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ON SOME APPLICATIONS OF THE ARCHIMEDEAN COPULAS IN THE PROOFS OF THE ALMOST SURE CENTRAL LIMIT THEOREMS FOR CERTAIN ORDER STATISTICS

Marcin Dudziński and Konrad Furmańczyk

Abstract. Our goal is to establish and prove the almost sure central limit theorems for some order statistics $\left\{M_n^{(k)}\right\}$, $k = 1, 2, \ldots$, formed by stochastic processes $(X_1, X_2, \ldots, X_n)$, $n \in N$, the distributions of which are defined by certain Archimedean copulas. Some properties of generators of such the copulas are intensively used in our proofs. The first class of theorems stated and proved in the paper concerns sequences of ordinary maxima $\{M_n\}$, the second class of the presented results and proofs applies for sequences of the second largest maxima $\left\{M_n^{(2)}\right\}$ and the third (and the last) part of our investigations is devoted to the proofs of the almost sure central limit theorems for the k-th largest maxima $\left\{M_n^{(k)}\right\}$ in general. The assumptions imposed in the first two of the mentioned groups of claims significantly differ from the conditions used in the last - the most general - case.

1. Introduction and preliminaries

Starting with the notable papers by Brosamler [2] and Schatte [23], the almost sure versions of limit theorems have been studied by a large number of authors. These types of limit theorems are commonly known as the almost sure central limit theorems (ASCLTs). The following property is investigated in the research concerning the ASCLTs. Namely, let: $X_1, X_2, \ldots, X_i, \ldots$ be some r.v.'s, $f_1, f_2, \ldots, f_i, \ldots$ denote some real-valued, measurable functions, defined on $\mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^i, \ldots$, respectively; we seek conditions under which the following property is satisfied for some nondegenerate cdf H

$$(1) \quad \lim_{N\to\infty} \frac{1}{D_N} \sum_{n=1}^{N} d_n I\left(f_n\left(X_1, \ldots, X_n\right) \le x\right) = H\left(x\right) \quad \text{a.s.} \quad \text{for all } x \in C_H,$$

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where: $\{d_n\}$ is some sequence of weights, $D_N = \sum_{n=1}^{N} d_n$, I denotes the indicator function, and: $a.s.$, C_H stand for the almost sure convergence and the set of continuity points of function H, respectively.

The subject matter regarding the ASCLTs has drawn an immense attention since the publication of the two above mentioned papers and a large amount of works devoted to the proofs of (1) for various classes of functions f_n and random sequences $\{X_i\}$ have been published over the last twenty-five years or so. We cite in this context the articles by: Berkes and Csáki [1], Chen and Lin [3], Cheng et al. [4], Csáki and Gonchigdanzan [5], Dudziński [6]-[7], Dudziński and Górka [8], Gonchigdanzan and Rempała [11], Lacey and Philipp [14], Matuła [16], Mielniczuk [19], Peligrad and Shao [20], and Zhao et al. [29], among others. Functions f_n included different kinds of functions of r.v.'s, e.g., partial sums (see [1], [6], [14], [16], [19], [20]), products of partial sums (see [11]), maxima (see [1], [3]-[5]), maxima of sums (see [1], [8]), and - jointly - maxima and sums as well (see [7], [29]). It is worth noticing that not only the indicator functions need to be considered with regard to this issue (see e.g. Fazekas and Rychlik [9]); we say about the functional almost sure central limit theorem in this case.

With reference to the ASCLT for order statistics (the k-th largest maxima), which we are mostly concerned with in our work, there are several papers dealing with this topic. We should mention the papers by: Hörmann [12], Peng L. and Qi [21], Peng Z. et al. [22], Stadtmüller [26], and Tan [27] in this place.

Our principal objective is to prove the property in (1) with: $d_n = 1/n$, $D_N \sim \log N$, $f_n(X_1, \ldots, X_n) = M_n^{(k)}$, $k = 1, 2$, where $M_n^{(k)}$ stands for the k-th largest maximum of $X_1, \ldots, X_n$. The assumptions imposed in our assertions are strictly connected with the notions of the so-called Archimedean copulas and their generators. For this reason, we shall introduce some definitions and properties related to copulas, and to the Archimedean copulas in particular. Let us begin with the general definition of copula.

Definition 1.1. A d-dimensional function C: $[0,1]^d \to [0,1]$, $d \geq 2$, defined on the unit cube $[0,1]^d$, is a d-dimensional copula if C is a joint cdf of a d-dimensional random vector with uniform-$[0,1]$ marginals, i.e.,

$$C(u_1, u_2, \ldots, u_d) = P(U_1 \leq u_1, U_2 \leq u_2, \ldots, U_d \leq u_d),$$

where all of U_i, $i = 1, 2, \ldots, d$, have an uniform-$[0,1]$ cdf.

The theoretical groundwork for an area concerning the applications of copulas has been laid in the papers by Sklar [24]-[25], where the following celebrated claim has been stated among some other valuable results.

