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EQUIVALENT CONDITIONS OF COMPLETE MOMENT CONVERGENCE AND COMPLETE INTEGRAL CONVERGENCE FOR NOD SEQUENCES

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Abstract. In this paper, seven equivalent conditions of complete moment convergence and complete integral convergence for negatively orthant dependent (NOD, in short) sequences are shown under two cases: identical distribution and stochastic domination. The results obtained in the paper improve and generalize the corresponding ones of Liang et al. [10]). In addition, an extension of the Baum-Katz complete convergence theorem: six equivalent conditions of complete convergence is established.

1. Introduction

Let $\{Z_n, n \in \mathbb{N}\}$ be a sequence of random variables and a_n, b_n, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \text{ for all } \varepsilon > 0, \tag{1.1}$$

then the above result was called the complete moment convergence by Chow [6]. It is easy to show that it is a more general concept than complete convergence.

Recently, based on the Baum-Katz complete convergence theorem (see Baum and Katz [2]) and complete moment convergence of Chow [6], Li and Spătaru [9] investigated the refinement of complete convergence and established the following result.

Theorem 1.1. *Set $\{X, X_n, n \geq 1\}$ be a sequence of independent random variables with identical distribution. Let $EX = 0$, and let $0 < p < 2$, $r \geq 1$ and $q > 0$. Set*

$$f(x) = \sum_{n=1}^{\infty} n^{r-2} P(|S_n| > xn^{\frac{1}{p}}),\ x > 0.$$

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Then the following are equivalent:

(i) $\int_{\varepsilon}^{\infty} f(x^q)dx < \infty$ *for all* $\varepsilon > 0$;

(ii) $\begin{cases} E|X|^{\frac{1}{q}} < \infty, & \textit{if } q < \frac{1}{pr}, \\ E|X|^{pr}\log^{+}|X| < \infty, & \textit{if } q = \frac{1}{pr}, \\ E|X|^{pr} < \infty, & \textit{if } q > \frac{1}{pr}. \end{cases}$

Chen and Wang [5] showed that the refinement of compete convergence and complete moment convergence are equivalent, that is to say, (1.1) is equivalent to

$$\int_{\varepsilon}^{\infty} \sum_{n=1}^{\infty} a_n P(|Z_n| > \varepsilon^{\frac{1}{q}} b_n)dx < \infty, \ \forall\, \varepsilon > 0. \tag{1.2}$$

Later, Liang et al. [10] referred to (1.2) as being a complete integral convergence. Obviously, it can exactly describe the convergence rate of a sequence of random variables than complete convergence.

Furthermore, Liang et al. [10] extended the i.i.d. assumption to identically distributed and negatively associated (NA, in short) random variables and obtained the following new version.

Theorem 1.2. *Let* $\{X, X_n, n \geq 1\}$ *be a sequence of identically distributed NA random variables. Let* $0 < p < 2$, $r > 1$, $q > 0$, *and let*

$$f(x) = \sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n} |S_k| > xn^{\frac{1}{p}}\right), \ x > 0.$$

Then the following are equivalent:

(i) $\sum_{i=1}^{\infty} n^{r-2-\frac{1}{pq}} E\left(\max_{1\leq k\leq n} |S_k| - \varepsilon n^{\frac{1}{pq}}\right)^{+} < \infty$ *for all* $\varepsilon > 0$;

(ii) $\int_{\varepsilon}^{\infty} f(x^q)dx < \infty$ *for all* $\varepsilon > 0$;

(iii) $\begin{cases} E|X|^{\frac{1}{q}} < \infty, & \textit{if } q < \frac{1}{pr}, \\ E|X|^{pr}\log^{+}|X| < \infty, & \textit{if } q = \frac{1}{pr}, \\ E|X|^{pr} < \infty, & \textit{if } q > \frac{1}{pr}, \\ EX = 0, & \textit{when } 1 \leq p < 2. \end{cases}$

Inspired by Theorem 1.2, we will establish seven equivalent conditions of complete moment convergence and complete integral convergence for negatively orthant dependent (NOD, in short) sequence under two cases: identical distribution and stochastic domination. These results will improve and extend Theorem 1.2 for NA random variables to a more general case: NOD random variables. In addition, we get six equivalent conditions of complete convergence, which can be regarded as the extension of the Baum-Katz complete convergence theorem in Baum and Katz [2].

Now let us recall the concept of NOD random variables as follows.

Definition 1.1. A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be negatively upper orthant dependent (NUOD) if for all numbers $x_1, x_2, \ldots, x_n$,

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i) \tag{1.3}$$

and negatively lower orthant dependent (NLOD) if for all numbers $x_1, x_2, \ldots, x_n$,

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i). \tag{1.4}$$

A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be negatively orthant dependent (NOD) if they are both NUOD and NLOD. An infinite sequence $\{X_n, n \geq 1\}$ is said to be NOD if every finite subcollection is NOD.

