive (or, negative) order ([3], [4]). About the summability of Cesàro means over various domains, refer to [1, 13, 14, 15]. Also, [7, 11] analyze Cesàro means over several kinds of function-spaces. Moreover, there are some results for orthonormality of Dirichlet kernels [8, 9]. In [10] the author relates Cesàro means to a Markov process.

In this paper, we introduce a weighted Fourier series motivated from the Cesàro mean of the Fourier partial sums and derive the convolution form of the suggested series, equipped with a family of proper signed kernels. In addition, we show that L^2 convergence of the weighted Fourier partial sums is much faster than that of the Cesàro means.

This article is organized as follows. Section 2 provides the definition and properties of a weighted Fourier partial sum equipped with second order derivatives of e^{ikx} 's, which are motivated from the Cesàro mean in first order derivatives. Also it is shown that the signed kernel derived from the weighted Fourier partial sum satisfies three properties which are used to prove the reproducing result for the weighted Fourier series in the next section. In Section 3, the first main theorem, pointwise convergence for weighted Fourier partial sums is stated and proved. In Section 4, it is proven that the L^2 -norm convergence of weighted Fourier partial sums is better compared to that of the Cesàro means. The last section presents some numerical results of two functions.

2. Kernel for the weighted Fourier series

We write the Cesàro mean as the first derivatives,

$$(4) \quad \begin{aligned} \sigma_n(f, x) &= (K_n * f)(x) \\ &= \sum_{|k| \leq n} c_k e^{ikx} - \frac{1}{n+1} \left(\frac{d}{d(ix)} \sum_{k=0}^n c_k e^{ikx} + \frac{d}{d(-ix)} \sum_{k=0}^n c_{-k} e^{-ikx} \right) \end{aligned}$$

and replace first order derivatives with second order derivatives in (4) as
(5)

$$\tilde{\sigma}_n(f, x) = \sum_{|k| \leq n} c_k e^{ikx} - \frac{1}{(n+1)^2} \left(\frac{d^2}{d(ix)^2} \sum_{k=0}^n c_k e^{ikx} + \frac{d^2}{d(-ix)^2} \sum_{k=0}^n c_{-k} e^{-ikx} \right).$$

And now we explore how the notion of $\tilde{\sigma}_n(f, x)$ applies to Fourier series readily.

First, put

$$(6) \quad F_n(x) = D_n(x) + \frac{1}{(n+1)^2} \frac{d^2}{dx^2} D_n(x) \quad (n \geq 0).$$

From the definition of $\tilde{\sigma}_n$, the next result follows:

Proposition 2.1. *For $n \geq 0$, we have*

$$(7) \quad \begin{aligned} \tilde{\sigma}_n(f, x) &= (F_n * f)(x) \\ &= \sum_{|k| \leq n} \left(1 - \frac{k^2}{(n+1)^2} \right) c_k e^{ikx}. \end{aligned}$$

Moreover, some elementary properties of $F_n(x)$ are stated below.

$$\textbf{Lemma 2.2.} \quad (a) \quad F_n(x) = \sum_{|k| \leq n} \left(1 - \frac{k^2}{(n+1)^2} \right) e^{ikx}.$$

$$(b) \quad F_n(x) = \frac{-(n + \frac{1}{2}) \sin(\frac{x}{2}) \cos((n+1)x) + \frac{1}{2} \sin((n + \frac{1}{2})x)}{(n+1)^2 \sin^3(\frac{x}{2})}.$$

Proof. From the definition of F_n , since

$$\frac{d^2}{dx^2} D_n(x) = - \sum_{|k| \leq n} k^2 e^{ikx},$$

we have

$$\begin{aligned} F_n(x) &= \sum_{|k| \leq n} e^{ikx} - \frac{1}{(n+1)^2} \sum_{|k| \leq n} k^2 e^{ikx} \\ &= \sum_{|k| \leq n} \left(1 - \frac{k^2}{(n+1)^2} \right) e^{ikx}. \end{aligned}$$

This finishes the proof of (a).

From (2), by differentiation, we have

$$D'_n(x) = \frac{(n + \frac{1}{2}) \cos((n + \frac{1}{2})x) \sin(\frac{x}{2}) - \frac{1}{2} \cos(\frac{x}{2}) \sin((n + \frac{1}{2})x)}{\sin^2(\frac{x}{2})}$$

and

$$D''_n(x) = \frac{-(n^2 + n) \sin((n + \frac{1}{2})x) \sin^2(\frac{x}{2})}{\sin^3(\frac{x}{2})}$$

$$- \frac{(n + \frac{1}{2}) \cos((n + \frac{1}{2})x) \sin(\frac{x}{2}) \cos(\frac{x}{2}) + \frac{1}{2} \cos^2(\frac{x}{2}) \sin((n + \frac{1}{2})x)}{\sin^3(\frac{x}{2})}.$$

