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CHARACTERISTIC FUNCTION OF SYMMETRIC DAMEK-RICCI SPACE

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Abstract. A Damek-Ricci space is a typical locally harmonic manifold, which is a generalization of the rank one symmetric space of the noncompact type. In this paper, we determine explicitly the characteristic function of a Damek-Ricci space by calculating the determinant of a Jacobi tensor.

1. Introduction

A Riemannian manifold (M, g) is locally harmonic at $p \in M$ if every volume density $\sqrt{\det(g_{ij})}$ is a function of the Riemannian distance from p on some neighborhood of p in M. There are several equivalent definitions for locally harmonic manifolds ([2], p. 156). One of them is the following:

Theorem 1. *A Riemannian manifold $M = (M, g)$ is locally harmonic at $p \in M$ if and only if the equality*

$$\triangle\Omega_p = f_p(\Omega_p) \qquad \left(\Omega_p = \frac{1}{2}r_p^2\right)$$

holds for a certain smooth function f_p on $[0, \varepsilon(p))$, where $\varepsilon(p)$ is the injectivity radius at $p \in M$ and r_p is the Riemannian distance from p.

It is known that the function f_p in Theorem 1 does not depend on the choice of $p \in M$ ([2], Proposition 6.16). The function $f = f_p$ $(p \in M)$ is called the characteristic function of a harmonic manifold $M = (M, g)$. The characteristic function plays an important role in the geometry of harmonic manifolds and there are many applications such as [5, 8, 9].

The characteristic function has been determined previously for all the rank one symmetric spaces except for the Cayley projective plane and the Cayley

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hyperbolic plane [8, 9]. Very recently, this function of the Cayley projective plane and the Cayley hyperbolic plane has just been determined explicitly [6].

Damek-Ricci spaces are important noncompact harmonic manifolds. Nonsymmetric Damek-Ricci spaces are counterexamples of the Lichnerowicz conjecture: "every locally harmonic manifold is a locally symmetric space." and there are many nonsymmetric Damek-Ricci spaces, in addition to symmetric Damek-Ricci spaces. Complex hyperbolic space $\mathbb{C}\mathrm{H}^n(-1)$, quaternion hyperbolic space $\mathrm{QH}^n(-1)$, and Cayley hyperbolic plane $\mathfrak{C}\mathrm{H}^2(-1)$ are all symmetric Damek-Ricci spaces.

Damek-Ricci spaces are harmonic, so they have the characteristic function. In [1], the characteristic function of a Damek-Ricci space was discovered but it is not the explicit form. In this paper, we can get the explicit form of the characteristic function of a Damek-Ricci space by using another method. In this article, we shall prove the following theorem by using the Jacobi tensor.

Theorem 2. *Let S be a symmetric Damek-Ricci space. Then, the characteristic function as a harmonic manifold is given by*

$$\triangle\Omega = 1 + \sqrt{\frac{\Omega}{2}}((n+m)\coth(\sqrt{\frac{\Omega}{2}}) + m\tanh(\sqrt{\frac{\Omega}{2}})). \tag{1.1}$$

We aimed our paper to be self-contained as much as possible. The authors thank to the referee for the kind suggestions.

2. Preliminaries

In this section, we prepare a brief review on the geometry of the Damek-Ricci space.

A Damek-Ricci space is a one-dimensional extension of a generalized Heisenberg group and a Lie group with the Lie algebra of Iwasawa type. It is a solvable Lie group with a left invariant metric, and is a Riemannian homogeneous Hadamard manifold which is harmonic (see [1] for details). Concretely, the definition of a Damek-Ricci space is the follwing [1, 4].

Definition. Let $\mathfrak{s}$ be a Lie algebra with inner product $\langle\cdot,\cdot\rangle$ satisfying

$$\mathfrak{s} = \mathfrak{o} \oplus \mathfrak{h} \oplus \mathfrak{a} \tag{2.1}$$

orthogonally where $\mathfrak{a}$ is a one-dimensional subspace of s, $\mathfrak{o} \oplus \mathfrak{h} = [\mathfrak{s}, \mathfrak{s}]$, and the linear maps

$$\begin{gathered} J : \mathfrak{h} \to \mathrm{End}(\mathfrak{o}),\ J_X := J(X), \\ J_X^2 = -Id_{\mathfrak{o}}, \forall X \in \mathfrak{h} \end{gathered} \tag{2.2}$$

are given. The simply connected Lie group S with the Lie algebra $\mathfrak{s}$ is called a *Damek-Ricci space*.