The concept of NOD random variables was introduced by Joag-Dev and Proschan [8]. Obviously, independent random variables are NOD. Joag-Dev and Proschan [8] pointed out that NA random variables are NOD, but neither being NUOD nor being NLOD implies being NA. Meanwhile, Hu [7] introduced the concept of negatively superadditive dependence (NSD, in short) and pointed out that NSD implies NOD (see Property 2 of Hu [7]). So we can see that NOD is much weaker than NA and NSD. A number of useful results for NOD random variables have been established by many authors. We refer to Volodin [19] for the Kolmogorov exponential inequality, Asadian et al. [1] for Rosenthal's type inequality, Zarei and Jabbari [28], Wu [24], Wang et al. [20], Sung [18], Yi et al. [27] and Chen and Sung [4] for complete convergence, Wang et al. [21] and Sung [17] for exponential inequalities, Wu and Jiang [25] for the strong consistency of M estimator in a linear model, Shen [12, 14] for strong limit theorems of weighted sums, Shen [15] for the asymptotic approximation of inverse moments, Wang and Si [22] for the complete consistency of estimator of nonparametric regression model, Qiu et al. [11] and Wu and Volodin [26] for the complete moment convergence, and so on.

The following concept of stochastic domination will be used in this work.

Definition 1.2. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

Throughout this paper, let $I(A)$ be the indicator function of the set A and let $\lfloor x \rfloor$ be the integer part of x. The symbol C denotes a positive constant which is not necessarily the same one in each appearance. Denote $X^+ = XI(X > 0)$, $X^- = -XI(X < 0)$ and $\log x = \ln \max\{x, e\}$, where ln is the natural logarithm. $a_n \approx b_n$ stands for $a_n \le C_1 b_n$ and $a_n \ge C_2 b_n$, where C_1 and C_2 are positive real numbers.

2. Preliminaries

In this section, we will present some important lemmas which will be used to prove the main results of the paper.

The first one is a basic property for NOD random variables, which was established by Bozorgnia et al. [3].

Lemma 2.1. *Let random variables $X_1, X_2, \ldots, X_n$ be NOD, and $f_1, f_2, \ldots, f_n$ be all nondecreasing (or nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are NOD.*

The next one plays an important role to prove the main results of the paper. The details of the proof could be referred to Lemma 1.10 of Wu [24].

Lemma 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of NOD random variables. Then there exists a positive constant C such that, for any $\varepsilon \ge 0$ and all $n \ge 1$,*

$$\left[1 - P\left(\max_{1\le i\le n} |X_i| > \varepsilon\right)\right]^2 \sum_{i=1}^{n} P(|X_i| > \varepsilon) \le CP\left(\max_{1\le i\le n} |X_i| > \varepsilon\right).$$

The following one is the Rosential-type maximal inequality which can be found in Wu [24].

Lemma 2.3. *Let $\{X_n, n \ge 1\}$ be a sequence of NOD random variables, and satisfy $EX_n = 0$, $E|X_n|^q < \infty$, $q \ge 2$. Then there exists a positive constant C_q depending only on q such that*

$$E \max_{1\le j\le n} \left|\sum_{i=1}^{j} X_i\right|^q \le C \log^q n \left\{\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right\}.$$

Lemma 2.4. *Let $\alpha \ge -1$, $\beta \ge 0$, and let $\{Y_n, n \ge 1\}$ be a nondecreasing sequence of nonnegative random variables. If*

$$\sum_{n\ge 1} n^\alpha P(Y_n > xn^\beta) < \infty \text{ for all } x > 0,$$

then

$$\lim_{n\to\infty} P(Y_n > xn^\beta) = 0 \text{ for all } x > 0.$$

The details of the above lemma can refer to Lemma 3.4 in Liang et al. [10].

The last one is a basic property for stochastic domination. For the proof, one can refer to Wu [23] or Shen et al. [16].

Lemma 2.5. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold*:

$$E|X_n|^\alpha I\left(|X_n| \leq b\right) \leq C_1\left[E|X|^\alpha I\left(|X| \leq b\right) + b^\alpha P\left(|X| > b\right)\right],$$

$$E|X_n|^\alpha I\left(|X_n| > b\right) \leq C_2 E|X|^\alpha I\left(|X| > b\right),$$

where C_1 and C_2 are positive constants.

3. Main results and their proofs

Theorem 3.1. *Let $0 < p < 2$, $r > 1$, $q > 0$ and let $\{X, X_n, n \geq 1\}$ be a sequence of NOD random variables with identical distribution. Assume further that $EX = 0$ when $1 \leq p < 2$. Denote $S_n = \sum_{i=1}^n X_i$, $S_n^{(k)} = S_n - X_k$, $k = 1, 2, \ldots, n$, $n \geq 1$, and*

$$f(x) = \sum_{n=1}^\infty n^{r-2} P\left(\max_{1\leq k\leq n} |S_k| > x n^{\frac{1}{p}}\right), \ x > 0.$$

Then the following statements are equivalent:

(i) $\displaystyle\int_\varepsilon^\infty f(x^q)dx < \infty$ *for all* $\varepsilon > 0$;

(ii) $\displaystyle\sum_{n=1}^\infty n^{r-2-\frac{1}{pq}} E\left(\max_{1\leq k\leq n} |S_k|^{\frac{1}{q}} - \varepsilon n^{\frac{1}{pq}}\right)^+ < \infty$ *for all* $\varepsilon > 0$;

(iii) $\begin{cases} E|X|^{\frac{1}{q}} < \infty, & \text{if } q < \frac{1}{pr}, \\ E|X|^{pr} \log|X| < \infty, & \text{if } q = \frac{1}{pr}, \\ E|X|^{pr} < \infty, & \text{if } q > \frac{1}{pr}; \end{cases}$