Theorem 1.1 (Sklar's theorem)**.** *For a given multivariate* (*joint*) *cdf F of a random vector* $(X_1, \ldots, X_d)$ *with marginal cdf's $F_1, \ldots, F_d$, $d \geq 2$, there exists* *a unique copula C satisfying*

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)). \tag{2}$$

This copula is unique if the $F_i's$, $i = 1, \dots, d$, are continuous.

Conversely, for a given copula C: $[0,1]^d \to [0,1]$ and marginal cdf's $F_1, \dots$, F_d, the relation (2) defines a multivariate distribution of $(X_1, \dots, X_d)$ with margins X_i having some cdf's F_i, $i = 1, \dots, d$, respectively.

In view of Sklar's theorem, we may treat a copula as a structure describing the dependence between the coordinates of the random vector $(X_1, \dots, X_d)$. Indeed, (2) means that C couples the marginal cdf's F_i to the joint cdf F. Simultaneously, due to Sklar's proposition, we are also able to decouple the dependence structure into the corresponding marginals.

In our investigations leading to the proof of the ASCLT for some order statistics, we are concerned with a special class of copulas, commonly known as the Archimedean copulas. Before we define the Archimedean copula, we will introduce the notion of copula's generator.

Definition 1.2. Suppose that $d \geq 2$ and Ψ: $[0,1] \to [0,\infty]$ is a strictly decreasing, convex function satisfying the conditions $\Psi(0) = \infty$ and $\Psi(1) = 0$. Let for $x_i \in [0,1]$, $i = 1, \dots, d$,

$$C^{\Psi}(x_1, \dots, x_d) = \Psi^{-1}\left(\sum_{i=1}^{d} \Psi(x_i)\right). \tag{3}$$

The function Ψ is called a generator of the copula C^{Ψ}.

If $d \geq 3$, C^{Ψ} is on the whole not a copula. However, the following statement from Kimberling [13] gives a necessary and sufficient condition under which C^{Ψ} is a copula for all $d \geq 2$.

Theorem 1.2. *Choose $d \geq 2$. The function $C^{\Psi}(x_1, \dots, x_d)$ in (3) is a copula if and only if a generator Ψ has an inverse Ψ^{-1}, which is completely monotonic on $[0,\infty)$, i.e.,*

$$(-1)^j \frac{d^j}{dz^j} \Psi^{-1}(z) \geq 0 \quad \textit{for all } j \in \mathbb{N} \textit{ and } z \in [0,\infty).$$

We are now in a position to define the class of Archimedean copulas.

Definition 1.3. If Ψ^{-1} is completely monotonic on $[0,\infty)$, we say that C^{Ψ} given by (3) is the so-called Archimedean copula.

In our research, we study the situation when the investigated sequence of r.v.'s (X_i) is a stochastic process defined as follows. Namely, we assume that, for any $i \in \mathbb{N}$, a r.v. X_i has a marginal cdf F of the continuous type and that, for any sequence $(t_1, t_2, \dots, t_n)$, of natural numbers, the n-dimensional distribution of $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ is defined by a certain Archimedean copula

$C^{\Psi_n} = C^{\Psi}$ having a generator $\Psi_n = \Psi$, not depending on n. It means that, for any $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$,

$$P\left(X_{t_1} \leq x_1, X_{t_2} \leq x_2, \ldots, X_{t_n} \leq x_n\right) = \Psi^{-1}\left(\sum_{i=1}^{n} \Psi\left(x_i\right)\right).$$

Thus, the considered r.v.'s (X_i) form an exchangeable sequence of identically distributed r.v.'s with a common cdf F, such that, for any fixed $n \geq 2$, $n \in \mathbb{N}$, the family of r.v.'s $(X_1, \ldots, X_n)$ has the Archimedean copula $C^{\Psi_n} = C^{\Psi}$ with a generator $\Psi_n = \Psi$ (both the Archimedean copula and its generator do not depend on n).

It can be shown that under the assumption above, there exists a r.v. $\Theta_n = \Theta > 0$, not depending on n, such that $(\Psi)^{-1}$ is the Laplace transform of Θ, i.e.,

$$(\Psi)^{-1}(z) = E_{\Theta}\left\{\exp\left(-\Theta \cdot z\right)\right\} \quad \text{for any } z \in [0, \infty], \tag{4}$$

where, here and in further parts of our work, E_{Θ} denotes the expected value of appropriate r.v.

We also assume that, for any $x \in \mathbb{R}$ and $\theta \in \operatorname{supp} \Theta$,

$$P\left(X_i \leq x \middle| \Theta = \theta\right) = \left(G(x)\right)^{\theta}, \; i = 1, 2, \ldots, n, \tag{5}$$

for $G = G_n$, not depending on n and satisfying

$$G(x) = \exp\left\{-\Psi\left(F(x)\right)\right\}. \tag{6}$$

It is known (see Marshall and Olkin [15] and Frees and Valdez [10]) that under the conditions imposed above, $X_1, \ldots, X_n$ are conditionally independent given $\Theta = \Theta_n$ (which, for recollection, does not depend on n).