We will use the notations $n := \dim \mathfrak{o}$, $m := \dim \mathfrak{h}$, and the Jacobi operator $R_v(w) := R(w, v)v$ for all tangent vectors $v, w \in TS$ in [1] for a Damek-Ricci space S. For a symmetric Damek-Ricci space, the Jacobi operator has constant eigenvalues with multiplicities [1]:

Theorem 3. *Let S be a symmetric Damek-Ricci space. Let $V+Y+sA$ be a unit vector in $\mathfrak{s}$ where $V \in \mathfrak{v}, Y \in \mathfrak{h}, sA \in \mathfrak{a}$. The eigenvalues and multiplicities of R_{V+Y+sA} are $0, 1$; $-1/4, n$; $-1, m$.*

3. Proof of Theorem 2

This chapter uses the same method as [6]. First, we denote by $\gamma = \gamma(t)$ the normal geodesic in (S, g) through the identity $e = \gamma(0)$ with the initial direction $\gamma'(0) = y_0$. By Theorem 3, there is an orthonormal basis $\{y_0, y_1, \ldots, y_n, y_{\bar{1}}, \ldots, y_{\bar{m}}\}$ of s such that

$$R_{y_0}(y_j) = -\frac{1}{4} y_j, \quad R_{y_0}(y_{\bar{k}}) = -y_{\bar{k}} \tag{3.1}$$

for $1 \leqq j \leqq n$ and $1 \leqq k \leqq m$ when identifying $\mathfrak{s}$ and T_eS. Then there is a parallel frame field $\{y_1(t), \ldots, y_n(t), y_{\bar{1}}(t), \ldots, y_{\bar{m}}(t)\}$ along γ such that

$$y_j(0) = y_j (1 \leqq j \leqq n), \quad y_{\bar{k}}(0) = y_{\bar{k}} (1 \leqq k \leqq m). \tag{3.2}$$

Since S is locally symmetric, by (3.1) and (3.2),

$$R_{\gamma'(t)}(y_j(t)) = -\frac{1}{4} y_j(t), \quad R_{\gamma'(t)}(y_{\bar{k}}(t)) = -y_{\bar{k}}(t) \tag{3.3}$$

for $1 \leqq j \leqq n$ and $1 \leqq k \leqq m$.

Now, let $Y_i(t)$ $(1 \leqq i \leqq n)$ and $Y_{\bar{l}}(t)$ $(1 \leqq l \leqq m)$ be the Jacobi vector fields along γ satisfying the following conditions

$$Y_i(0) = 0, \ Y_{\bar{l}}(0) = 0, \quad Y_i'(0) = (\nabla_{\gamma'} Y_i)(0) = y_i, \ Y_{\bar{l}}'(0) = (\nabla_{\gamma'} Y_{\bar{l}})(0) = y_{\bar{l}} \tag{3.4}$$

for $1 \leqq i \leqq n$ and $1 \leqq l \leqq m$. We set as follows along γ:

$$Y_i(t) = \sum_{j=1}^{n} a_{ji}(t) y_j(t) + \sum_{k=1}^{m} a_{\bar{k}i}(t) y_{\bar{k}}(t), \quad Y_{\bar{l}}(t) = \sum_{j=1}^{n} a_{j\bar{l}}(t) y_j(t) + \sum_{k=1}^{m} a_{\bar{k}\bar{l}}(t) y_{\bar{k}}(t) \tag{3.5}$$

for $1 \leqq i \leqq n$ and $1 \leqq l \leqq m$.

Since $Y_i(t)$ $(1 \leqq i \leqq n)$ and $Y_{\bar{l}}(t)$ $(1 \leqq l \leqq m)$ are Jacobi vector fields along the geodesic, from (3.5), taking account of (3.3), we have the following system of differential equations along γ:

$$a''_{ji} - \frac{1}{4}a_{ji} = 0,\quad a''_{\bar{k}i} = a''_{j\bar{l}} = 0,\quad a''_{\bar{k}\bar{l}} - a_{\bar{k}\bar{l}} = 0 \tag{3.6}$$

for $1 \leqq i, j \leqq n$ and $1 \leqq k, l \leqq m$.

Solving (3.6) under the initial conditions (3.4), we have

$$a_{ji}(t) = 2\delta_{ji}\sinh(\frac{t}{2}),\quad a_{\bar{k}i}(t) = a_{j\bar{l}}(t) = 0,\quad a_{\bar{k}\bar{l}}(t) = \delta_{\bar{k}\bar{l}}\sinh t \tag{3.7}$$

for $1 \leqq i, j \leqq n$ and $1 \leqq k, l \leqq m$.