(iv) $\displaystyle\sum_{n=1}^\infty n^{r-2-\frac{1}{pq}} E\left(\max_{1\leq k\leq n} |S_n^{(k)}|^{\frac{1}{q}} - \varepsilon n^{\frac{1}{pq}}\right)^+ < \infty$ *for all* $\varepsilon > 0$;

(v) $\displaystyle\sum_{n=1}^\infty n^{r-2-\frac{1}{pq}} E\left(\max_{1\leq k\leq n} |X_k|^{\frac{1}{q}} - \varepsilon n^{\frac{1}{pq}}\right)^+ < \infty$ *for all* $\varepsilon > 0$;

(vi) $\displaystyle\sum_{n=1}^\infty n^{r-2} E\left(\sup_{k\geq n} k^{-\frac{1}{pq}} |S_k|^{\frac{1}{q}} - \varepsilon\right)^+ < \infty$ *for all* $\varepsilon > 0$;

(vii) $\displaystyle\sum_{n=1}^\infty n^{r-2} E\left(\sup_{k\geq n} k^{-\frac{1}{pq}} |X_k|^{\frac{1}{q}} - \varepsilon\right)^+ < \infty$ *for all* $\varepsilon > 0$.

Proof. In order to prove the equivalence of the above seven statements, we will give proving process as the order in the following chart:

$$\begin{tikzcd}
\text{(iv)} \arrow[rr, Rightarrow] & & \text{(v)} \arrow[d, Rightarrow] \\
\text{(ii)} \arrow[u, Rightarrow] \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{(i)} & \text{(iii)} \arrow[l, Rightarrow] \\
\text{(vi)} \arrow[rr, Rightarrow] & & \text{(vii)} \arrow[u, Rightarrow]
\end{tikzcd}$$

(ii)⇒(iv): Note that

$$|S_n^{(k)}| = |S_n - X_k| \le |S_n| + |X_k| \le |S_n| + |S_k| + |S_{k-1}| \le 3 \max_{1\le k\le n} |S_n|.$$

Thus, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} E\left(\max_{1\le k\le n} |S_n^{(k)}|^{\frac{1}{q}} - \varepsilon n^{\frac{1}{pq}}\right)^+ \\
&= \sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} \int_0^\infty P\left(\max_{1\le k\le n} |S_n^{(k)}|^{\frac{1}{q}} - \varepsilon n^{\frac{1}{pq}} > x\right) dx \\
&\le \sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} \int_0^\infty P\left(\max_{1\le k\le n} |S_n|^{\frac{1}{q}} > 3^{-\frac{1}{q}}\varepsilon n^{\frac{1}{pq}} + 3^{-\frac{1}{q}} x\right) dx \\
&= 3^{\frac{1}{q}} \sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} \int_0^\infty P\left(\max_{1\le k\le n} |S_n|^{\frac{1}{q}} - 3^{-\frac{1}{q}}\varepsilon n^{\frac{1}{pq}} > y\right) dy \\
&\qquad (\text{letting } x = 3^{\frac{1}{q}} y) \\
&= 3^{\frac{1}{q}} \sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} E\left(\max_{1\le k\le n} |S_n|^{\frac{1}{q}} - \varepsilon_0 n^{\frac{1}{pq}}\right)^+ (\varepsilon_0 = 3^{-\frac{1}{q}}\varepsilon) < \infty. \qquad (3.1)
\end{aligned}$$

(iv)⇒(v): Note that

$$|X_k| = |S_n - S_n^{(k)}| \le \left|\frac{1}{n-1}\sum_{k=1}^{n} S_n^{(k)}\right| + |S_n^{(k)}| \le 3 \max_{1\le k\le n} |S_n^{(k)}|,$$

which together with the similar proof of (3.1) yields (v).
(v)⇒(iii): It is easy to see that

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} E\left(\max_{1\le k\le n} |X_k|^{\frac{1}{q}} - \varepsilon n^{\frac{1}{pq}}\right)^+ \\
&= \sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} \int_0^\infty P\left(\max_{1\le k\le n} |X_k|^{\frac{1}{q}} - \varepsilon n^{\frac{1}{pq}} > x\right) dx \\
&\ge \sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} \int_0^{\varepsilon n^{\frac{1}{pq}}} P\left(\max_{1\le k\le n} |X_k|^{\frac{1}{q}} - \varepsilon n^{\frac{1}{pq}} > x\right) dx
\end{aligned}$$

$$\geq \varepsilon \sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n} |X_k|^{\frac{1}{q}} > 2\varepsilon n^{\frac{1}{pq}}\right), \tag{3.2}$$

which implies that

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n} |X_k|^{\frac{1}{q}} > \varepsilon n^{\frac{1}{pq}}\right) < \infty,\ \forall\ \varepsilon > 0. \tag{3.3}$$

Then by Lemma 2.4, we get

$$P\left(\max_{1\leq k\leq n} |X_k|^{\frac{1}{q}} > \varepsilon n^{\frac{1}{pq}}\right) \to 0,\ n\to\infty,\ \forall\ \varepsilon > 0. \tag{3.4}$$

From (3.4), we can see that for $\forall\ x \geq 0,\ \varepsilon > 0$,

$$P\left(\max_{1\leq k\leq n} |X_k|^{\frac{1}{q}} > \varepsilon n^{\frac{1}{pq}} + x\right) \to 0,\ n\to\infty.$$

