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INFINITELY MANY SOLUTIONS FOR A CLASS OF MODIFIED NONLINEAR FOURTH-ORDER ELLIPTIC EQUATIONS ON $\mathbb{R}^N$

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Abstract. This paper is concerned with the following fourth-order elliptic equations

$$\triangle^2 u - \Delta u + V(x)u - \frac{\kappa}{2}\Delta(u^2)u = f(x,u), \quad \text{in } \mathbb{R}^{\mathrm{N}},$$

where $N \leq 6$, $\kappa \geq 0$. Under some appropriate assumptions on $V(x)$ and $f(x,u)$, we prove the existence of infinitely many negative-energy solutions for the above system via the genus properties in critical point theory. Some recent results from the literature are extended.

1. Introduction

Consider the following fourth-order elliptic equations of the form

$$\alpha\triangle^2 u - \Delta u + V(x)u - \frac{\kappa}{2}\Delta(u^2)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\triangle^2 := \triangle(\triangle)$ is the biharmonic operator, $\alpha,\ \kappa \in \mathbb{R}$.

When $\alpha = 1$, $\kappa = 0$, we get the following fourth-order elliptic equation

$$\triangle^2 u - \Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Many authors studied Eq. (1.2) on a bounded domain as follows

$$\begin{cases} \triangle^2 u - \Delta u = f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^N$. In [1], An and Liu used the Mountain Pass Theorem to get the existence results for Eq. (1.3). In [23], by using the sign-changing critical theorems that if $f(x,t)$ is odd in t and satisfies some additional conditions, Zhou got infinitely many sign-changing solutions. While without symmetry, Wang and Shen in [15] obtained the multiplicity result by perturbation theory. In [22], Zhang and Wei obtained the existence of infinitely many solutions via variant fountain theorem established in Zou [24] when the

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nonlinearity $f(x,u)$ involves a combination of superlinear and asymptotically linear terms.

Fourth-order elliptic equation on unbounded domains also attract a lot of attention. For instance, see [2, 17, 18, 19] and the references therein. In [19], by using the Mountain Pass Theorem and symmetric Mountain Pass Theorem, Yin and Wu obtained infinitely many high energy solutions for problem (1.2) under the condition that $f(x,u)$ is superlinear at infinity in u. However, for the whole space $\mathbb{R}^N$ case, the main difficulty of this problem is the lack of compactness for the Sobolev's embedding theorem. In order to overcome this difficulty, they assumed that the potential $V(x)$ satisfies

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf\limits_{x\in\mathbb{R}^N} V(x) \geq a > 0$, where $a > 0$ is a constant. Moreover, for any $M > 0$, $meas\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^N$.

Later, under the condition (V_1), when $f(x,u)$ satisfies more weaker and generic conditions, Ye and Tang [18] obtained the existence of infinitely many large-energy and small-energy solutions, which unified and generalized the results in [19], besides, the sublinear case was also considered by them.

Eq. (1.1) with $\alpha = 0$ is a modified nonlinear Schrödinger equation (also called quasilinear Schrödinger equation), whose solutions are related to the existence of solitary wave solutions for the following quasilinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\triangle\psi + V(x)\psi - \kappa\triangle(\rho(|\psi|^2))\rho'(|\psi|^2) - f(x,\psi), \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $V(x)$ is a given potential, κ is a real constant, ρ and f are real functions. We would like to mention that quasilinear equation of the form (1.4) arises in various branches of mathematical physics and has been derived as models of several physics phenomenon corresponding to various types of nonlinear terms ρ, such as see [3, 4, 8].

The semilinear case ($\kappa = 0$) has been studied extensively in recent years with a huge variety of conditions on the potential $V(x)$ and the nonlinearity f, see for example [10, 13] and the references therein. Compared to the semilinear problem, the quasilinear case ($\kappa \neq 0$) becomes more complicated since the effects of the quasilinear and non-convex term $\triangle(u^2)u$. One of the main difficulties of the quasilinear problem is that there is no suitable space on which the energy functional is well defined and belongs to C^1-class except for $N = 1$ (see [9]). There has been several ideas and approaches used in recent years to overcome the difficulties such as by minimizations [5, 9], the Nehari or Pohozaev manifold [6, 12] and change of variables [20, 21].

Inspired by the above facts, the aim of this paper is to study the existence of nontrivial solution and infinitely many negative-energy solutions for problem (1.1) with $\alpha = 1$ via the genus theory in critical point theory. To the best of our knowledge, there has been few works concerning this case up to now.

We assume that $V(x)$ satisfies (V_1) and $f(x,u)$ satisfy the following hypotheses.

(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exist $1 < \alpha_1, \alpha_2 < 2$ and positive functions $c_1 \in L^{\frac{2}{2-\alpha_1}}(\mathbb{R}^N, \mathbb{R})$, $c_2 \in L^{\frac{2}{2-\alpha_2}}(\mathbb{R}^N, \mathbb{R})$ such that

$$|f(x,u)| \leq \alpha_1 c_1(x)|u|^{\alpha_1 - 1} + \alpha_2 c_2(x)|u|^{\alpha_2 - 1}, \ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

(f_2) There exist a bounded open set $J \subset \mathbb{R}^N$ and three constants $a_1, a_2 > 0$ and $a_3 \in (1,2)$ such that

$$F(x,u) \geq a_2|u|^{a_3}, \ \forall (x,u) \in J \times [-a_1, a_1],$$

where $F(x,u) = \int_0^u f(x,s)ds$.