The remainder of the paper is structured as follows. In Sections 2-3, we formulate our major results, which are the corresponding ASCLTs for maxima $\{M_n\}$ or for the second largest maxima $\{M_n^{(2)}\}$ (see the statements in Section 2) and the ASCLT for the k-th largest maxima $\{M_n^{(k)}\}$ as well (see the statement in Section 3 for this general case). In Section 4, some auxiliary results necessary for the proofs of the ASCLTs for ordinary maxima and for the second largest maxima are stated and proved. The mentioned proofs are given in Section 5. Furthermore, in Section 6, the proofs of the ASCLTs for the k-th largest maxima - the assertions established in Section 3 - are given. Appendix containing comments on some of the assumed conditions has been added at the end of our work.

2. Main results I (the ASCLTs for: $\{M_n\}$, $\{M_n^{(2)}\}$)

Our first principal result is the following ASCLT for ordinary maxima $\{M_n\}$ and for the second largest maxima $\{M_n^{(2)}\}$.

Theorem 2.1. (i) *Suppose that* $\{X_i\}$ *is a stochastic process defined as above, i.e., it is a sequence of identically distributed r.v.'s of the continuous type, with a common cdf* F*, such that, for any fixed* $n \geq 2$, $n \in \mathbb{N}$*, the family of r.v.'s* $(X_1, \ldots, X_n)$ *has the Archimedean copula* C^Ψ *with a generator* Ψ*. Furthermore, assume that:* C^Ψ *is the Clayton copula, i.e., the copula with a generator of the form* $\Psi(t) = \frac{1}{\alpha}(t^{-\alpha} - 1)$ *for some* $\alpha > 0$*, and that a numerical sequence* $\{u_n\}$ *fulfills one of the following conditions:*

$$n(1 - F(u_n)) \sim 1/n^\varepsilon \quad \text{for some } \varepsilon > 0, \tag{7}$$

or

$$n(1 - F(u_n)) \sim n^\varepsilon \quad \text{for some } \varepsilon \in (1 - 1/(1+\alpha); 1), \tag{8}$$

where, here and in subsequent parts of the paper, $a_n \sim b_n$ *stands for the property that* $\lim_{n\to\infty} a_n/b_n = 1$.

Additionally, suppose that the property in (5) *holds true with* Θ *and* G*, such as in* (4) *and* (6)*, respectively, as well as that* $\Lambda(\Theta)$ *is a r.v. satisfying*

$$\lim_{n\to\infty} n\left\{1 - (G(u_n))^\Theta\right\} = \Lambda(\Theta) \quad a.s. \tag{9}$$

Then, we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I(M_n \leq u_n) = E_\Theta\left(e^{-\Lambda(\Theta)}\right) \quad a.s., \tag{10}$$

where, here and throughout the whole paper, $\log x = \ln(\max(x, e))$.

(ii) *Suppose that:* $\{X_i\}$ *is a stochastic process defined earlier,* (9) *holds true, a numerical sequence* $\{u_n\}$ *satisfies*

$$n(1 - F(u_n)) \sim 1/n^\varepsilon \quad \text{for some } \varepsilon > 1, \tag{11}$$

and a r.v. $\Lambda(\Theta)$ *fulfills* (9).

Then, we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(M_n^{(2)} \leq u_n\right) = E_\Theta\left\{e^{-\Lambda(\Theta)}(1 + \Lambda(\Theta))\right\} \quad a.s. \tag{12}$$

We also prove the following ASCLT for ordinary maxima $\{M_n\}$.

Theorem 2.2. *Suppose that* $\{X_i\}$ *is a stochastic process defined earlier, i.e., it is a sequence of identically distributed r.v.'s of the continuous type, with a common cdf* F*, such that, for any fixed* $n \geq 2$, $n \in \mathbb{N}$*, the family of r.v.'s* $(X_1, \ldots, X_n)$ *has the Archimedean copula* C^Ψ *with a generator* Ψ*. Moreover, assume that:* C^Ψ *is the Gumbel copula, i.e., the copula with a generator of the form* $\Psi(t) = (-\ln t)^\alpha$ *for some* $\alpha > 1$*, the condition in* (9) *holds true, and a numerical sequence* $\{u_n\}$ *satisfies*

$$n(1 - F(u_n)) \sim n^\varepsilon \quad \text{for some } \varepsilon \in (1 - 1/\alpha; 1), \tag{13}$$

as well as that: (5) *is valid with* Θ *and* G*, such as in* (4) *and* (6)*, respectively, and a r.v.* $\Lambda(\Theta)$ *fulfils* (9)*. Then,* (10) *holds.*

Theorems 2.1 and 2.2 together with Corollary 5.2 and Example 5.3, stated by Wüthrich [28], straightforwardly imply the following two claims:

Corollary 2.1. *Under the assumptions of Theorem 2.1 on* $\{X_i\}$*, we have*:

(i) *if* $F = \mathit{uniform}(0,1)$ *and* $x_n \sim 1/n^{\varepsilon}$ *for some* $\varepsilon > 0$*, then*:

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(M_n \leq 1 - x_n/n\right) = 1 \quad a.s., \tag{14}$$

(ii) *if* $F = \mathit{uniform}(0,1)$ *and* $x_n \sim n^{\varepsilon}$ *for some* $\varepsilon \in (1 - 1/(1+\alpha); 1)$*, where* $\alpha > 0$ *is the corresponding parameter of the Clayton copula generator* $\Psi(t) = \dfrac{1}{\alpha}\left(t^{-\alpha} - 1\right)$*, then*:

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(M_n \leq 1 - x_n/n\right) = 0 \quad a.s.; \tag{15}$$

Corollary 2.2. *Under the assumptions of Theorem 2.2 on* $\{X_i\}$*, we have that if* $F = \mathit{uniform}(0,1)$ *and* $x_n \sim n^{\varepsilon}$ *for some* $\varepsilon \in (0; 1/\alpha)$*, where* $\alpha > 1$ *is the parameter of the Gumbel copula generator* $\Psi(t) = (-\ln t)^{\alpha}$*, we have*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(M_n \leq 1 - x_n/n^{1/\alpha}\right) = 0 \quad a.s. \tag{16}$$

Remark 2.1. As has already been mentioned above, some explanations concerning the form of assumptions (7), (8), (11) and (13) have been placed at the end of our note.

3. Main results II (the ASCLTs for $\{M_n^{(k)}\}$)

For the general case of order statistics of rank k, we may prove the following assertions.

Theorem 3.1. *Suppose that* $\{X_i\}$ *is a stochastic process defined in Introduction and preliminaries, i.e., it is a sequence of identically distributed r.v.'s of the continuous type, with a common cdf* F*, such that, for any fixed* $n \geq 2$*,* $n \in \mathbb{N}$*, the family of r.v.'s* $(X_1, \ldots, X_n)$ *has the Archimedean copula* C^{Ψ} *with a generator* Ψ*. In addition, assume that*: (5) *holds true with* Θ *and* G*, such as in* (4) *and* (6)*, respectively,* k *is a fixed natural number satisfying the property*

$$E_{\Theta}\Theta^{2(k-1)} < \infty, \tag{17}$$

as well as (9) *is fulfilled for some r.v.* $\Lambda(\Theta)$ *and a numerical sequence* $\{u_n\}$ *obeys the condition*

$$u_n \geq F^{-1}\left(\Psi^{-1}\left(C/n^{\beta}\right)\right) \quad \textit{for some generic constants } C > 0 \textit{ and } \beta > 1. \tag{18}$$

Then,

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left(M_n^{(k)}\le u_n\right)=E_\Theta\left\{e^{-\Lambda(\Theta)}\sum_{s=0}^{k-1}\frac{(\Lambda(\Theta))^s}{s!}\right\}\quad a.s. \tag{19}$$

Corollary 3.1. *If* $\Psi(t)=(-\ln t)^\alpha$ *for some* $\alpha>1$ (C^Ψ *is the Gumbel copula*) *and conditions*: (4), (6), (17) *are satisfied, as well as assumption* (9) *is fulfilled for some r.v.* $\Lambda(\Theta)$ *and the following assumption is imposed on a numerical sequence* $\{u_n\}$

$$\lim_{n\to\infty} n\,(1-F(u_n))=\tau\quad \text{for some } 0\le\tau<\infty, \tag{20}$$

we have

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left(M_n^{(k)}\le u_n\right)=E_\Theta\left\{e^{-\Lambda(\Theta)}\sum_{s=0}^{k-1}\frac{(\Lambda(\Theta))^s}{s!}\right\}\quad a.s.,$$

where a r.v. $\Lambda(\Theta)$ *has an expected value* $E_\Theta\{\Lambda(\Theta)\}=\tau$.

Remark 3.1. It is worthwhile to mention that assumption (17) is satisfied if, e.g., the following condition holds true

$$\left(\Psi^{-1}(v)\right)^{(2(k-1))}\Big|_{v=0}<\infty, \tag{21}$$

where $(\Psi^{-1}(v))^{(j)}$ stands for the j-th derivative of the inverse function $(\Psi^{-1}(v))$.

This fact is justified in Appendix.

4. Auxiliary results necessary for the proofs of the ASCLTs for ordinary maxima and for the second largest maxima

The objective of this section is to state and prove some lemmas, which will be needed in the proofs of Theorems 2.1-2.2. First, we shall prove the following result.

Lemma 4.1. *Under the assumptions of Theorem 2.1*(i) *on* $\{X_i\}$, Ψ, $\{u_n\}$, Θ *and* G, *we have for* $m<n$

$$|Cov\,(I(M_m\le u_m),I(M_n\le u_n))|\ll m/n+1/n^\delta\quad \text{for some } \delta>0, \tag{22}$$

where, here as well as in subsequent relations and derivations, $a(m,n)\ll b(m,n)$ ($a(n)\ll b(n)$) *stands for* $a(m,n)=\mathcal{O}(b(m,n))$ ($a(n)=\mathcal{O}(b(n))$) *as* $m,n\to\infty$ ($n\to\infty$).