From (6) and according to elementary properties of $\sin(x)$ and $\cos(x)$, we have

$$\begin{aligned} F_n(x) &= \frac{(n+1)^2 \sin((n + \frac{1}{2})x) \sin^2(\frac{x}{2}) - (n^2 + n) \sin((n + \frac{1}{2})x) \sin^2(\frac{x}{2})}{(n+1)^2 \sin^3(\frac{x}{2})} \\ &\quad - \frac{(n + \frac{1}{2}) \cos((n + \frac{1}{2})x) \sin(\frac{x}{2}) \cos(\frac{x}{2}) + \frac{1}{2} \cos^2(\frac{x}{2}) \sin((n + \frac{1}{2})x)}{(n+1)^2 \sin^3(\frac{x}{2})} \\ &= \frac{(n+1) \sin((n + \frac{1}{2})x) \sin^2(\frac{x}{2})}{(n+1)^2 \sin^3(\frac{x}{2})} - \frac{(n + \frac{1}{2}) \cos((n + \frac{1}{2})x) \sin(\frac{x}{2}) \cos(\frac{x}{2})}{(n+1)^2 \sin^3(\frac{x}{2})} \\ &\quad + \frac{\frac{1}{2} \cos^2(\frac{x}{2}) \sin((n + \frac{1}{2})x)}{(n+1)^2 \sin^3(\frac{x}{2})} \\ &= \frac{-(n + \frac{1}{2}) \sin(\frac{x}{2}) \cos((n + \frac{1}{2})x + \frac{x}{2}) + \frac{1}{2} \sin((n + \frac{1}{2})x)}{(n+1)^2 \sin^3(\frac{x}{2})} \\ &= \frac{-(n + \frac{1}{2}) \sin(\frac{x}{2}) \cos((n+1)x) + \frac{1}{2} \sin((n + \frac{1}{2})x)}{(n+1)^2 \sin^3(\frac{x}{2})}, \end{aligned}$$

which proves (b). Therefore, the proof is complete. \square

Corollary 2.3. (a) $|F_n(x)| \leq \frac{5(2n+1)}{6}$.

$$(b) \quad |F_n(x)| \leq \frac{5\pi^3}{4(n+1)|x|^3} \quad (0 < |x| < \pi).$$

Proof. By (a) of Lemma 2.2,

$$\begin{aligned} |F_n(x)| &\leq \sum_{|k| \leq n} \left(1 - \frac{k^2}{(n+1)^2} \right) \\ &= (2n+1) \left(1 - \frac{n}{3(n+1)} \right), \end{aligned}$$

where the first inequality comes from $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$ and the last equality holds from the fact that $1 - \frac{n}{3(n+1)}$ is decreasing. Hence, we prove (a).

Next, let $0 < |x| < \pi$. We may assume $0 < x < \pi$. By (b) of Lemma 2.2,

$$\begin{aligned} |F_n(x)| &\leq \frac{(n + \frac{1}{2})\pi^2}{(n+1)^2 x^2} + \frac{\pi^3}{2(n+1)^2 x^3} \\ &\leq \frac{\pi^2}{(n+1)x^3} \left(x + \frac{\pi}{2(n+1)} \right). \end{aligned}$$

By substituting $\pi, 1$ for x, n , respectively, we have (b). Therefore, the proof is complete. \square

A family of kernels $\{G_n(x)\}_{n=1}^\infty$ on $[-\pi, \pi]$ is said to be a family of good kernels if it satisfies the following properties:

(i) For all $n \geq 1$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} G_n(x) dx = 1.$$

(ii) There exists $M > 0$ such that for all $n \geq 1$,

$$\int_{-\pi}^{\pi} |G_n(x)| dx \leq M.$$

(iii) For every $\delta > 0$,

$$\int_{\delta \leq |x| \leq \pi} |G_n(x)| dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Here note that $G_n(x)$ may not be nonnegative. Properties (i), (ii) and (iii) mean the normalizability, the uniform integrability and vanishment of tails in the L^1 -limit, respectively.

As basic results of this paper, it is shown that $\{F_n(x)\}_{n=1}^\infty$ is a family of good kernels through the following lemmas.

Lemma 2.4. *The family of $\{F_n(x)\}_{n=1}^\infty$ is normalized.*

Proof. It follows readily that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x) dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{|k| \leq n} e^{ikx} - \frac{1}{(n+1)^2} \sum_{|k| \leq n} k^2 e^{ikx} \right) dx \\ &= \sum_{|k| \leq n} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} dx - \frac{1}{(n+1)^2} \sum_{|k| \leq n} \frac{k^2}{2\pi} \int_{-\pi}^{\pi} e^{ikx} dx \\ &= 1. \end{aligned} \quad \square$$

Lemma 2.5. *The family of $\{F_n(x)\}_{n=1}^\infty$ is uniformly integrable.*

Proof. We begin by splitting the integral into two parts:

$$\int_{-\pi}^{\pi} |F_n(x)| dx = 2 \int_0^{\frac{1}{n}} |F_n(x)| dx + 2 \int_{\frac{1}{n}}^{\pi} |F_n(x)| dx \equiv 2I + 2II.$$

Estimate of I : From (a) of Corollary 2.3, we get

$$(8) \quad I = \int_0^{\frac{1}{n}} |F_n(x)| dx \leq \frac{5(2n+1)}{6n} \leq \frac{5}{2}.$$

Estimate of II : It follows from (b) of Lemma 2.2 that

$$\begin{aligned} F_n(x) &= \frac{-(n+1) \sin(\frac{x}{2}) \cos(n+1)x + \frac{1}{2} \sin((n+1)x) \cos(\frac{x}{2})}{(n+1)^2 \sin^3(\frac{x}{2})} \\ &= \frac{-\sin(\frac{x}{2}) \frac{d}{dx} \sin((n+1)x) + \frac{1}{2} \sin((n+1)x) \cos(\frac{x}{2})}{(n+1)^2 \sin^3(\frac{x}{2})}. \end{aligned}$$