(f_3) $f(x,u) = -f(x,-u)$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.

Now, we state our main results.

Theorem 1.1. *Assume conditions* (V_1) *and* (f_1)-(f_2) *hold, then problem* (1.1) *possesses at least one nontrivial solution.*

Theorem 1.2. *Assume conditions* (V_1) *and* (f_1)-(f_3) *hold, then problem* (1.1) *possesses infinitely many solutions* (u_k) *such that*

$$\frac{1}{2}\int_{\mathbb{R}^N}(|\triangle u_k|^2 + |\nabla u_k|^2 + V(x)u_k^2)dx + \frac{\kappa}{2}\int_{\mathbb{R}^N} u_k^2|\nabla u_k|^2 dx - \int_{\mathbb{R}^N} F(x,u_k)dx \to 0^-$$

as $k \to \infty$.

Evidently, the assumption (f_2) holds if the following conditions holds:

(f_2') There exist a bounded open set $J \subset \mathbb{R}^N$ and three constants $a_1, a_2 > 0$ and $a_3 \in (1,2)$ such that

$$f(x,u)u \geq a_2 a_3 |u|^{a_3}, \ \forall (x,u) \in J \times [-a_1, a_1].$$

Therefore, by Theorems 1.1 and 1.2, we have the following corollary.

Corollary 1.1. *Assume conditions* (V_1), (f_1) *and* (f_2') *hold, then problem* (1.1) *possesses at least one nontrivial solution. If additionally,* (f_3) *holds, then problem* (1.1) *possesses infinitely many solutions* (u_k) *such that*

$$\frac{1}{2}\int_{\mathbb{R}^N}(|\triangle u_k|^2 + |\nabla u_k|^2 + V(x)u_k^2)dx + \frac{\kappa}{2}\int_{\mathbb{R}^N} u_k^2|\nabla u_k|^2 dx - \int_{\mathbb{R}^N} F(x,u_k)dx \to 0^-$$

as $k \to \infty$.

By Theorems 1.1 and 1.2, we also have the following corollaries.

Corollary 1.2. *Assume* (V_1) *and the following conditions hold*:

(f_4) $F(x,u) = b(x)G(u)$,

where $G(u) \in C^1(\mathbb{R}, \mathbb{R})$, $b \in C^1(\mathbb{R}^N, \mathbb{R}) \bigcap L^{\frac{2}{2-\gamma_1}}(\mathbb{R}^N, \mathbb{R})$ *for the constant* $\gamma_1 \in (1,2)$, *and some* $x_0 > 0$ *such that* $b(x_0) > 0$.

(f_5) *There exist constants* M, $m > 0$ *and* $\gamma_0 \in (1,2)$ *such that*

$$m|u|^{\gamma_0} \leq G(u) \leq M|u|^{\gamma_1}, \ \forall \ u \in \mathbb{R},$$

Then problem (1.1) *possesses at least one nontrivial solution.*

Corollary 1.3. *Assume* (V_1), (f_4)-(f_5) *and* $G(-u)=G(u)$ *hold for any* $u\in\mathbb{R}$, *then problem* (1.1) *possesses infinitely many solutions* (u_k) *such that*

$$\frac{1}{2}\int_{\mathbb{R}^N}(|\triangle u_k|^2+|\nabla u_k|^2+V(x)u_k^2)dx+\frac{\kappa}{2}\int_{\mathbb{R}^N}u_k^2|\nabla u_k|^2dx-\int_{\mathbb{R}^N}F(x,u_k)dx\to 0^-$$

as $k\to\infty$.

Remark 1.1. It is well known that for the quasilinear Schrödinger equation problem (1.1), we must overcome the difficulty that the energy functional is not well defined due to the non-convex term $\triangle(u^2)u$, while in this paper, under the assumptions (V_1) and $N\le 6$, we prove $\int_{\mathbb{R}^N}\triangle(u^2)u^2<\infty$, which implies the energy functional of problem (1.1) is well defined on our working space.

Remark 1.2. It is not difficult to find the function $f(x,u)$ satisfy all the conditions of Theorem 1.2. For example, let

$$f(x,u)=\frac{7\sin^2 x_1}{6(1+e^{|x|})}|u|^{\frac{-5}{6}}u+\frac{3\cos^2 x_1}{2(1+e^{|x|})}|u|^{\frac{-1}{2}}u,$$

where $x=\{x_1,x_2,\ldots,x_{\mathbb{N}}\}$. Then,

$$|f(x,u)|\le\frac{7\sin^2 x_1}{6(1+e^{|x|})}|u|^{\frac{1}{6}}+\frac{3\cos^2 x_1}{2(1+e^{|x|})}|u|^{\frac{1}{2}},\ \forall\ (x,u)\in(\mathbb{R}^N\times\mathbb{R}),$$

and

$$\begin{aligned}F(x,u)&=\frac{\sin^2 x_1}{1+e^{|x|}}|u|^{\frac{7}{6}}+\frac{\cos^2 x_1}{1+e^{|x|}}|u|^{\frac{3}{2}}\\&\ge\frac{\cos^2 1}{1+e}|u|^{\frac{3}{2}},\ \forall(x,u)\in J\times[-1.1],\end{aligned}$$

where

$$\frac{7}{6}=\alpha_1<\alpha_2=\frac{3}{2},\ c_1(x)=\frac{\sin^2 x_1}{1+e^{|x|}},\ c_2(x)=\frac{\cos^2 x_1}{1+e^{|x|}},$$

and

$$a_1=1,\ a_2=\frac{\cos^2 1}{1+e},\ a_3=\frac{3}{2},\ J=B(0,1).$$

The remainder of this paper is as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proofs of our main results.