Proof of Lemma 4.1. Let, for $m<n$, $M_{m,n}:=\max(X_{m+1},\dots,X_n)$. Observe that:

$$\begin{aligned}
&|Cov\,(I(M_m\le u_m),I(M_n\le u_n))|\\
=\ &|Cov\,(I(M_m\le u_m),I(M_n\le u_n)-I(M_{m,n}\le u_n))|\\
&+|Cov\,(I(M_m\le u_m),I(M_{m,n}\le u_n))|\\
\le\ &2E\,|I(M_n\le u_n)-I(M_{m,n}\le u_n)|
\end{aligned}$$

$$+ \left| Cov \left(I \left(M_m \leq u_m \right), I \left(M_{m,n} \leq u_n \right) \right) \right|,$$

and therefore,

$$\begin{aligned} & \left| Cov \left(I \left(M_m \leq u_m \right), I \left(M_n \leq u_n \right) \right) \right| \\ & \ll E \left| I \left(M_n \leq u_n \right) - I \left(M_{m,n} \leq u_n \right) \right| \\ & \quad + \left| Cov \left(I \left(M_m \leq u_m \right), I \left(M_{m,n} \leq u_n \right) \right) \right| \\ & =: A + B. \end{aligned} \tag{23}$$

First, we shall estimate the component A in (23). We have

$$\begin{aligned} & E \left| I \left(M_n \leq u_n \right) - I \left(M_{m,n} \leq u_n \right) \right| = P \left(M_{m,n} \leq u_n \right) - P \left(M_n \leq u_n \right) \\ & = E_\Theta P \left(M_{m,n} \leq u_n \right| \Theta) - E_\Theta P \left(M_n \leq u_n \right| \Theta), \end{aligned} \tag{24}$$

where Θ is a r.v. satisfying (4)-(5).

Due to the condition in (5) and the fact that $X_1, \ldots, X_n$ are conditionally independent given Θ, we obtain:

$$E_\Theta P \left(M_{m,n} \leq u_n \right| \Theta) = E_\Theta \left\{ \left(G \left(u_n \right) \right)^\Theta \right\}^{n-m}, \tag{25}$$

$$E_\Theta P \left(M_n \leq u_n \right| \Theta) = E_\Theta \left\{ \left(G \left(u_n \right) \right)^\Theta \right\}^{n}. \tag{26}$$

Consequently, it follows from (24)-(26) that

$$A = E_\Theta \left[\left\{ \left(G \left(u_n \right) \right)^\Theta \right\}^{n-m} - \left\{ \left(G \left(u_n \right) \right)^\Theta \right\}^{n} \right]. \tag{27}$$

By (27), the relation $0 \leq \left(G \left(u_n \right) \right)^\Theta \leq 1$ and the property that $z^{n-m} - z^n < m/n$, if $0 \leq z \leq 1$ and $1 \leq m < n$, we immediately get

$$A < m/n. \tag{28}$$

Thus, it remains to estimate the term B in (23). Obviously, we have

$$\begin{aligned} B = & \left| E \left[I \left(M_m \leq u_m \right) I \left(M_{m,n} \leq u_n \right) \right] \right. \\ & \left. - EI \left(M_m \leq u_m \right) EI \left(M_{m,n} \leq u_n \right) \right| \\ = & \left| E_\Theta P \left(M_m \leq u_m, M_{m,n} \leq u_n \right| \Theta) \right. \\ & \left. - E_\Theta P \left(M_m \leq u_m \right| \Theta) E_\Theta P \left(M_{m,n} \leq u_n \right| \Theta) \right|. \end{aligned} \tag{29}$$

In view of (5) and the fact that $X_1, \ldots, X_n$ are conditionally independent given a r.v. Θ, we obtain

$$E_\Theta P \left(M_m \leq u_m, M_{m,n} \leq u_n \right| \Theta) = E_\Theta \left[\left\{ G \left(u_m \right)^\Theta \right\}^m \left\{ G \left(u_n \right)^\Theta \right\}^{n-m} \right]. \tag{30}$$

The relations in (25), (26) and (30) together with (29) imply

$$B = \left| E_\Theta \left[\left\{ G \left(u_m \right)^\Theta \right\}^m \left\{ G \left(u_n \right)^\Theta \right\}^{n-m} \right] \right. \tag{31}$$

$$- E_\Theta \left\{ G\left(u_m\right)^\Theta \right\}^m E_\Theta \left\{ G\left(u_n\right)^\Theta \right\}^{n-m} \Bigg| .$$