Since $(n+1)^2 > n^2$ for all $n \geq 0$, we get

$$\begin{aligned} |F_n(x)| &\leq \left| \frac{-\sin(\frac{x}{2}) \frac{d}{dx} \sin((n+1)x) + \frac{1}{2} \sin((n+1)x) \cos(\frac{x}{2})}{n^2 \sin^3(\frac{x}{2})} \right| \\ &\leq \left| \frac{\frac{d}{dx} \sin((n+1)x)}{n^2 \sin^2(\frac{x}{2})} \right| + \left| \frac{\sin((n+1)x) \cos(\frac{x}{2})}{2n^2 \sin^3(\frac{x}{2})} \right| \\ &\equiv A + B. \end{aligned}$$

Then

$$II \leq \int_{1/n}^{\pi} A dx + \int_{1/n}^{\pi} B dx \equiv III + IV.$$

Estimate of IV : It follows that

$$\begin{aligned} (9) \quad IV &\leq \frac{1}{n^2} \int_{\frac{1}{n}}^{\pi} \frac{|\sin((n+1)x)|}{\sin^3(\frac{x}{2})} dx \\ &\leq \frac{\pi^3}{n^2} \int_{\frac{1}{n}}^{\pi} \frac{1}{x^3} dx \\ &= \frac{\pi^3}{2} \left(1 - \frac{1}{\pi^2 n^2} \right) \\ &\leq \frac{\pi^3}{2}. \end{aligned}$$

Estimate of III : We will estimate III by taking a partition of $[1/n, \pi]$. The inequality, $\sin(x/2) \geq x/\pi$ implies

$$\begin{aligned} III &= \frac{1}{n^2} \left(\int_{\frac{1}{n}}^{\frac{\pi}{2(n+1)}} + \sum_{k=1}^n \int_{\frac{(2k-1)\pi}{2(n+1)}}^{\frac{(2k+1)\pi}{2(n+1)}} + \int_{\frac{(2n+1)\pi}{2(n+1)}}^{\pi} \right) \frac{|\frac{d}{dx} \sin((n+1)x)|}{\sin^2(\frac{x}{2})} dx \\ &\leq \frac{\pi^2}{n^2} \left(\int_{\frac{1}{n}}^{\frac{\pi}{2(n+1)}} + \sum_{k=1}^n \int_{\frac{(2k-1)\pi}{2(n+1)}}^{\frac{(2k+1)\pi}{2(n+1)}} + \int_{\frac{(2n+1)\pi}{2(n+1)}}^{\pi} \right) \frac{|\frac{d}{dx} \sin((n+1)x)|}{x^2} dx \\ &= \frac{\pi^2}{n^2} (III_1 + III_2 + III_3), \quad \text{say.} \end{aligned}$$

Estimate of III_1 : Since $\frac{d}{dx} \sin((n+1)x) \geq 0$ on $[\frac{1}{n}, \frac{\pi}{2(n+1)}]$, we have

$$\begin{aligned} (10) \quad III_1 &\leq \int_{\frac{1}{n}}^{\frac{\pi}{2(n+1)}} \frac{|\frac{d}{dx} \sin((n+1)x)|}{x^2} dx \\ &= \int_{\frac{1}{n}}^{\frac{\pi}{2(n+1)}} \frac{\frac{d}{dx} \sin((n+1)x)}{x^2} dx. \end{aligned}$$

Performing integration by parts, the last integral of (10) is equal to

$$\frac{4(n+1)^2}{\pi^2} - n^2 \sin \frac{n+1}{n} + \int_{\frac{1}{n}}^{\frac{\pi}{2(n+1)}} \frac{2 \sin((n+1)x)}{x^3} dx$$

$$\begin{aligned}
&\leq \frac{4(n+1)^2}{\pi^2} - n^2 \sin \frac{n+1}{n} + \int_{\frac{1}{n}}^{\frac{\pi}{2(n+1)}} \frac{2}{x^3} dx \\
&= n^2 - n^2 \sin \frac{n+1}{n}.
\end{aligned}$$

Thus

$$(11) \quad III_1 \leq n^2.$$

Estimate of III_2 : From integration by parts, we get

$$\begin{aligned}
(12) \quad III_2 &= \sum_{k=1}^n (-1)^k \int_{\frac{(2k-1)\pi}{2(n+1)}}^{\frac{(2k+1)\pi}{2(n+1)}} \frac{\frac{d}{dx} \sin((n+1)x)}{x^2} dx \\
&= \sum_{k=1}^n \left(\frac{4(n+1)^2}{(2k+1)^2 \pi^2} + \frac{4(n+1)^2}{(2k-1)^2 \pi^2} + (-1)^k \int_{\frac{(2k-1)\pi}{2(n+1)}}^{\frac{(2k+1)\pi}{2(n+1)}} \frac{2 \sin((n+1)x)}{x^3} dx \right) \\
&\leq \sum_{k=1}^n \left(\frac{4(n+1)^2}{(2k+1)^2 \pi^2} + \frac{4(n+1)^2}{(2k-1)^2 \pi^2} + \int_{\frac{(2k-1)\pi}{2(n+1)}}^{\frac{(2k+1)\pi}{2(n+1)}} \frac{2}{x^3} dx \right) \\
&= \frac{8(n+1)^2}{\pi^2} \sum_{k=1}^n \frac{1}{(2k-1)^2} \\
&\leq \frac{8(n+1)^2}{\pi^2} \left(1 + \frac{1}{2} \sum_{k=2}^n \frac{1}{(k-1)^2} \right) \\
&\leq \frac{8(n+1)^2}{\pi^2} \left(1 + \frac{\pi^2}{12} \right).
\end{aligned}$$