Therefore, it follows from Lemma 2.2 that, for sufficiently large n,

$$\begin{aligned} &P\left(\max_{1\leq k\leq n} |X_k|^{\frac{1}{q}} > \varepsilon n^{\frac{1}{pq}} + x\right) \\ \geq\ &C\sum_{k=1}^{n} P\left(|X_k|^{\frac{1}{q}} > \varepsilon n^{\frac{1}{pq}} + x\right),\ \forall\ x\geq 0 \text{ and } \varepsilon > 0. \end{aligned}$$

Taking $\varepsilon = 1$ in (v) and combining with the above inequality, we can obtain that

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} E\left(\max_{1\leq k\leq n} |X_k|^{\frac{1}{q}} - n^{\frac{1}{pq}}\right)^{+} \\ &= \sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} \int_0^{\infty} P\left(\max_{1\leq k\leq n} |X_k|^{\frac{1}{q}} > n^{\frac{1}{pq}} + x\right) dx \\ &\geq C\sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} \int_0^{\infty} \sum_{k=1}^{n} P\left(|X_k|^{\frac{1}{q}} > n^{\frac{1}{pq}} + x\right) dx \\ &= C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{pq}} \int_0^{\infty} P\left(|X|^{\frac{1}{q}} > n^{\frac{1}{pq}} + x\right) dx \\ &= C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{pq}} \int_{n^{\frac{1}{pq}}}^{\infty} P\left(|X| > y^q\right) dy \text{ (letting } y = x + n^{\frac{1}{pq}}) \\ &= C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{pq}} \sum_{m=n}^{\infty} \int_{m^{\frac{1}{pq}}}^{(m+1)^{\frac{1}{pq}}} P(|X| > y^q) dy \\ &\approx C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{pq}} \sum_{m=n}^{\infty} m^{\frac{1}{pq}-1} P(|X| > m^{\frac{1}{p}}) \end{aligned}$$

$$
\begin{aligned}
&= C\sum_{m=1}^{\infty} m^{\frac{1}{pq}-1}P(|X|>m^{\frac{1}{p}})\sum_{n=1}^{m} n^{r-1-\frac{1}{pq}}\\
&\approx \begin{cases} C\sum\limits_{m=1}^{\infty} m^{\frac{1}{pq}-1}P(|X|>m^{\frac{1}{p}}), & \text{if } q<\frac{1}{pr}\\ C\sum\limits_{m=1}^{\infty} m^{\frac{1}{pq}-1}\log m P(|X|>m^{\frac{1}{p}}), & \text{if } q=\frac{1}{pr}\\ C\sum\limits_{m=1}^{\infty} m^{r-1}P(|X|>m^{\frac{1}{p}}), & \text{if } q>\frac{1}{pr}\end{cases}\\
&= \begin{cases} C\sum\limits_{m=1}^{\infty} m^{\frac{1}{pq}-1}\sum\limits_{j=m}^{\infty} P(j^{\frac{1}{p}}<|X|\le (j+1)^{\frac{1}{p}}), & \text{if } q<\frac{1}{pr}\\ C\sum\limits_{m=1}^{\infty} m^{\frac{1}{pq}-1}\log m\sum\limits_{j=m}^{\infty} P(j^{\frac{1}{p}}<|X|\le (j+1)^{\frac{1}{p}}), & \text{if } q=\frac{1}{pr}\\ C\sum\limits_{m=1}^{\infty} m^{r-1}\sum\limits_{j=m}^{\infty} P(j^{\frac{1}{p}}<|X|\le (j+1)^{\frac{1}{p}}), & \text{if } q>\frac{1}{pr}\end{cases}\\
&= \begin{cases} C\sum\limits_{j=1}^{\infty} P(j^{\frac{1}{p}}<|X|\le (j+1)^{\frac{1}{p}})\sum\limits_{m=1}^{j} m^{\frac{1}{pq}-1}, & \text{if } q<\frac{1}{pr}\\ C\sum\limits_{j=1}^{\infty} P(j^{\frac{1}{p}}<|X|\le (j+1)^{\frac{1}{p}})\sum\limits_{m=1}^{j} m^{\frac{1}{pq}-1}\log m, & \text{if } q=\frac{1}{pr}\\ C\sum\limits_{j=1}^{\infty} P(j^{\frac{1}{p}}<|X|\le (j+1)^{\frac{1}{p}})\sum\limits_{m=1}^{j} m^{r-1}, & \text{if } q>\frac{1}{pr}\end{cases}\\
&\approx \begin{cases} C\sum\limits_{j=1}^{\infty} j^{\frac{1}{pq}}P(j^{\frac{1}{p}}<|X|\le (j+1)^{\frac{1}{p}}), & \text{if } q<\frac{1}{pr}\\ C\sum\limits_{j=1}^{\infty} j^{r}\log j P(j^{\frac{1}{p}}<|X|\le (j+1)^{\frac{1}{p}}), & \text{if } q=\frac{1}{pr}\\ C\sum\limits_{j=1}^{\infty} j^{r}P(j^{\frac{1}{p}}<|X|\le (j+1)^{\frac{1}{p}}), & \text{if } q>\frac{1}{pr}\end{cases}
\end{aligned}
$$

$$
\approx \begin{cases} CE|X|^{\frac{1}{q}}, & \text{if } q<\frac{1}{pr}\\ CE|X|^{pr}\log|X|, & \text{if } q=\frac{1}{pr}\\ CE|X|^{pr}, & \text{if } q>\frac{1}{pr}.\end{cases} \tag{3.5}
$$

Due to the above proof, (iii) holds immediately.