Clearly, (31) may be rewritten as follows

$$\begin{aligned} B \le & \left| E_\Theta \left[\left\{ G\left(u_m\right)^\Theta \right\}^m \left(\left\{ G\left(u_n\right)^\Theta \right\}^{n-m} - \left\{ G\left(u_n\right)^\Theta \right\}^n \right) \right] \right| \\ & + \left| E_\Theta \left[\left\{ G\left(u_m\right)^\Theta \right\}^m \left(\left\{ G\left(u_n\right)^\Theta \right\}^n - E_\Theta \left\{ G\left(u_n\right)^\Theta \right\}^n \right) \right] \right| \\ & + \left| E_\Theta \left\{ G\left(u_m\right)^\Theta \right\}^m \left(E_\Theta \left\{ G\left(u_n\right)^\Theta \right\}^n - E_\Theta \left\{ G\left(u_n\right)^\Theta \right\}^{n-m} \right) \right| \\ & =: B_1 + B_2 + B_3. \end{aligned} \tag{32}$$

This and the properties used in the estimation of A immediately yield

$$B_1 + B_3 < 2\left(m/n\right). \tag{33}$$

Thus, we need to find the bound for B_2 in (32).

Assume first that the assumption in (7) holds. Using the facts that both $\left\{ G\left(u_m\right)^\Theta \right\}^m$ and $\left\{ G\left(u_n\right)^\Theta \right\}^n - E_\Theta \left\{ G\left(u_n\right)^\Theta \right\}^n$ are bounded r.v.'s with finite moments of any finite orders, together with the Schwarz inequality and the relation $0 \le E_\Theta \left\{ G\left(u_m\right)^\Theta \right\}^m \le 1$, we obtain

$$\begin{aligned} B_2 & \le \sqrt{E_\Theta \left[\left\{ G\left(u_n\right)^\Theta \right\}^n - E_\Theta \left\{ G\left(u_n\right)^\Theta \right\}^n \right]^2} \\ & = \sqrt{E_\Theta \left\{ G\left(u_n\right)^\Theta \right\}^{2n} - \left[E_\Theta \left\{ G\left(u_n\right)^\Theta \right\}^n \right]^2}. \end{aligned} \tag{34}$$

In addition, it follows from the relations on G in (6) and on Ψ^{-1} in (4) that:

$$\begin{aligned} E_\Theta \left\{ G\left(u_n\right)^\Theta \right\}^{2n} & = E_\Theta \left\{ G\left(u_n\right) \right\}^{\Theta 2n} = E_\Theta \left[\exp\left\{ -\theta 2n \Psi\left(F\left(u_n\right)\right) \right\} \right] \\ & = \Psi^{-1}\left(2n\Psi\left(F\left(u_n\right)\right)\right), \end{aligned} \tag{35}$$

$$\begin{aligned} \left[E_\Theta \left\{ G\left(u_n\right)^\Theta \right\}^n \right]^2 & = \left(E_\Theta \left\{ G\left(u_n\right)^{\Theta n} \right\} \right)^2 = \left[E_\Theta \left\{ \exp\left(-\theta n \Psi\left(F\left(u_n\right)\right)\right) \right\} \right]^2 \\ & = \left[\Psi^{-1}\left(n\Psi\left(F\left(u_n\right)\right)\right) \right]^2. \end{aligned} \tag{36}$$

Hence, by virtue of (34)-(36),

$$\begin{aligned} & \sqrt{E_\Theta \left[\left\{ G\left(u_n\right)^\Theta \right\}^n - E_\Theta \left\{ G\left(u_n\right)^\Theta \right\}^n \right]^2} \\ & = \sqrt{\Psi^{-1}\left(2n\Psi\left(F\left(u_n\right)\right)\right) - \left[\Psi^{-1}\left(n\Psi\left(F\left(u_n\right)\right)\right) \right]^2} \\ & \le \sqrt{\begin{array}{c} \left| \Psi^{-1}\left(2n\Psi\left(F\left(u_n\right)\right)\right) - \Psi^{-1}\left(n\Psi\left(F\left(u_n\right)\right)\right) \right| \Psi^{-1}\left(n\Psi\left(F\left(u_n\right)\right)\right) \\ + \left| \Psi^{-1}\left(n\Psi\left(F\left(u_n\right)\right)\right) - \Psi^{-1}(0) \right| \Psi^{-1}\left(2n\Psi\left(F\left(u_n\right)\right)\right) \end{array}}, \end{aligned} \tag{37}$$

where the last relation follows from the fact that $|z-xy| \leq |z-x|\,|y| + |y-1|\,|z|$ for any real-valued x, y and z.