Estimate of III_3 : First, applying differentiation, we get

$$\begin{aligned}
(13) \quad III_3 &\leq (n+1) \int_{\frac{(2n+1)\pi}{2(n+1)}}^{\pi} \frac{1}{x^2} dx \\
&= \frac{n+1}{(2n+1)\pi}.
\end{aligned}$$

From (11), (12) and (13),

$$\begin{aligned}
(14) \quad III &= \frac{\pi^2}{n^2} (III_1 + III_2 + III_3) \\
&\leq \pi^2 + \frac{8(n+1)^2}{n^2} \left(1 + \frac{\pi^2}{12} \right) + \frac{(n+1)\pi}{n^2(2n+1)} \\
&\leq \pi^2 + 32 \left(1 + \frac{\pi^2}{12} \right) + \frac{2\pi}{3}.
\end{aligned}$$

Finally, by summing up (8), (14) and (9), we see that $2I + 2II$ is uniformly bounded:

$$(15) \quad 2I + 2III + 2IV \leq \pi^3 + \frac{22}{3}\pi^2 + \frac{4}{3}\pi + 69.$$

Hence, we prove that $F_n(x)$ satisfies (ii) of properties of good kernels. Therefore, the proof is complete. \square

Lemma 2.6. *The tail of F_n vanishes in L^1 as $n \rightarrow \infty$.*

Proof. Let $0 < \delta < \pi$. By (b) of Corollary 2.3,

$$(16) \quad \int_{\delta}^{\pi} |F_n(x)| dx \leq \frac{5\pi^3(\pi - \delta)}{4(n+1)\delta^3},$$

consequently, (16) vanishes as $n \rightarrow \infty$. Therefore, the proof is complete. \square

3. Pointwise convergence of $\tilde{\sigma}_n(f, x)$

The importance of good kernels is highlighted by their use in connection with convolutions. More precisely, the convolutions of a continuous function and a family of good kernels produce a continuous function as their limit.

We can now prove the pointwise convergence of $\tilde{\sigma}_n(f, x)$.

Theorem 3.1. *Let $f \in L^1$. If the limits $f(a+0)$ and $f(a-0)$ exist, then*

$$\lim_{n \rightarrow \infty} \tilde{\sigma}_n(f, a) = \frac{f(a^+) + f(a^-)}{2}.$$

In particular, if f is continuous in a closed interval $I \subset [-\pi, \pi]$, then the convergence is uniform over I .

Proof. We may assume that $f(a) = (f(a+0) + f(a-0))/2$. On account of Lemma 2.4 and the fact that $F_n(x)$ is even, we have

$$\tilde{\sigma}_n(f, a) - f(a) = \frac{1}{\pi} \int_0^{\pi} \left(\frac{f(a+x) + f(a-x)}{2} - f(a) \right) F_n(x) dx.$$

For $0 < \eta < \pi$, we see that

$$(17) \quad \begin{aligned} |\tilde{\sigma}_n(f, a) - f(a)| &\leq \frac{1}{\pi} \left(\int_0^{\eta} + \int_{\eta}^{\pi} \right) \left| \frac{f(a+x) + f(a-x)}{2} - f(a) \right| |F_n(x)| dx \\ &= A + B, \quad \text{say.} \end{aligned}$$

Estimate of A : Put $C = \max_n \|F_n\|_1$. Then $0 < C < \infty$ by Lemma 2.5. By continuity, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(a+x) + f(a-x)}{2} - f(a) \right| < \frac{\epsilon}{C}$$

provided $0 \leq x < \delta$. We set $\eta = \delta$ in (17) and observe that

$$(18) \quad |A| \leq \frac{\epsilon}{C} \frac{1}{\pi} \int_0^{\delta} |F_n(x)| dx \leq \epsilon.$$

To bound B , we note that

$$\max_{\delta < x < \pi} |F_n(x)| \leq \frac{5\pi^3}{4(n+1)\delta^3}$$

by (b) of Corollary 2.3. Thus

$$(19) \quad |B| \leq \frac{5\pi^3}{4(n+1)\delta^3} (2\|f\|_1 + |f(a)|),$$

where L^1 -norm is defined as the normalized integral. Hence, (19) vanishes as $n \rightarrow \infty$. This completes the first assertion.

If now f is continuous in a closed interval of $[-\pi, \pi]$, then the uniform convergence follows directly from (19).

Therefore, the proof is complete. \square

Corollary 3.2. *Let f be an integrable function on $[-\pi, \pi]$. Then*

$$\lim_{n \rightarrow \infty} (F_n * f)(x) = f(x)$$

whenever f is continuous at x . If f is continuous everywhere, then the above limit is uniform.