Notation 1.1. Throughout this paper, we shall denote by $|\cdot|$ the L^r-norm and C various positive generic constants, which may vary from line to line. $2_*=\frac{2N}{N-4}$ for $N\ge 5$ and $2_*=+\infty$ for $N\le 4$, is the critical Sobolev exponent. Also if we take a subsequence of a sequence $\{u_n\}$ we shall denote it again $\{u_n\}$.

2. Variational setting and preliminaries

Let

$$H^2(\mathbb{R}^N):=\{u\in L^2(\mathbb{R}^N):\nabla u,\triangle u\in L^2(\mathbb{R}^N)\},$$
$$E:=\{u\in H^2(\mathbb{R}^N)|\int_{\mathbb{R}^N}V(x)u^2dx<+\infty\}.$$

Then, under the conditions (V_1), E is a Hilbert space with the following inner product and norm

$$\langle u,v\rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \nabla v + V(x)uv)dx,$$

$$\|u\| = (\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2)dx)^{\frac{1}{2}}.$$

Moreover, we have the following compactness lemma from [2].

Lemma 2.1 ([2, Lemma 2.1]). *Under the assumption* (V_1), *the embedding* $E \hookrightarrow L^r(\mathbb{R}^N)$ *is continuous for* $2 \le r \le 2_*$ *and* $E \hookrightarrow L^r(\mathbb{R}^N)$ *is compact for* $2 \le r < 2_*$.

Lemma 2.2. *Under assumption* (V_1), (f_1) *and* $N \le 6$, *the functional* $I : E \to \mathbb{R}$ *defined by*

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{\kappa}{2}\int_{\mathbb{R}^N} u^2|\nabla u|^2 dx - \int_{\mathbb{R}^N} F(x,u)dx \tag{2.1}$$

is well defined and of class $C^1(E,\mathbb{R})$ *and*

$$\langle I'(u), v\rangle = (u,v) + \kappa \int_{\mathbb{R}^N} (uv|\nabla u|^2 + u^2 \nabla u \nabla v)dx - \int_{\mathbb{R}^N} f(x,u)vdx. \tag{2.2}$$

Moreover, the critical points of I *in* E *are solutions of problem* (1.1).

Proof. From (f_1), one has

$$|F(x,u)| \le c_1(x)|u|^{\alpha_1} + c_2(x)|u|^{\alpha_2},\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.3}$$

Then, for any $u \in E$, it follows from (V_1), (2.3) and the Hölder inequality that

$$\begin{aligned} \int_{\mathbb{R}^N} |F(x,u)|dx &\le \int_{\mathbb{R}^N} \left[c_1(x)|u|^{\alpha_1} + c_2(x)|u|^{\alpha_2}\right]dx \\ &\le \sum_{i=1}^{2} a^{\frac{-\alpha_i}{2}} \Big(\int_{\mathbb{R}^N} |c_i(x)|^{\frac{2}{2-\alpha_i}}dx\Big)^{\frac{2-\alpha_i}{2}} \Big(\int_{\mathbb{R}^N} V(x)|u|^2 dx\Big)^{\frac{\alpha_i}{2}} \\ &\le \sum_{i=1}^{2} a^{\frac{-\alpha_i}{2}} ||c_i||_{\frac{2}{2-\alpha_i}} ||u||^{\alpha_i}. \end{aligned} \tag{2.4}$$

Next, we prove $\int_{\mathbb{R}^N} u^2|\nabla u|^2 dx < +\infty$ for every $u \in E$. Firstly, we choose two numbers $p = 3$ and $t = \frac{p}{p-1}$. Then $\frac{1}{p} + \frac{1}{t} = 1$, $2 \le 2p \le 2_*$ and $2 \le 2t \le 2^*$ for $N \le 6$. Then by Lemma 2.1 and the assumption of (V_1), we have

$$\begin{aligned} \|u\|_{H^2}^2 &= \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + |u|^2)dx \\ &\le C\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2)dx = C\|u\|^2, \end{aligned}$$

where $C = \max\{1, \frac{1}{a_0}\}$.

Since $H^2(\mathbb{R}^N) = W^{2,2}(\mathbb{R}^N) \hookrightarrow W^{1,r}(\mathbb{R}^N),\ 2 \le r \le 2^*$ and $H^2(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N),\ 2 \le r \le 2_*$, we have

$$\int_{\mathbb{R}^N} u^{2p} dx < +\infty,\ \int_{\mathbb{R}^N} |\nabla u|^{2t} dx < +\infty.$$

By Hölder inequality and Lemma 2.1, we have

$$\int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx \le \big(\int_{\mathbb{R}^N} u^{2p} dx\big)^{\frac{1}{p}} \big(\int_{\mathbb{R}^N} |\nabla u|^{2t} dx\big)^{\frac{1}{t}} < +\infty, \tag{2.5}$$

It follows from (2.4) and (2.5) that I is well defined on E.