This and the property that the inverse of $\Psi(t) = \frac{1}{\alpha}(t^{-\alpha}-1)$ (i.e., $\Psi^{-1}(t) = (1+\alpha t)^{-1/\alpha}$) is Lipschitz with the Lipschitz constant 1, as well as the facts that it is a decreasing, positive and bounded by 1 function, imply

$$\sqrt{E_\Theta\left[\left\{G(u_n)^\Theta\right\}^n - E_\Theta\left\{G(u_n)^\Theta\right\}^n\right]^2} \leq \sqrt{2n\Psi(F(u_n))}. \tag{38}$$

Let, here and throughout the whole paper, $\tau_n := n(1-F(u_n))$. Observe that

$$\begin{aligned} n\Psi(F(u_n)) &= n\left[\frac{1}{\alpha}\left((F(u_n))^{-\alpha}-1\right)\right] = n\left[\frac{1}{\alpha}\left(1-\frac{\tau_n}{n}\right)^{-\alpha}\right] \\ &\sim n\left[\frac{1}{\alpha}\alpha\left(\frac{\tau_n}{n}\right)\right] = \tau_n = n(1-F(u_n)). \end{aligned} \tag{39}$$

It stems from (39) and assumption (7) that

$$n\Psi(F(u_n)) \ll 1/n^\varepsilon \quad \text{for some } \varepsilon > 0. \tag{40}$$

Thus, in view of (34), (38) and (40), we get

$$B_2 \ll 1/n^{\varepsilon/2} \quad \text{for some } \varepsilon > 0, \text{ if (7) holds.} \tag{41}$$

Assume now that assumption (8) is satisfied. Due to (34)-(36), we have

$$\begin{aligned} B_2 &\leq \sqrt{E_\Theta\left\{G(u_n)^\Theta\right\}^{2n} - \left[E_\Theta\left\{G(u_n)^\Theta\right\}^n\right]^2} \\ &= \sqrt{\Psi^{-1}(2n\Psi(F(u_n))) - \left[\Psi^{-1}(n\Psi(F(u_n)))\right]^2} \\ &\leq \sqrt{\Psi^{-1}(2n\Psi(F(u_n)))}. \end{aligned} \tag{42}$$

Furthermore, by virtue of (39) and (8), we obtain:

$$2n\Psi(F(u_n)) = 2\tau_n = 2n(1-F(u_n)) \sim 2n^\varepsilon \tag{43}$$

for some $\varepsilon \in (1-1/(1+\alpha);1)$, and

$$\begin{aligned} \Psi(1-F(u_n)) &= \frac{1}{\alpha}\left((1-F(u_n))^{-\alpha}-1\right) = \frac{1}{\alpha}\left((1-F(u_n))^{-\alpha}-1\right) \\ &\leq \frac{1}{\alpha}\frac{1}{(1-F(u_n))^\alpha} = \frac{n^\alpha}{\alpha}\frac{1}{[n(1-F(u_n))]^\alpha} \sim \frac{n^\alpha}{\alpha}\frac{1}{n^{\varepsilon\alpha}} = \frac{1}{\alpha}n^{\alpha(1-\varepsilon)}. \end{aligned} \tag{44}$$

In addition, it follows from (8) that $\varepsilon > 1-1/(1+\alpha)$, and equivalently that: $\varepsilon > \alpha/(1+\alpha)$, $1/(1+\alpha) > 1-\varepsilon$. Consequently, $\varepsilon > \alpha(1-\varepsilon)$. This and the relations in (43)-(44) yield

$$2n\Psi(F(u_n)) > \Psi(1-F(u_n)) \quad \text{for all sufficiently large } n.$$

The inequality above and the fact that Ψ^{-1} is a decreasing function imply

$$\Psi^{-1}(2n\Psi(F(u_n))) \ll \Psi^{-1}(\Psi(1-F(u_n))) = 1-F(u_n). \tag{45}$$

Therefore, by (42), (45) and assumption (8), we have

$$B_2 \ll \sqrt{1 - F(u_n)} \ll \sqrt{\frac{n^\varepsilon}{n}} = 1/n^{(1-\varepsilon)/2} \quad \text{for some } \varepsilon \in (1 - 1/(1+\alpha);1).$$

Since $\varepsilon < 1$, we get $\varepsilon_1 := (1-\varepsilon)/2 > 0$ and hence,

$$B_2 \ll 1/n^{\varepsilon_1} \quad \text{for some } \varepsilon_1 > 0, \text{ if (8) holds.} \tag{46}$$

Thus, due to (32), (33), (41) and (46), we conclude

$$B \ll m/n + 1/n^\delta \quad \text{for some } \delta > 0. \tag{47}$$

Finally, the relations in (23), (28) and (47) imply a desired result in (22). □

The following claim will be employed in the proof of Theorem 2.2.