4. Norm convergences

We start this section by comparing $\tilde{\sigma}_n(f, x)$ with $\sigma_n(f, x)$ in L^2 -norm, which are corresponding to the Fourier series of f , where $L^2 = L^2([-\pi, \pi])$.

Theorem 4.1. *Let $f \in L^2$. Then*

$$(20) \quad \|\sigma_n(f)\|_2 \leq \|\tilde{\sigma}_n(f)\|_2.$$

Proof. Let $f \sim \sum_k c_k e^{ikx}$ and fix n . By Parseval's identity we have

$$\begin{aligned} \|\sigma_n(f)\|_2^2 &= \sum_{|k| \leq n} |c_k|^2 \left(1 - \frac{|k|}{n+1}\right)^2 \\ &\leq \sum_{|k| \leq n} |c_k|^2 \left(1 - \frac{k^2}{(n+1)^2}\right)^2 \\ &= \|\tilde{\sigma}_n(f)\|_2^2, \end{aligned}$$

where the second inequality holds since $\frac{k^2}{(n+1)^2} \leq \frac{|k|}{n+1}$ for every k ($|k| \leq n$). Therefore, the proof is complete. \square

Let n, m be positive integers. From [17], the Cesàro (C, m) -mean $\sigma_n^m(f, x)$ corresponding to the Fourier series of f is derived as

$$(21) \quad \sigma_n^m(f, x) = \sum_{|k| \leq n} \prod_{\ell=1}^m \left(1 - \frac{|k|}{n+\ell}\right) c_k e^{ikx}.$$

When $m = 1$, $\sigma_n^1(f, x) = \sigma_n(f, x)$. The identity (21) is announced with real numbers m in [17].

Corollary 4.2. *Let $f \in L^2$. Then*

$$\|\sigma_n^{m'}(f)\|_2 \leq \|\sigma_n^m(f)\|_2 \leq \|\tilde{\sigma}_n(f)\|_2 \leq \|s_n(f)\|_2 \quad (m' \geq m \geq 1).$$

Proof. Let $1 \leq m \leq m'$. The first inequality follows readily. Indeed,

$$\prod_{\ell=1}^m \left(1 - \frac{|k|}{n+\ell}\right) \geq \prod_{\ell=1}^{m'} \left(1 - \frac{|k|}{n+\ell}\right)$$

and apply Parseval's identity to L^2 -norms. The first inequality and Theorem 4.1 yield the second inequality. The last inequality is also derived by

$$\left(1 - \frac{k^2}{(n+1)^2}\right) \leq 1 \quad (|k| \leq n)$$

and by Parseval's identity. Therefore, the proof is complete. \square

By Hölder's inequality and by Theorem 4.1, we conclude the next result.

Corollary 4.3. *Let $f \in L^1 \cap L^2$. Then*

$$\|\sigma_n(f)\|_1 \leq \|\tilde{\sigma}_n(f)\|_2.$$

We will show L^p -norm convergences for $\tilde{\sigma}_n(f, x)$, where $1 \leq p < \infty$, where $L^p = L^p([-\pi, \pi])$.

Theorem 4.4. *Let $1 \leq p < \infty$. Then for each $f \in L^p$,*

$$\|\tilde{\sigma}_n(f) - f\|_p \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. By the property (i) of the good kernels,

$$\tilde{\sigma}_n(f, x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-y) - f(x)) F_n(y) dy.$$

Let $0 < \eta < \pi$. By Minkowski's inequality for integrals,

$$\begin{aligned} \|\tilde{\sigma}_n(f) - f\|_p &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(\cdot - y) - f\|_p |F_n(y)| dy \\ &= \frac{1}{2\pi} \left(\int_{|y| < \eta} + \int_{\eta \leq |y| \leq \pi} \right) \|f(\cdot - y) - f\|_p |F_n(y)| dy \\ &= I + II, \end{aligned}$$

say.

Let $C = \max_n \|F_n\|_1$. By the property (ii) of the good kernels, $0 < C < \infty$. From the L^p -continuity, for given $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(\cdot - y) - f\|_p < \frac{\epsilon}{C}$ for every y ($|y| < \delta$). Thus

$$I \leq \frac{\epsilon}{C} \int_{-\pi}^{\pi} |F_n(y)| \frac{dy}{2\pi} \leq \epsilon.$$

Next, II vanishes as $n \rightarrow \infty$. Indeed, by the property (ii) of the good kernels,

$$\begin{aligned} II &\leq \frac{1}{2\pi} \int_{\delta \leq |y| \leq \pi} \|f(\cdot - y) - f\|_p |F_n(y)| dy \\ &\leq 2\|f\|_p \int_{\delta \leq |y| \leq \pi} |F_n(y)| \frac{dy}{2\pi} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Therefore, the proof is complete. \square

We now state and prove the convergence of a weighted Fourier series is faster than that of the Cesàro mean in L^2 .

Theorem 4.5. *Let $f \in L^2$. Then*

$$\|f - \tilde{\sigma}_n(f)\|_2 \leq \|f - \sigma_n(f)\|_2$$

for any n .