Now, we prove that $I \in C^1(E, \mathbb{R})$. Set

$$\Phi_1(u) := \frac{1}{2} \int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx,\ \Phi_2(u) := \int_{\mathbb{R}^N} F(x, u) dx.$$

Then $I(u) = \frac{1}{2}\|u\|^2 + \kappa \Phi_1(u) - \Phi_2(u)$. In order to prove $I \in C^1(E, \mathbb{R})$, we only to prove that $\Phi_i \in C^1(E, \mathbb{R})$, $i = 1, 2$. By the proof of Lemma 2.2 in [2], it is easy to verify that $\Phi_1 \in C^1(E, \mathbb{R})$. Next, we prove (2.2) and $\Phi_2 \in C^1(E, \mathbb{R})$.

For any function $\theta : \mathbb{R}^N \to (0, 1)$, by (f_1) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \max_{t\in[0,1]} |f(x, u(x) + t\theta(x)v(x))v(x)| dx \\
= &\int_{\mathbb{R}^N} \max_{t\in[0,1]} |f(x, u(x) + t\theta(x)v(x))||v(x)| dx \\
\le &\sum_{i=1}^{2} \alpha_i \int_{\mathbb{R}^N} (c_i(x)|u(x) + t\theta(x)v(x)|^{\alpha_i - 1})|v(x)| dx \\
\le &\sum_{i=1}^{2} \alpha_i \int_{\mathbb{R}^N} (c_i(x)(|u(x)|^{\alpha_i - 1} + |v(x)|^{\alpha_i - 1})|v(x)| dx \\
\le &\sum_{i=1}^{2} \alpha_i a^{\frac{-\alpha_i}{2}} \big(\int_{\mathbb{R}^N} |c_i(x)|^{\frac{2}{2-\alpha_i}} dx\big)^{\frac{2-\alpha_i}{2}} \big(\int_{\mathbb{R}^N} V(x)|u(x)|^2 dx\big)^{\frac{\alpha_i - 1}{2}} \\
&\times \big(\int_{\mathbb{R}^N} V(x)|v(x)|^2 dx\big)^{\frac{1}{2}} \\
&+ \sum_{i=1}^{2} \alpha_i a^{\frac{-\alpha_i}{2}} \big(\int_{\mathbb{R}^N} |c_i(x)|^{\frac{2}{2-\alpha_i}} dx\big)^{\frac{2-\alpha_i}{2}} \big(\int_{\mathbb{R}^N} V(x)|v(x)|^2 dx\big)^{\frac{\alpha_i}{2}} \\
\le &\sum_{i=1}^{2} \alpha_i a^{\frac{-\alpha_i}{2}} ||c_i||_{\frac{2}{2-\alpha_i}} (||u||^{\alpha_i - 1} + ||v||^{\alpha_i - 1})||v|| \\
< &\ +\infty.
\end{aligned} \tag{2.6}$$

Then, by (2.1), (2.6) and Lebesgue's Dominated Convergence Theorem, we have

(2.7)
$$\begin{aligned}&\langle I'(u),v\rangle\\ =\ &\lim_{t\to0^+}\frac{I(u+tv)-I(u)}{t}\\ =\ &\lim_{t\to0^+}\Big[(u,v)+\frac{t}{2}||v||^2+\frac{\kappa}{2}\int_{\mathbb{R}^N}\big(t^3v^2|\nabla v|^2+2t^2v^2\nabla u\nabla v+2t^2uv|\nabla v|^2\\ &+4tuv\nabla u\nabla v+tu^2|\nabla v|^2+tv^2|\nabla u|^2+2u^2\nabla u\nabla v+2uv|\nabla u|^2dx\big)\\ &-\int_{\mathbb{R}^N}f(x,u+\theta tv)vdx\Big]\\ =\ &(u,v)+\kappa\int_{\mathbb{R}^N}(uv|\nabla u|^2+u^2\nabla u\nabla v)dx-\int_{\mathbb{R}^N}f(x,u)vdx,\end{aligned}$$

which implies (2.2) holds. Now, we show that $\Phi_2\in C^1(E,\mathbb{R})$. Let $u_n\to u$ in E, then $u_n\to u$ in $L^2(\mathbb{R}^N)$ and

$$\lim_{n\to\infty}u_n=u\quad\text{a.e. } x\in\ \mathbb{R}^N. \tag{2.8}$$

Now, we claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)|^2dx=0. \tag{2.9}$$

Otherwise, there exist a constant $\varepsilon_0>0$ and a sequence $\{u_{ni}\}$ such that

$$\int_{\mathbb{R}^N}|f(x,u_{ni})-f(x,u)|^2dx\geq\varepsilon_0,\ \forall i\in\mathbb{N}. \tag{2.10}$$

In fact, since $u_n\to u$ in $L^2(\mathbb{R}^N)$, passing to a subsequence if necessary, it can be assumed that $\sum\limits_{i=1}^{\infty}||u_{ni}-u||_2^2<+\infty$. Set $\omega(x)=(\sum\limits_{i=1}^{\infty}||u_{ni}-u||_2^2)^{\frac{1}{2}}$, then $\omega\in L^2(\mathbb{R}^N)$. Evidently