(ii)⇒(vi): It follows from (ii) that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2}E\left(\sup_{k\ge n} k^{-\frac{1}{pq}}|S_k|^{\frac{1}{q}}-\varepsilon\right)^{+}\\
&= \sum_{n=1}^{\infty} n^{r-2}\int_0^{\infty} P\left(\sup_{k\ge n} k^{-\frac{1}{pq}}|S_k|^{\frac{1}{q}}>\varepsilon+t\right)dt\\
&= \sum_{i=1}^{\infty}\sum_{n=2^{i-1}}^{2^i-1} n^{r-2}\int_0^{\infty} P\left(\sup_{k\ge n} k^{-\frac{1}{pq}}|S_k|^{\frac{1}{q}}>\varepsilon+t\right)dt
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{i=1}^{\infty}2^{i(r-1)}\int_0^{\infty}P\left(\sup_{k\ge 2^{i-1}}k^{-\frac{1}{pq}}|S_k|^{\frac{1}{q}}>\varepsilon+t\right)dt\\
&\le C\sum_{i=1}^{\infty}2^{i(r-1)}\sum_{j=i}^{\infty}\int_0^{\infty}P\left(\max_{2^{j-1}\le k<2^j}k^{-\frac{1}{pq}}|S_k|^{\frac{1}{q}}>\varepsilon+t\right)dt\\
&= C\sum_{j=1}^{\infty}\int_0^{\infty}P\left(\max_{2^{j-1}\le k<2^j}k^{-\frac{1}{pq}}|S_k|^{\frac{1}{q}}>\varepsilon+t\right)dt\sum_{i=1}^{j}2^{i(r-1)}\\
&\le C\sum_{j=1}^{\infty}2^{j(r-1)}\int_0^{\infty}P\left(\max_{2^{j-1}\le k<2^j}k^{-\frac{1}{pq}}|S_k|^{\frac{1}{q}}>\varepsilon+t\right)dt\\
&\le C\sum_{j=1}^{\infty}2^{j(r-1)}\int_0^{\infty}P\left(\max_{1\le k\le 2^j}|S_k|^{\frac{1}{q}}>2^{\frac{j-1}{pq}}(\varepsilon+t)\right)dt\\
&\le C\sum_{j=1}^{\infty}2^{j(r-1-\frac{1}{pq})}\int_0^{\infty}P\left(\max_{1\le k\le 2^j}|S_k|^{\frac{1}{q}}>2^{\frac{j-1}{pq}}\varepsilon+x\right)dx \text{ (letting } x=2^{\frac{j-1}{pq}}t)\\
&\le C\sum_{j=0}^{\infty}\sum_{n=2^j}^{2^{j+1}}2^{j(r-2-\frac{1}{pq})}\int_0^{\infty}P\left(\max_{1\le k\le 2^j}|S_k|^{\frac{1}{q}}>2^{-\frac{2}{pq}}\cdot 2^{\frac{j+1}{pq}}\varepsilon+x\right)dx\\
&\le C\sum_{n=1}^{\infty}n^{r-2-\frac{1}{pq}}\int_0^{\infty}P\left(\max_{1\le k\le n}|S_k|^{\frac{1}{q}}>2^{-\frac{2}{pq}}n^{\frac{1}{pq}}\varepsilon+x\right)dx\\
&= C\sum_{n=1}^{\infty}n^{r-2-\frac{1}{pq}}E\left(\max_{1\le k\le n}|S_k|^{\frac{1}{q}}-\varepsilon_0 n^{\frac{1}{pq}}\right)^{+}\ (\varepsilon_0=2^{-\frac{2}{pq}}\varepsilon)\\
&<\infty.
\end{aligned}
$$

(vi)⇒(vii): Noting that

$$
\begin{aligned}
k^{-\frac{1}{p}}|X_k| &= k^{-\frac{1}{p}}|S_k-S_{k-1}|\\
&\le k^{-\frac{1}{p}}(|S_k|+|S_{k-1}|)\\
&\le k^{-\frac{1}{p}}|S_k|+(k-1)^{-\frac{1}{p}}|S_{k-1}|\\
&\le 2\sup_{j\ge k-1}j^{-\frac{1}{p}}|S_j|,\ k\ge 2
\end{aligned}
$$

and similar to the proof of (3.1), we can get (vii) immediately.