Proof. Let $f \in L^2$. We may assume that $f = \sum_k c_k e^{ikx}$ in L^2 , where $f \sim \sum_k c_k e^{ikx}$ and also note $\{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$. By Parseval's identity,

$$\begin{aligned} \|f - \tilde{\sigma}_n(f)\|_2^2 &= \sum_{|k| \leq n} \frac{k^4}{(n+1)^4} |c_k|^2 + \sum_{|k| > n} |c_k|^2 \\ &\leq \sum_{|k| \leq n} \frac{k^2}{(n+1)^2} |c_k|^2 + \sum_{|k| > n} |c_k|^2 \\ &= \|f - \sigma_n(f)\|_2^2. \end{aligned}$$

Therefore, the proof is complete. \square

The next conclusion follows from the orthogonality of $\{e^{ikx}\}_{k \in \mathbb{Z}}$ and from Theorem 4.5.

Corollary 4.6. *Let $f \in L^2$. Then*

$$\|f - s_n(f)\|_2 \leq \|f - \tilde{\sigma}_n(f)\|_2 \leq \|f - \sigma_n(f)\|_2$$

for any n .

Remark. In Theorem 4.5, one may conceive that

$$(22) \quad \frac{\|f - \sigma_n(f)\|_2}{\|f - \tilde{\sigma}_n(f)\|_2}$$

is bounded in L^2 . However, we show that there exists $f \in L^2$ such that (22) approaches ∞ as $n \rightarrow \infty$, which is precisely $f(x) = \sum_k e^{-|k|} e^{ikx}$. Indeed,

$$\begin{aligned} \frac{\|f - \sigma_n(f)\|_2^2}{\|f - \tilde{\sigma}_n(f)\|_2^2} &= (n+1)^2 \frac{\sum_{|k| \leq n} k^2 e^{-2|k|} + \frac{2(n+1)^2}{e^2-1} e^{-2n}}{\sum_{|k| \leq n} k^4 e^{-2|k|} + \frac{2(n+1)^4}{e^2-1} e^{-2n}} \\ &= (n+1)^2 A, \quad \text{say.} \end{aligned}$$

By choosing n_0 such that $\frac{2(n+1)^4}{e^2-1}e^{-2n} \leq 1$ for $n \geq n_0$, we have

$$(23) \quad \begin{aligned} A &\geq \frac{\sum_{|k| \leq n} k^2 e^{-2|k|} + \frac{2(n+1)^2}{e^2-1} e^{-2n}}{\sum_{|k| \leq n} k^4 e^{-2|k|} + 1} \\ &\geq \frac{2 \int_1^{n+1} t^2 e^{-2t} dt + \frac{2(n+1)^2}{e^2-1} e^{-2n}}{2e^{-2} + 2 \int_2^\infty t^4 e^{-2t} dt + 1} \end{aligned}$$

for any $n \geq n_0$. Now take the limit as $n \rightarrow \infty$ on the numerator of the last term of (23). It follows that

$$A \geq \frac{2 \int_1^\infty t^2 e^{-2t} dt}{2e^{-2} + 2 \int_2^\infty t^4 e^{-2t} dt + 1} = \frac{\frac{5}{2}e^{-2}}{2e^{-2} + \frac{103}{2}e^{-4} + 1},$$

which is a positive constant. Therefore, $(n+1)^2 A \rightarrow \infty$ as $n \rightarrow \infty$.

5. A characterization of Fourier series

Fejér's characterization theorem asserts that a necessary and sufficient condition for the trigonometric series

$$(24) \quad \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

to be Fourier series of an $f \in L^p$ ($1 < p \leq \infty$) is its Ceàro means $\sigma_n(x)$ be bounded in L^p , where

$$\sigma_n(x) = \sum_{|k| \leq n} \left(1 - \frac{|k|}{n+1}\right) c_k e^{ikx}.$$

In this section, we prove that $\sigma_n(x)$ is bounded in L^p if and only if $\tilde{\sigma}_n(x)$ does, where

$$\tilde{\sigma}_n(x) = \sum_{|k| \leq n} \left(1 - \frac{k^2}{(n+1)^2}\right) c_k e^{ikx}.$$

Theorem 5.1. *Let $1 < p \leq \infty$. Then $\sum c_n e^{inx}$ is the Fourier series of an L^p -function f if and only if there exists $C < \infty$ such that*

$$\|\tilde{\sigma}_n\|_p \leq C.$$

In this case $\|f\|_p \leq C$.

Proof. By Lemma 2.5 and Corollary 2.3(a), we have

$$\begin{aligned} \|\tilde{\sigma}_n(f)\|_p &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(\cdot - t)\|_p |F_n(t)| dt \\ &\leq \|f\|_p \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{5(2n+1)}{6} dt \\ &= \frac{5(2n+1)}{6} \|f\|_p. \end{aligned}$$

Thus necessity holds.

By Weak compactness in L^p ($1 < p \leq \infty$), there is a sequence $n_k \rightarrow \infty$ and $f \in L^p$ such that $\tilde{\sigma}_{n_k} \rightarrow f$ weakly as $n_k \rightarrow \infty$. Since $e^{-int} \in L^\infty$, we have

$$\begin{aligned} \lim_{n_k \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\sigma}_{n_k}(t) e^{-int} dt &= \lim_{n_k \rightarrow \infty} \left(1 - \frac{n^2}{(n_k + 1)^2} \right) c_n \\ &= c_n \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \\ &= c_n(f). \end{aligned} \quad \square$$

Remark. Theorem does not hold for $p = 1$: Let $c_n = 1$ for all n . It follows that $\|\tilde{\sigma}_n\|_1 = \|F_n\|_1$ is uniformly bounded by Lemma 2.5. On the other hand, $\sum e^{inx}$ is not a Fourier series of any L^1 -function, since it does not satisfy Riemann Lebesgue Lemma.