(2.11)
$$\begin{aligned}&|f(x,u_{ni})-f(x,u)|^2\\ \leq\ &2|f(x,u_{ni})|^2+2|f(x,u)|^2\\ \leq\ &4\alpha_1^2|c_1(x)|^2\big[|u_{ni}|^{2(\alpha_1-1)}+|u|^{2(\alpha_1-1)}\big]\\ &\quad+4\alpha_2^2|c_2(x)|^2\big[|u_{ni}|^{2(\alpha_2-1)}+|u|^{2(\alpha_2-1)}\big]\\ \leq\ &\sum_{j=1}^{2}(4^{\alpha_j}+4)\alpha_j^2|c_j(x)|^2\big[|u_{ni}-u|^{2(\alpha_j-1)}+|u|^{2(\alpha_j-1)}\big]\\ \leq\ &\sum_{j=1}^{2}(4^{\alpha_j}+4)\alpha_j^2|c_j(x)|^2\big[|\omega(x)|^{2(\alpha_j-1)}+|u|^{2(\alpha_j-1)}\big]\\ :=\ &h(x),\ \forall\ i\in\ \mathbb{N},\ x\in\mathbb{R}^N\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} h(x)dx \\
&= \sum_{j=1}^{2}(4^{\alpha_j}+4)\alpha_j^2\int_{\mathbb{R}^N}|c_j(x)|^2\big[|\omega(x)|^{2(\alpha_j-1)}+|u|^{2(\alpha_j-1)}\big]dx \\
&\le \sum_{j=1}^{2}(4^{\alpha_j}+4)\alpha_j^2||c_j||^2_{\frac{2}{2-\alpha_j}}\big(||\omega||_2^{2(\alpha_j-1)}+||u||_2^{2(\alpha_j-1)}\big) \\
&< +\infty.
\end{aligned} \tag{2.12}$$

It follows from (2.11), (2.12) and the Lebesgue's Dominated Convergence Theorem that (2.9) holds.

Then, by (2.2), (2.9) and $\Phi_1 \in C^1(E,\mathbb{R})$, we have

$$\begin{aligned}
&|\langle I'(u_n)-I'(u),v\rangle| \\
&= \Big|(u_n-u,v)+\kappa\int_{\mathbb{R}^N}\big(|u_n|^2\nabla u_n-|u|^2\nabla u\big)\cdot\nabla v dx \\
&\quad +\kappa\int_{\mathbb{R}^N}\big(|\nabla u_n|^2u_n-|\nabla u|^2u\big)\cdot v dx-\int_{\mathbb{R}^N}[f(x,u_n)-f(x,u)]v dx\Big| \\
&\le ||u_n-u||||v||+|\kappa\int_{\mathbb{R}^N}\big(|u_n|^2\nabla u_n-|u|^2\nabla u\big)\cdot\nabla v dx \\
&\quad +\kappa\int_{\mathbb{R}^N}\big(|\nabla u_n|^2u_n-|\nabla u|^2u\big)v dx|+a^{-\frac{1}{2}}\Big(\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)|^2dx\Big)^{\frac{1}{2}}||v|| \\
&\to 0 \text{ as } n\to\infty,
\end{aligned}$$

which implies that $I \in C^1(E,\mathbb{R})$. Moreover, by a standard argument, it is easy to verify that the critical points of I in E are solutions of problem (1.1) (see [16]). The proof is complete. □

Lemma 2.3. *Assume that (V_1), (f_1) and $N \le 6$ hold. Then I is bounded from below and satisfies the (PS) condition.*

Proof. By Lemma 2.1, (f_1), the Sobolev embedding theorem and the Hölder inequality, we have

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2+\frac{\kappa}{2}\int_{\mathbb{R}^N}u^2|\nabla u|^2dx-\int_{\mathbb{R}^N}F(x,u)dx \\
&\ge \frac{1}{2}\|u\|^2-\int_{\mathbb{R}^N}F(x,u)dx \\
&\ge \frac{1}{2}\|u\|^2-\int_{\mathbb{R}^N}c_1(x)|u|^{\alpha_1}dx-\int_{\mathbb{R}^N}c_2(x)|u|^{\alpha_2}dx \\
&\ge \frac{1}{2}\|u\|^2-\sum_{i=1}^{2}a^{\frac{-\alpha_i}{2}}||c_i||_{\frac{2}{2-\alpha_i}}||u||^{\alpha_i},
\end{aligned} \tag{2.13}$$

which implies that $I(u) \to +\infty$, as $||u|| \to \infty$, since $\alpha_1,\ \alpha_2 \in (1,2)$. Consequently, I is bounded from below.

Next, we prove that I satisfies the (PS) condition. Assume that $\{u_n\}$ is a (PS) sequence of I such that $I(u_n)$ is bounded and $||I'(u_n)|| \to 0$ as $n \to \infty$. Then, it follows from (2.13) that there exists a constant $C > 0$ such that

$$||u_n||_2 \le a^{\frac{-1}{2}}||u_n|| \le C,\ n \in \mathbb{N}. \tag{2.14}$$

Then by Lemma 2.1, there exists $u \in E$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } E,\\ &u_n \to u \text{ in } L^s(\mathbb{R}^N),\ s \in [2, 2_*),\\ &u_n \to u \text{ a.e. } \mathbb{R}^N. \end{aligned} \tag{2.15}$$