(vii)⇒(iii): Similar to the proofs of (3.2) and (3.3), we have by (vii) that

$$
\sum_{n=1}^{\infty}n^{r-2}P\left(\sup_{k\ge n}k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}}>\varepsilon\right)<\infty,\ \forall\ \varepsilon>0. \tag{3.6}
$$

Thus,

$$
\lim_{n\to\infty}P\left(\sup_{k\ge n}k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}}>\varepsilon\right)=0,\ \forall\ \varepsilon>0. \tag{3.7}
$$

Otherwise, there exist $\varepsilon_0 > 0$, $\delta > 0$ and a positive sequence $\{n_j, j \geq 1\}$ with $n_j \uparrow \infty$, such that

$$P\left(\sup_{k\geq n_j} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > \varepsilon_0\right) \geq \delta,\ \forall\ j \geq 1.$$

Without loss of generality, we assume $n_j + 1 < n_{j+1}^{C_0}$ as j sufficiently large, where $0 < C_0 < 1$. Thereby, we have by $r > 1$ that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} P\left(\sup_{k\geq n} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > \varepsilon_0\right)\\
&\geq \sum_{j=1}^{\infty}\sum_{n=\lfloor\log(n_j+1)\rfloor}^{\lfloor\log n_{j+1}\rfloor} n^{r-2} P\left(\sup_{k\geq n} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > \varepsilon_0\right)\\
&\geq \sum_{j=1}^{\infty}(\lfloor\log(n_j+1)\rfloor)^{r-1}\left(1-\frac{\lfloor\log(n_j+1)\rfloor}{\lfloor\log n_{j+1}\rfloor}\right) P\left(\sup_{k\geq\lfloor\log n_{j+1}\rfloor} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > \varepsilon_0\right)\\
&\geq C\sum_{j=1}^{\infty}(\log n_j)^{r-1} P\left(\sup_{k\geq\lfloor\log n_{j+1}\rfloor} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > \varepsilon_0\right)\\
&\geq C\sum_{j=1}^{\infty}(\log n_j)^{r-1} P\left(\sup_{k\geq n_{j+1}} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > \varepsilon_0\right) = \infty,
\end{aligned}$$

which is contradictory with (3.6). Hence (3.7) holds. Note that for $\forall\ \varepsilon > 0$ and $n \geq 1$,

$$P\left(\max_{n\leq k<2n} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > \varepsilon\right) \leq P\left(\sup_{k\geq n} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > \varepsilon\right),$$

which together with (3.7) yields that

$$\lim_{n\to\infty} P\left(\max_{n\leq k<2n} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > \varepsilon\right) = 0,\ \forall\ \varepsilon > 0.$$

Taking $\varepsilon = 2^{-\frac{1}{pq}} + x$, $x \geq 0$ in the above equality, we have by Lemma 2.2 that, for sufficiently large n,

$$\begin{aligned}
P\left(\max_{n\leq k<2n} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > 2^{-\frac{1}{pq}} + x\right) &\geq C\sum_{k=n}^{2n-1} P\left(k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > 2^{-\frac{1}{pq}} + x\right)\\
&\geq C\sum_{k=n}^{2n-1} P(|X_k|^{\frac{1}{q}} > (2n)^{\frac{1}{pq}}(2^{-\frac{1}{pq}} + x))\\
&= CnP(|X|^{\frac{1}{q}} > (2n)^{\frac{1}{pq}}(2^{-\frac{1}{pq}} + x)). \qquad (3.8)
\end{aligned}$$

Consequently, taking $\varepsilon = 2^{-\frac{1}{pq}}$ in (vii), we have by (3.8) that,

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{r-2} E\left(\sup_{k\geq n} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} - 2^{-\frac{1}{pq}}\right)^{+} \\
&= \sum_{n=1}^{\infty} n^{r-2} \int_0^{\infty} P\left(\sup_{k\geq n} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > 2^{-\frac{1}{pq}} + x\right) dx \\
&\geq \sum_{n=1}^{\infty} n^{r-2} \int_0^{\infty} P\left(\max_{n\leq k<2n} k^{-\frac{1}{pq}}|X_k|^{\frac{1}{q}} > 2^{-\frac{1}{pq}} + x\right) dx \\
&\geq C\sum_{n=1}^{\infty} n^{r-1} \int_0^{\infty} P\left(|X|^{\frac{1}{q}} > (2n)^{\frac{1}{pq}}(2^{-\frac{1}{pq}} + x)\right) dx \\
&= C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{pq}} \int_{n^{\frac{1}{pq}}}^{\infty} P\left(|X| > y^q\right) dy \text{ (letting } y = (2n)^{\frac{1}{pq}}(2^{-\frac{1}{pq}} + x)).
\end{aligned}$$

The rest proof is similar to (3.5). Hence, (iii) is proved by (vii).

(iii)⇒(i): The proof can refer to Theorem 2.1 in Liang et al. [10]. But there are several differences, which will be listed as follows.

(1) In the process of proving $\sum_{n=1}^{\infty} n^{r-2}\int_1^{\infty} P\left(\max_{1\leq k\leq n}|S_k| > x^q n^{\frac{1}{p}}\right) dx < \infty$:

(a) According to the definition of $X_{ni}(4)$, we can get

$$\begin{aligned}
I_4 &= \sum_{n=1}^{\infty} n^{r-2} \int_1^{\infty} P\left(\max_{1\leq k\leq n}\left|\sum_{1\leq i\leq k} X_{ni}(4)\right| > \frac{x^q n^{\frac{1}{p}}}{4}\right) dx \\
&\leq \sum_{n=1}^{\infty} n^{r-2} \int_1^{\infty} P\left(\bigcup_{i=1}^{n}(|X_i| > x^q n^{\frac{1}{p}}/4N)\right) dx \\
&\leq \sum_{n=1}^{\infty} n^{r-1} \int_1^{\infty} P\left(|X| > \frac{x^q n^{\frac{1}{p}}}{4N}\right) dx \\
&= \sum_{n=1}^{\infty} n^{r-1-\frac{1}{pq}} \int_{n^{\frac{1}{pq}}}^{\infty} P\left(|X| > \frac{y^q}{4N}\right) dy \text{ (letting } x = n^{-\frac{1}{pq}}y), \qquad (3.9)
\end{aligned}$$

which together with proof of (3.5) yields that $I_4 < \infty$.