For a finite measure μ on $[-\pi, \pi]$, we define

$$c_n(\mu) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} d\mu(t) \quad \text{for all } n.$$

If $\sum c_n e^{int}$ is the Fourier series of μ , then it is clear that

$$(25) \quad \tilde{\sigma}_n(\mu, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x - t) d\mu(t)$$

and there exists C such that

$$(26) \quad \|\tilde{\sigma}_n(\mu)\|_1 \leq \frac{C}{2\pi} \int_{-\pi}^{\pi} |d(\mu(t))|.$$

The quality (25) holds, since we have

$$\begin{aligned} \tilde{\sigma}_n(\mu, x) &= \sum_{|k| \leq n} \left(1 - \frac{k^2}{(n+1)^2} \right) c_k(\mu) e^{ikx} \\ &= \sum_{|k| \leq n} \left(1 - \frac{k^2}{(n+1)^2} \right) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} d\mu(t) e^{ikx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{|k| \leq n} \left(1 - \frac{k^2}{(n+1)^2} \right) e^{ik(x-t)} d\mu(t) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x - t) d\mu(t). \end{aligned}$$

From (25), it follows that

$$\begin{aligned} \|\tilde{\sigma}_n(\mu)\|_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{\sigma}_n(\mu, x)| d(x) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x - t) d\mu(t) \right| d(x) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(x-t)| |d\mu(t)| dx \\
&\leq \frac{C}{2\pi} \int_{-\pi}^{\pi} |d(\mu(t))|.
\end{aligned}$$

Thus the inequality (26) holds.

Theorem 5.2. $\sum c_n e^{inx}$ is the Fourier series of a finite measure μ if and only if there exists $C < \infty$ such that

$$\|\tilde{\sigma}_n\|_1 \leq C.$$

In this case $\int_{-\pi}^{\pi} |d\mu| \leq C$.

Proof. From (26), it is enough to show the sufficiency. Let $h_n(x) = \int_{-\pi}^x \tilde{\sigma}_n(t) dt$. The function h_n is uniformly bounded variation over $[-\pi, \pi]$. Since $h_n(-\pi) = 0$ for each n , $\{|h_n|\}$ can not diverge uniformly to ∞ in $[-\pi, \pi]$. By Helly's selection Lemma, there exist a sequence $n_k \rightarrow \infty$ and h of bounded variation over $[-\pi, \pi]$ such that

$$\lim_{n_k \rightarrow \infty} \int_{-\pi}^x \tilde{\sigma}_{n_k}(t) dt = h(x), \quad x \in [-\pi, \pi].$$

Fix n . For $n_k > |n|$ we have

$$\begin{aligned}
\left(1 - \frac{n^2}{(n_k + 1)^2}\right) c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\sigma}_{n_k}(t) e^{-int} dt \\
&= \frac{1}{2\pi} h_{n_k}(\pi) + \frac{in}{2\pi} \int_{-\pi}^{\pi} h_{n_k}(t) e^{int} dt.
\end{aligned}$$

By letting $n_k \rightarrow \infty$, we set

$$c_n = \frac{1}{2\pi} h(\pi) + \frac{in}{2\pi} \int_{-\pi}^{\pi} h(t) e^{-int} dt.$$

Therefore $d\mu = dh$. □

6. Numerical experiments

In the last section, we compare $\tilde{\sigma}_n(f, x)$ with $\sigma_n(f, x)$ and $s_n(f, x)$ in numerical simulation. The following two typical examples are considered on $[-\pi, \pi]$, which are given by

$$\begin{aligned}
f(x) &= \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{otherwise,} \end{cases} \\
g(x) &= \begin{cases} 1 - \frac{1}{\pi}x & \text{if } x \geq 0 \\ -1 - \frac{1}{\pi}x & \text{otherwise.} \end{cases}
\end{aligned}$$

Let us note that f has simple discontinuities at 0, at the end-points, and that g is discontinuous only at 0 as a periodic function on \mathbb{R} .

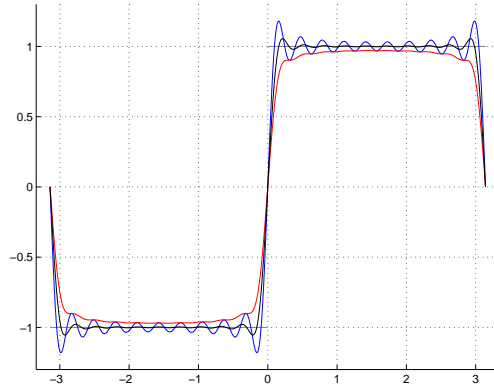


FIGURE 1. When $n = 20$, the blue, red, black colored lines denote $s_{20}(f, x)$, $\sigma_{20}(f, x)$, $\tilde{\sigma}_{20}(f, x)$, respectively.