Therefore

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(|u_n|^2\nabla u_n - |u|^2\nabla u\big)\cdot \nabla(u_n - u)dx\\ = &\int_{\mathbb{R}^N} \big(|u_n|^2 - |u|^2\big)\nabla u_n \nabla(u_n - u)dx + \int_{\mathbb{R}^N} |u|^2|\nabla(u_n-u)|^2 dx\\ \ge &\int_{\mathbb{R}^N} \big(|u_n|^2 - |u|^2\big)\nabla u_n \nabla(u_n - u)dx\\ \ge & -\int_{\mathbb{R}^N} (|u_n - u|(u_n| + |u|)|\nabla u_n||\nabla(u_n-u)|)dx\\ \ge & -\Big(\int_{\mathbb{R}^N}|u_n-u|^6dx\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}^N}(|u_n|+|u|)^6dx\Big)^{\frac{1}{6}}\\ &\times\Big(\int_{\mathbb{R}^N}|\nabla u_n|^3dx\Big)^{\frac{1}{3}}\Big(\int_{\mathbb{R}^N}|\nabla(u_n-u)|^3dx\Big)^{\frac{1}{3}}\\ \ge & -C||u_n-u||_6 \to 0 \end{aligned} \tag{2.16}$$

as $n \to \infty$. Analogously, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(|\nabla u_n|^2 u_n - |\nabla u|^2 u\big)\cdot (u_n - u)dx\\ = &\int_{\mathbb{R}^N} \big(|\nabla u_n|^2 - |\nabla u|^2\big)u(u_n - u)dx + \int_{\mathbb{R}^N} |\nabla u_n|^2|(u_n-u)|^2 dx\\ \ge & -\int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + |\nabla u|^2\big)|u_n||u_n-u|dx\\ \ge & -\Big(\int_{\mathbb{R}^N}|u_n-u|^6dx\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}^N}|u_n|^6dx\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}^N}|\nabla u_n|^3dx\Big)^{\frac{2}{3}}\\ & -\Big(\int_{\mathbb{R}^N}|u_n-u|^6dx\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}^N}|u_n|^6dx\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}^N}|\nabla u|^3dx\Big)^{\frac{2}{3}}\\ \ge & -C||u_n-u||_6 \to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.17}$$

On the other hand, for any given $\varepsilon > 0$, by (f_1), we can choose $R_\varepsilon > 0$ such that

$$(\int_{|x|>R_\varepsilon} |c_i(x)|^{\frac{2}{2-\alpha_i}} dx\big)^{\frac{2-\alpha_i}{2}} < \varepsilon,\ i = 1, 2. \tag{2.18}$$

It follows from (2.15) that there exists $n_0 > 0$ such that

$$\int_{|x|\leq R_\varepsilon} |u_n - u|^2 dx < \varepsilon^2 \ \text{ for } \ n \geq n_0. \tag{2.19}$$

Therefore, by (f_1), (2.14), (2.19) and the Hölder inequality, for any $n \geq n_0$, one has

$$\begin{aligned}
&\int_{|x|\leq R_\varepsilon} |f(x,u_n) - f(x,u)||u_n - u|dx \\
\leq&\big(\int_{|x|\leq R_\varepsilon} |f(x,u_n) - f(x,u)|^2 dx\big)^{\frac{1}{2}}\big(\int_{|x|\leq R_\varepsilon} |u_n - u|^2 dx\big)^{\frac{1}{2}} \\
\leq&\varepsilon\big[\int_{|x|\leq R_\varepsilon} 2(|f(x,u_n)|^2 + |f(x,u)|^2)dx\big]^{\frac{1}{2}} \\
\leq&\varepsilon\big[4\sum_{i=1}^{2}\alpha_i^2\int_{|x|\leq R_\varepsilon} |c_i(x)|^2(|u_n|^{2(\alpha_i-1)} + |u|^{2(\alpha_i-1)})dx\big]^{\frac{1}{2}} \\
\leq&C\varepsilon\big[\sum_{i=1}^{2}\alpha_i^2||c_i||^2_{\frac{2}{2-\alpha_i}}\big(||u_n||_2^{2(\alpha_i-1)} + ||u||_2^{2(\alpha_i-1)}\big)\big]^{\frac{1}{2}} \\
\leq&C\varepsilon\big[\sum_{i=1}^{2}\alpha_i^2||c_i||^2_{\frac{2}{2-\alpha_i}}\big(C^{2(\alpha_i-1)} + ||u||_2^{2(\alpha_i-1)}\big)\big]^{\frac{1}{2}}.
\end{aligned} \tag{2.20}$$

For another, for $n \in \mathbb{N}$, it follows from (f_1), (2.14), (2.18) and Hölder inequality that

$$\begin{aligned}
&\int_{|x|>R_\varepsilon} |f(x,u_n) - f(x,u)||u_n - u|dx \\
\leq& \sum_{i=1}^{2}\alpha_i\int_{|x|>R_\varepsilon} |c_i(x)|\big(|u_n|^{\alpha_i-1} + |u|^{\alpha_i-1}\big)\big(|u_n| + |u|\big)dx \\
\leq& 2\sum_{i=1}^{2}\alpha_i\int_{|x|>R_\varepsilon} |c_i(x)|\big(|u_n|^{\alpha_i} + |u|^{\alpha_i}\big)dx \\
\leq& 2\sum_{i=1}^{2}\alpha_i\big(\int_{|x|>R_\varepsilon} |c_i(x)|^{\frac{2}{2-\alpha_i}} dx\big)^{\frac{2-\alpha_i}{2}}\big(||u_n||_2^{\alpha_i} + ||u||_2^{\alpha_i}\big) \\
\leq& 2\sum_{i=1}^{2}\alpha_i\big(\int_{|x|>R_\varepsilon} |c_i(x)|^{\frac{2}{2-\alpha_i}} dx\big)^{\frac{2-\alpha_i}{2}}\big(C^{\alpha_i} + ||u||_2^{\alpha_i}\big)
\end{aligned} \tag{2.21}$$

$$\leq 2\varepsilon \sum_{i=1}^{2} \alpha_i \big(C^{\alpha_i} + ||u||_2^{\alpha_i}\big).$$