(b) Applying Lemma 2.3, C_r inequality and Jensen's inequality to I_1^*, we have

$$\begin{aligned}
I_1^* &\leq C\sum_{n=1}^{\infty} n^{r-2-\frac{M}{p}}(\log n)^M \int_1^{\infty} x^{-Mq}\left\{\left(\sum_{i=1}^{n} E(X_{ni}(1) - EX_{ni}(1))^2\right)^{M/2}\right\} dx \\
&\quad + C\sum_{n=1}^{\infty} n^{r-2-\frac{M}{p}}(\log n)^M \int_1^{\infty} x^{-Mq}\left\{\sum_{i=1}^{n} E|X_{ni}(1) - EX_{ni}(1)|^M\right\} dx
\end{aligned}$$

$$\leq C \sum_{n=1}^{\infty} n^{r-2-\frac{M}{p}} (\log n)^M$$
$$\int_1^{\infty} x^{-Mq} \left\{ \left(\sum_{i=1}^{n} EX_{ni}^2(1) \right)^{M/2} + \sum_{i=1}^{n} E|X_{ni}(1)|^M \right\} dx$$
$$=: I_{11} + I_{12}.$$

By the choice of M in Liang et al. [10], we can still get $I_1^* < \infty$.

(2) There are no detailed proof in Liang et al. [10] for

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n} |S_k| > \varepsilon^q n^{\frac{1}{p}} \right) < \infty.$$

Here we give the details.

To begin with, by the truncation in Liang et al. [10], we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n} |S_k| > \varepsilon^q n^{\frac{1}{p}} \right) \\
\leq &\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_{ni}(1) \right| > \frac{\varepsilon^q n^{\frac{1}{p}}}{4} \right) \\
&+ \sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_{ni}(2) \right| > \frac{\varepsilon^q n^{\frac{1}{p}}}{4} \right) \\
&+ \sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_{ni}(3) \right| > \frac{\varepsilon^q n^{\frac{1}{p}}}{4} \right) \\
&+ \sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_{ni}(4) \right| > \frac{\varepsilon^q n^{\frac{1}{p}}}{4} \right) \\
=: &J_1 + J_2 + J_3 + J_4. \qquad (3.10)
\end{aligned}$$

Analogously to (3.9) and $E|X|^{pr} < \infty$ (implied by (iii)), we have

$$\begin{aligned}
J_4 &\leq \sum_{n=1}^{\infty} n^{r-1} P\left(|X| > \frac{\varepsilon^q n^{\frac{1}{p}}}{4N} \right) \\
&= \sum_{n=1}^{\infty} n^{r-1} \sum_{i=n}^{\infty} P\left(\frac{\varepsilon^q i^{\frac{1}{p}}}{4N} < |X| \leq \frac{\varepsilon^q (i+1)^{\frac{1}{p}}}{4N} \right) \\
&= \sum_{i=1}^{\infty} P\left(\frac{\varepsilon^q i^{\frac{1}{p}}}{4N} < |X| \leq \frac{\varepsilon^q (i+1)^{\frac{1}{p}}}{4N} \right) \sum_{n=1}^{i} n^{r-1} \\
&\leq C \sum_{i=1}^{\infty} i^r P\left(\frac{\varepsilon^q i^{\frac{1}{p}}}{4N} < |X| \leq \frac{\varepsilon^q (i+1)^{\frac{1}{p}}}{4N} \right) \leq CE|X|^{pr} < \infty.
\end{aligned}$$

Replacing x by ε and removing the sign of integral in the proof of $I_1 < \infty$, $I_2 < \infty$ and $I_3 < \infty$ in Liang et al. [10], we can easily get $J_1 < \infty$, $J_2 < \infty$ and $J_3 < \infty$, respectively. This completes the proof for (iii)$\Rightarrow$(i).

(ii)$\Leftrightarrow$(i): Actually,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} E\left(\max_{1\le k\le n} |S_k|^{\frac{1}{q}} - \varepsilon n^{\frac{1}{pq}}\right)^+ \\
=& \sum_{n=1}^{\infty} n^{r-2-\frac{1}{pq}} \int_0^{\infty} P\left(\max_{1\le k\le n} |S_k|^{\frac{1}{q}} - \varepsilon n^{\frac{1}{pq}} > t\right) dt \\
=& \sum_{n=1}^{\infty} n^{r-2} \int_{\varepsilon}^{\infty} P\left(\max_{1\le k\le n} |S_k|^{\frac{1}{q}} > x n^{\frac{1}{pq}}\right) dx \ (\text{letting } t = n^{\frac{1}{pq}}(x-\varepsilon)) \\
=& \int_{\varepsilon}^{\infty} f(x^q) dx.
\end{aligned}
$$

Hence we get that (ii) is equivalent to (i). $\square$

It is well known that complete moment convergence can imply complete convergence, therefore we can get the following corollary.