Even though L^2 -norm difference $\|f - s_n(f)\|_2$ is the smallest since $\{e^{inx}\}$ are orthonormal system, $\|f - \tilde{\sigma}_n(f)\|_2$ is much smaller than $\|f - \sigma_n(f)\|_2$. (See Table 1 and Table 2). In comparing pointwise convergences, $\|f - \tilde{\sigma}_n(f)\|_2$ is the smallest, particularly near the deleted neighborhood of discontinuities. (Refer to Figures 1, 2, 3 and 4.)

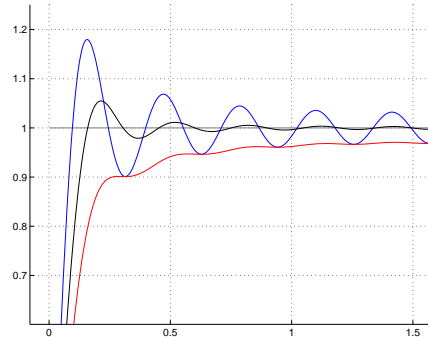
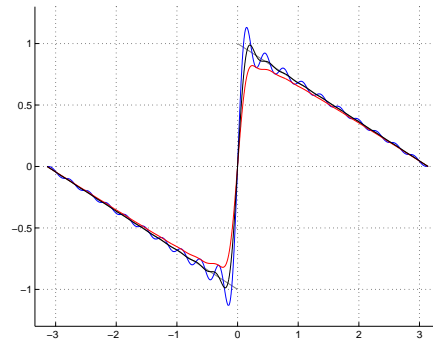
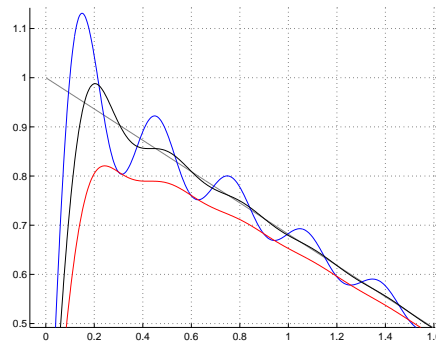


FIGURE 2. Local picture of Figure 1. The blue, red, black colored lines denote $s_{20}(f, x)$, $\sigma_{20}(f, x)$, $\tilde{\sigma}_{20}(f, x)$, respectively.

At the end-points g is continuous as a periodic function in \mathbb{R} . From Figures 3 and 4 show that $|g(x) - \tilde{\sigma}_n(g, x)|$ is the smallest in the neighborhood of π and $-\pi$.

TABLE 1. The estimated results are rounded off to 4 decimal places.

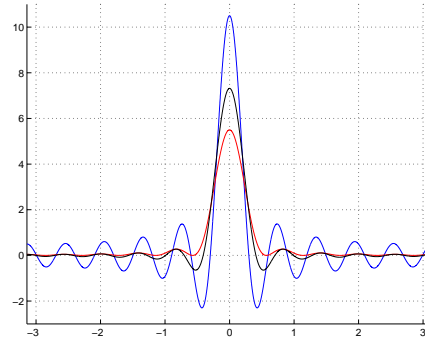
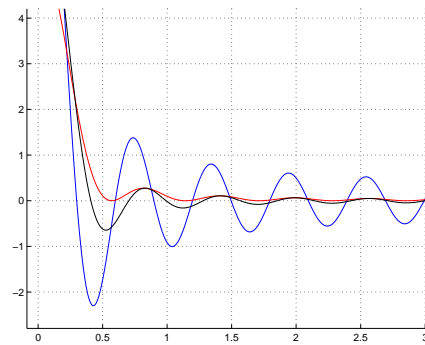
	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$\ f - s_n(f)\ _2$	0.2538	0.1272	0.0848	0.0636
$\ f - \sigma_n(f)\ _2$	0.4642	0.2427	0.1643	0.1242
$\ f - \tilde{\sigma}_n(f)\ _2$	0.3112	0.1620	0.1096	0.0828

FIGURE 3. When $n = 20$, the blue, red, black colored lines denote $s_{20}(g, x)$, $\sigma_{20}(g, x)$, $\tilde{\sigma}_{20}(g, x)$, respectively.FIGURE 4. Local picture of Figure 3. The blue, red, black colored lines denote $s_{20}(g, x)$, $\sigma_{20}(g, x)$, $\tilde{\sigma}_{20}(g, x)$, respectively.

Finally, we compare three kernels $D_n(x)$, $K_n(x)$, $F_n(x)$ in Figures 5 and 6 when $n = 10$. Let us note that $F_n(x)$ is not a positive kernel.

TABLE 2. The estimated results are rounded off to 4 decimal places.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$\ g - s_n(g)\ _2$	0.1211	0.0620	0.0417	0.0314
$\ g - \sigma_n(g)\ _2$	0.2263	0.1198	0.0814	0.0617
$\ g - \tilde{\sigma}_n(g)\ _2$	0.1546	0.0808	0.0547	0.0414

FIGURE 5. When $n = 10$, the blue, red, black colored lines denote $D_{10}(x)$, $K_{10}(x)$, $F_{10}(x)$, respectively.FIGURE 6. Local picture of Figure 5. The blue, red, black colored lines denote $D_{10}(x)$, $K_{10}(x)$, $F_{10}(x)$, respectively.

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