Since ε is arbitrary, combining (2.20) and (2.21), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))(u_n - u)dx = 0. \tag{2.22}$$

Then by (2.2), (2.16), (2.17), (2.22) and the weak convergence of $\{u_n\}$, one has

$$\begin{aligned}
o_n(1) &= \langle I'(u_n) - I'(u), u_n - u\rangle \\
&= \int_{\mathbb{R}^N} |\triangle(u_n - u)|^2 dx + \int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 dx + \int_{\mathbb{R}^N} V(x)(u_n - u)^2 dx \\
&\quad + \kappa \int_{\mathbb{R}^N} \big(|u_n|^2 \nabla u_n - |u|^2 \nabla u\big) \cdot \nabla(u_n - u)dx \\
&\quad + \kappa \int_{\mathbb{R}^N} \big(|\nabla u_n|^2 u_n - |\nabla u|^2 u\big) \cdot (u_n - u)dx \\
&\quad - \int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))(u_n - u)dx \\
&\geq ||u_n - u||^2 + \kappa \int_{\mathbb{R}^N} (u_n^2 - u^2)\nabla u \nabla(u_n - u)dx \\
&\quad + \kappa \int_{\mathbb{R}^N} (|\nabla u_n|^2 - |\nabla u|^2) u(u_n - u)dx \\
&\quad - \int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))(u_n - u)dx \\
&= ||u_n - u||^2 + o_n(1),
\end{aligned}$$

which implies that $u_n \to u$ in E. Therefore, I satisfies the (PS) condition. The proof is complete. □

Theorem 2.1 ([7]). *Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ satisfy the (PS) condition. If I is bounded from below, then $c = \inf_E I$ is a critical value of I.*

In order to find the multiplicity of nontrivial critical points of I, we will use the "genus" properties, so we recall the following definitions and results (see [11]).

Let E be a Banach space, $c \in \mathbb{R}$ and $I \in C^1(E,\mathbb{R})$. Set

$$\Sigma = \{A \subset E \setminus \{0\} : A \text{ is closed in } E \text{ and symmetric with respect to } 0\},$$

$$K_c = \{u \in E : I(u) = c, I'(u) = 0\},\ I^c = \{u \in E : I(u) \leq c\}.$$

Definition 2.1. For $A \in \Sigma$, we say genus of A is n (denoted by $\gamma(A) = n$) if there is an odd map $\varphi \in C(A, \mathbb{R}^N \setminus \{0\})$ and n is the smallest integer with this property.

Theorem 2.2. *Let E be an even C^1 functional on E and satisfy the (PS) condition. For any $n \in \mathbb{N}$, set*

$$\Sigma_n = \{A \in \Sigma : \gamma(A) \geq n\},\ c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} I(u).$$

(i) *If $\Sigma_n \neq \emptyset$ and $c_n \in \mathbb{R}$, then c_n is a critical value of I.*

(ii) *If there exists $r \in \mathbb{N}$ such that $c_n = c_{n+1} = \cdots = c_{n+r} = c \in \mathbb{R}$ and $c \neq I(0)$, then $\gamma(K_c) \geq r+1$.*

3. Proofs of main results

Proof of Theorem 1.1. By Lemma 2.2 and Lemma 2.3, the conditions of Theorem 2.1 are satisfied. Thus, $c = \inf_E I(u)$ is a critical value of I, that is, there exists a critical point u^* such that $I(u^*) = c$. Now, we show that $u^* \neq 0$.

Let $u \in (W_0^{1,2}(J) \bigcap E) \setminus \{0\}$ and $||u||_\infty \leq 1$, then by (2.1) and (f_2), we have

$$\begin{aligned} I(tu) &= \frac{t^2}{2}\|u\|^2 + \frac{\kappa t^4}{2}\int_{\mathbb{R}^N} u^2|\nabla u|^2 dx - \int_{\mathbb{R}^N} F(x,tu)dx \\ &= \frac{t^2}{2}\|u\|^2 + \frac{\kappa t^4}{2}\int_{\mathbb{R}^N} u^2|\nabla u|^2 dx - \int_J F(x,tu)dx \\ &\leq \frac{t^2}{2}\|u\|^2 + \frac{\kappa t^4}{2}\int_{\mathbb{R}^N} u^2|\nabla u|^2 dx - a_2 t^{a_3}\int_J |u|^{a_3}dx, \end{aligned} \tag{3.1}$$

where $0 < t < a_1$, a_1 be given in (f_2). Since $1 < a_3 < 2$, it follows from (3.1) that $I(tu) < 0$ for $t > 0$ small enough. Therefore, $I(u^*) = c < 0$, that is, u^* is a nontrivial critical point of I, and so u^* is a nontrivial solution of problem (1.1). The proof is complete. □