Corollary 3.1. *Let $0 < p < 2$, $r > 1$, and let $\{X, X_n, n \ge 1\}$ be a sequence of NOD random variables with identical distribution. Assume further that $EX = 0$ when $1 \le p < 2$. Denote $S_n = \sum_{i=1}^{n} X_i$, $S_n^{(k)} = S_n - X_k$, $k = 1, 2, \ldots, n$, $n \ge 1$. Then the following statements are equivalent*:

(I) $E|X|^{pr} < \infty$;

(II) $\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n} |S_k| > \varepsilon n^{\frac{1}{p}}\right) < \infty$ *for all* $\varepsilon > 0$;

(III) $\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n} |S_n^{(k)}| > \varepsilon n^{\frac{1}{p}}\right) < \infty$ *for all* $\varepsilon > 0$;

(IV) $\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n} |X_k| > \varepsilon n^{\frac{1}{p}}\right) < \infty$ *for all* $\varepsilon > 0$;

(V) $\sum_{n=1}^{\infty} n^{r-2} P\left(\sup_{k\ge n} k^{-\frac{1}{p}} |S_k| > \varepsilon\right) < \infty$ *for all* $\varepsilon > 0$;

(VI) $\sum_{n=1}^{\infty} n^{r-2} P\left(\sup_{k\ge n} k^{-\frac{1}{p}} |X_k| > \varepsilon\right) < \infty$ *for all* $\varepsilon > 0$.

Proof. The proof can be shown as the order in the following chart:

$$\begin{array}{ccc} \text{(III)} & \Longrightarrow & \text{(IV)} \\ \Uparrow & & \Downarrow \\ \text{(II)} & \Longleftarrow & \text{(I)} \\ \Downarrow & & \Uparrow \\ \text{(V)} & \Longrightarrow & \text{(VI)} \end{array}$$

At first, we prove (IV) $\Rightarrow$ (I). By (3.3), (3.4) and Lemma 2.2, we get that for sufficiently large n,

$$P\left(\max_{1\le k\le n}|X_k| > \varepsilon n^{\frac{1}{p}}\right) \ge C\sum_{k=1}^{n} P\left(|X_k| > \varepsilon n^{\frac{1}{p}}\right),\ \forall\,\varepsilon > 0. \tag{3.11}$$

Thus, taking $\varepsilon = 1$ in (IV) and combining with (3.11) and $r > 1$, we have

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n}|X_k| > n^{\frac{1}{p}}\right) \\
&\ge C\sum_{n=1}^{\infty} n^{r-2}\sum_{k=1}^{n} P\left(|X_k| > n^{\frac{1}{p}}\right) \\
&= C\sum_{n=1}^{\infty} n^{r-1} P\left(|X| > n^{\frac{1}{p}}\right) \\
&= C\sum_{n=1}^{\infty} n^{r-1}\sum_{m=n}^{\infty} P\left(m < |X|^p \le m+1\right) \\
&= C\sum_{m=1}^{\infty} P\left(m < |X|^p \le m+1\right)\sum_{n=1}^{m} n^{r-1} \\
&\ge C\sum_{m=1}^{\infty} (m+1)^r P\left(m < |X|^p \le m+1\right) \\
&\ge C\sum_{m=1}^{\infty} E|X|^{pr} I\left(m < |X|^p \le m+1\right) \\
&\ge CE|X|^{pr}. \qquad (3.12)
\end{aligned}$$

Next, we prove (VI) $\Rightarrow$ (I). By taking $\varepsilon = 2^{-\frac{1}{p}}$ in (VI), (3.6), (3.7) and (3.8), we can obtain

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{r-2} P\left(\sup_{k\ge n} k^{-\frac{1}{p}}|X_k| > 2^{-\frac{1}{p}}\right) \\
&\ge \sum_{n=1}^{\infty} n^{r-2} P\left(\max_{n\le k<2n} k^{-\frac{1}{p}}|X_k| > 2^{-\frac{1}{p}}\right)
\end{aligned}$$

$$\geq C \sum_{n=1}^{\infty} n^{r-1} P(|X| > n^{\frac{1}{p}}).$$

The rest proof is the same as (3.12). Hence (I) follows from (VI).

(I) $\Rightarrow$ (II) can be verified by the similar process of proving (3.10). The rest proof can be easily obtained by Theorem 3.1. This completes the proof of corollary. □

Remark 3.1. We point out that Corollary 3.1 can be regarded as an extension of the Baum-Katz complete convergence theorem (see Baum and Katz [2]).

By Definition 1.2, Lemma 2.5 and the similar proof of Theorem 3.1, we can get the following result.

Theorem 3.2. *Let $0 < p < 2$, $r > 1$, $q > 0$ and let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables, which is stochastically dominated by a random variable X. Assume further that $EX_n = 0$ when $1 \leq p < 2$, and*

$$P(|X_n| > x) \geq CP(|X| > x)$$

for any $x \geq 0$, where C is a positive number. Then (i)-(vii) *in Theorem 3.1 are equivalent.*

Remark 3.2. Theorems 3.1 and 3.2 also hold for other dependent sequences, such as negatively associated (NA) sequence, negatively superadditive dependent (NSD) sequence and extended negatively dependent (END, see Shen [13]) sequence. The keys to the proofs of Theorems 3.1 and 3.2 are Lemmas 2.1-2.3, which are satisfied for NA sequence, NSD sequence and END sequence.

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