Now, we define $(n+m)\times(n+m)$-matrix $A(t)$ by

$$A(t) = \begin{pmatrix} a_{ij}(t) & a_{i\bar{k}}(t) \\ a_{\bar{l}j}(t) & a_{\overline{lk}}(t) \end{pmatrix} \tag{3.8}$$

for $1 \leqq i, j \leqq n$ and $1 \leqq k, l \leqq m$. Let $\Theta_e(q)$ be the volume density function of the geodesic sphere centered at e through q for each point q in a normal neighborhood centered at e. Then, it is well-known that the following equality

$$\Theta_e(\gamma(t)) = \det A(t) \tag{3.9}$$

holds along the geodesic γ for small t. From (3.8) with (3.7), we have

$$\det A(t) = \left(2\sinh\frac{1}{2}t\right)^n (\sinh t)^m. \tag{3.10}$$

Thus, from (3.9) and (3.10), we have

$$\ln\Theta_e(\gamma(t)) = n\ln 2 + n\ln\sinh\frac{1}{2}t + m\ln\sinh t. \tag{3.11}$$

Here, since a Damek-Ricci space S is a harmonic manifold, the volume density function θ_e (and hence, the function Θ_e) is a radial function on a normal neighborhood U_e centered at the identity e. Thus, Θ_e is determined by its value along the geodesic γ. Thus, from (3.11), we easily see that the function Θ_e is given by

$$\ln\Theta_e(q) = n\ln 2 + n\ln\sinh\frac{1}{2}t + m\ln\sinh t, \tag{3.12}$$

where $q = \gamma(t) \in U_e - \{e\}$ ([3], p. 269).

Now, let $\phi(t)$ be a smooth function of t ($0 < t < \epsilon$, $\epsilon > 0$), and consider the function $f(q)$ on U_e defined by $f(q) = \phi(t)$, $t = d(e, q)$, $q \in U_e$. Then, the following equality holds as in [7] with the sign difference:

$$\triangle f = \phi''(t) + \frac{(\Theta_e(\gamma(t)))'}{\Theta_e(\gamma(t))}\phi'(t), \quad q \in \gamma(t), \tag{3.13}$$

where $\triangle$ denotes the Laplace-Beltrami operator of (S, g). Here, from (3.12), we get

$$\begin{aligned} \frac{(\Theta_e(\gamma(t)))'}{\Theta_e(\gamma(t))} &= (\ln \Theta_e(\gamma(t)))' \\ &= \frac{n}{2} \coth \frac{1}{2} t + m \coth t \\ &= \frac{n+m}{2} \coth \frac{1}{2} t + \frac{m}{2} \tanh \frac{1}{2} t. \end{aligned} \tag{3.14}$$

We here consider the special case where $\phi(t) = \frac{1}{2} t^2$ $(t > 0)$. Then, from (3.13) and (3.14), by direct calculation, we see that

$$\triangle \Omega = 1 + \sqrt{\frac{\Omega}{2}} \left\{ (n+m) \coth \sqrt{\frac{\Omega}{2}} + m \tanh \sqrt{\frac{\Omega}{2}} \right\} \tag{3.15}$$

holds on $U_e - \{e\}$. This completes the proof of Theorem 2.

Remark 1. From Theorem 2, taking account of the discussion in [6], we may reconfirm the statement that symmetric Damek-Ricci spaces are isometric to the complex hyperbolic space, the quaterninonic hyperbolic space or a Cayley hyperbolic plane with the canonical Riemannian metrics, respectively ([1], p. 79).

Remark 2. In the proof, $y_j(t)$ or $y_{\bar{k}}(t)$ are eigenvectors of the Jacobi operator at each point along the geodesic since they are eigenvectors at the initial point and the Damek-Ricci space S is locally symmetric. But, if S is nonsymmetric, usually, neither $R_{\gamma'(t)}(y_j(t))$ nor $R_{\gamma'(t)}(y_{\bar{k}}(t))$ are parallel, so we need to calculate $R_{\gamma'(t)}(y_j(t))$ and $R_{\gamma'(t)}(y_{\bar{k}}(t))$. Thus, the following question remains.

Question. How can we determine the characteristic function of a nonsymmetric Damek-Ricci space by the Jacobi tensor in the explicit form?

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