Proof of Theorem 1.2. By Lemma 2.2 and Lemma 2.3, $I \in C^1(E,\mathbb{R})$ is bounded from below and satisfies the (PS)-condition. It follows from (2.1) and (f_3) that I is even and $I(0) = 0$. In order to apply Theorem 2.2, we now show that for any $n \in \mathbb{N}$, there exists $\varepsilon > 0$ such that

$$\gamma(I^{-\varepsilon}) \geq n. \tag{3.2}$$

For any $n \in \mathbb{N}$, we take n disjoint open sets J_i such that

$$\bigcup_{i=1}^n J_i \subset J.$$

For $i = 1,2,\ldots,n$, let $u_i \in (W_0^{1,2}(J_i) \bigcap E) \setminus \{0\}$, $||u_i||_\infty \leq \infty$ and $||u_i|| = 1$, and

$$E_n = span\{u_1,u_2,\ldots,u_n\},\ S_n = \{u \in E_n : ||u|| = 1\}.$$

Then, for any $u \in E_n$, there exist $\lambda_i \in \mathbb{R}, i = 1,2,\ldots,n$ such that

$$u(x) = \sum_{i=1}^n \lambda_i u_i(x),\ x \in \mathbb{R}^N. \tag{3.3}$$

Then we get

$$||u||_{a_3} = \big(\int_{\mathbb{R}^N} |u|^{a_3}dx\big)^{\frac{1}{a_3}} = \big(\sum_{i=1}^{n} |\lambda_i|^{a_3} \int_{J_i} |u|^{a_3}dx\big)^{\frac{1}{a_3}}, \tag{3.4}$$

and

$$\begin{aligned}
\|u\|^2 &= \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2)dx \\
&= \sum_{i=1}^{n} \lambda_i^2 \int_{J_i} (|\Delta u_i|^2 + |\nabla u_i|^2 + V(x)|u_i|^2)dx \\
&\leq \sum_{i=1}^{n} \lambda_i^2 \int_{\mathbb{R}^N} (|\Delta u_i|^2 + |\nabla u_i|^2 + V(x)|u_i|^2)dx \\
&= \sum_{i=1}^{n} \lambda_i^2 ||u_i||^2 \\
&= \sum_{i=1}^{n} \lambda_i^2.
\end{aligned} \tag{3.5}$$

Since all norms are equivalent in a finite dimensional normed space, so there exists $d_1 > 0$ such that

$$d_1||u|| \leq ||u||_{a_3} \quad \text{for} \quad u \in E_n. \tag{3.6}$$

Then by (2.1), (2.5), (f_2), (3.3)-(3.6) and Sobolev embedding inequality, for $u \in S_n$, we have

$$\begin{aligned}
I(tu) &= \frac{t^2}{2}\|u\|^2 + \frac{\kappa t^4}{2} \int_{\mathbb{R}^N} u^2|\nabla u|^2dx - \int_{\mathbb{R}^N} F(x,tu)dx \\
&= \frac{t^2}{2}\|u\|^2 + \frac{\kappa t^4}{2} \int_{\mathbb{R}^N} u^2|\nabla u|^2dx - \sum_{i=1}^{n} \int_{J_i} F(x,t\lambda_i u_i)dx \\
&\leq \frac{t^2}{2}\|u\|^2 + \frac{\kappa t^4}{2}\tau_{2p}^2\tau_{2t}^2\|u\|^4 - a_2t^{a_3} \sum_{i=1}^{n} |\lambda_i|^{a_3} \int_{J_i} |u_i|^{a_3}dx \\
&= \frac{t^2}{2}\|u\|^2 + \frac{\kappa t^4}{2}\tau_{2p}^2\tau_{2t}^2\|u\|^4 - a_2t^{a_3}||u||_{a_3}^{a_3} \\
&\leq \frac{t^2}{2}\|u\|^2 + \frac{\kappa t^4}{2}\tau_{2p}^2\tau_{2t}^2\|u\|^4 - a_2(d_1t)^{a_3}||u||^{a_3} \\
&= \frac{t^2}{2} + \frac{\kappa t^4}{2}\tau_{2p}^2\tau_{2t}^2 - a_2(d_1t)^{a_3},
\end{aligned} \tag{3.7}$$

where τ_r is the best constant for the embedding of E into $L^r(\mathbb{R}^N)$, $r \in [2, 2_*]$. Since $0 < t \leq a_1$ and $1 < a_3 < 2$, then it follows from (3.7) that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$I(\delta u) < -\varepsilon \quad \text{for} \quad u \in S_n. \tag{3.8}$$

Let

$$S_n^\delta = \{\delta u : u \in S_n\},\ \Omega = \big\{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^N : \sum_{i=1}^{n} \lambda_i^2 < \delta^2\big\}.$$

It follows from (3.8) that

$$I(u) < -\varepsilon \quad \text{for} \quad u \in S_n^\delta,$$

which, together with the fact that $I \in C^1(E, \mathbb{R})$ and is even, implies that

$$S_n^\delta \subset I^{-\varepsilon} \in \Sigma. \tag{3.9}$$

On the other hand, by (3.3) and (3.5), there exists an odd homeomorphism mapping $\phi \in C(S_n^\delta, \partial\Omega)$. By some properties of the genus (see 3^0 of Propositions of 7.5 and 7.7 in [11]), we have

$$\gamma(I^{-\varepsilon}) \geq \gamma(S_n^\delta) = n. \tag{3.10}$$

Thus, the proof of (3.2) holds. Set

$$c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} I(u). \tag{3.11}$$

It follows from (3.10) and the fact I is bounded from below on E that $-\infty < c_n \leq -\varepsilon < 0$, that is to say, for any $n \in \mathbb{N}$, c_n is a real negative number. By Theorem 2.2, I has infinitely many nontrivial critical points, therefore, problem (1.1) possesses infinitely many nontrivial solutions. The proof is complete. □

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