Lemma 4.2. *Under the assumptions of Theorem 2.2 on* $\{X_i\}$, Ψ, $\{u_n\}$, Θ *and* G, *we have for* $m < n$

$$|Cov(I(M_m \le u_m), I(M_n \le u_n))| \ll m/n + 1/n^\delta \quad \textit{for some } \delta > 0. \tag{48}$$

Proof of Lemma 4.2. The idea of the proof is similar to the idea used in the proof of Lemma 4.1. The only difference concerns the estimation of component B_2. Recall that, by (42), we may bound B_2 as follows

$$B_2 \le \sqrt{\Psi^{-1}(2n\Psi(F(u_n)))}. \tag{49}$$

As previously, we set $\tau_n := n(1 - F(u_n))$. By virtue of assumption (13), we get:

$$\begin{aligned} 2n\Psi(F(u_n)) &= 2n(-\ln F(u_n))^\alpha \sim 2n\left(-\ln\left(1 - \frac{\tau_n}{n}\right)\right)^\alpha \\ &= 2n\frac{1}{n^\alpha}\left(-n\ln\left(1-\frac{\tau_n}{n}\right)\right)^\alpha \sim 2n^{1-\alpha}\left(-\ln\left(1-\frac{\tau_n}{n}\right)^n\right)^\alpha \\ &\sim 2n^{1-\alpha}(-\ln\exp(-\tau_n))^\alpha \sim 2n^{1-\alpha}(\tau_n)^\alpha \sim 2n^{1-\alpha}(n^\varepsilon)^\alpha \\ &= 2n^{1-\alpha(1-\varepsilon)}, \end{aligned} \tag{50}$$

and

$$\begin{aligned} \Psi(1 - F(u_n)) &= \left(-\ln(1-F(u_n))^{-\alpha}\right) = \left(-\ln\frac{\tau_n}{n}\right)^\alpha = \left(\ln\frac{n}{\tau_n}\right)^\alpha \\ &\sim \left(\ln n^{1-\varepsilon}\right)^\alpha = (1-\varepsilon)^\alpha(\ln n)^\alpha. \end{aligned} \tag{51}$$

Since $1 - 1/\alpha < \varepsilon < 1$, we have $\alpha(1-\varepsilon) < 1$, and consequently, $1 - \alpha(1-\varepsilon) > 0$. This and the relations in (50)-(51) yield

$$2n\Psi(F(u_n)) > \Psi(1 - F(u_n)) \quad \text{for all sufficiently large } n.$$

Therefore, as Ψ^{-1} is decreasing, we get

$$\Psi^{-1}(2n\Psi(F(u_n))) \ll \Psi^{-1}(\Psi(1-F(u_n))) = 1 - F(u_n). \tag{52}$$

The relations in (49) and (52) together with assumption (13) imply

$$B_2 \le \sqrt{\Psi^{-1}(2n\Psi(F(u_n)))} \ll \sqrt{\Psi^{-1}(\Psi(1-F(u_n)))} = \sqrt{1-F(u_n)}$$

$$\ll \sqrt{\frac{n^{\varepsilon}}{n}} = 1/n^{(1-\varepsilon)/2} \quad \text{for some } \varepsilon \in (0;1)\,. \tag{53}$$

Putting $\varepsilon_2 := (1-\varepsilon)/2$, we obtain, due to (53),

$$B_2 \leq 1/n^{\varepsilon_2} \quad \text{for some } \varepsilon_2 \in (0;1)\,. \tag{54}$$

As relation (54) - for B_2 - holds true and the other needed estimations - of components A and B_1 (see the notations in the proof of Lemma 4.1) - are identical as in the proof of Lemma 4.1, we obtain that the relation in (48), i.e., the result we wish to establish, is satisfied. □

The statement below will be needed in order to prove both Theorem 2.1 and Theorem 2.2.

Lemma 4.3. *Under the assumptions of Theorem 2.1*(i) *or Theorem 2.2, on* $\{X_i\}$*,* Ψ*,* $\{u_n\}$*,* Θ*,* G *and* $\Lambda(\Theta)$*, we obtain*

$$\lim_{n\to\infty} P(M_n \leq u_n) = E_\Theta\left(e^{-\Lambda(\Theta)}\right). \tag{55}$$

Proof. Following derivation (26) from the proof of Lemma 4.1 and the fact that $P(M_n \leq u_n) = E_\Theta P(M_n \leq u_n|\,\Theta)$, we immediately get

$$\lim_{n\to\infty} P(M_n \leq u_n) = \lim_{n\to\infty} E_\Theta \left\{(G(u_n))^\Theta\right\}^n. \tag{56}$$

In addition, since $0 \leq \left\{(G(u_n))^\Theta\right\}^n \leq 1$, it follows from the Lebesgue theorem on passing to the limit under the integral sign that

$$\lim_{n\to\infty} E_\Theta \left\{(G(u_n))^\Theta\right\}^n = E_\Theta\left(\lim_{n\to\infty}\left\{(G(u_n))^\Theta\right\}^n\right). \tag{57}$$

Furthermore, a Poisson approximation to the binomial distribution with $np_n = nP(X_1 > u_n|\,\Theta) = n\left\{1-(G(u_n))^\Theta\right\}$ and the property that, due to assumption (9), $n\left\{1-(G(u_n))^\Theta\right\} \stackrel{a.s.}{\to} \Lambda(\Theta)$ as $n\to\infty$, imply

$$\lim_{n\to\infty}\left\{(G(u_n))^\Theta\right\}^n = e^{-\Lambda(\Theta)} \quad \text{a.s.} \tag{58}$$

In view of (56)-(58), we have

$$\lim_{n\to\infty} P(M_n \leq u_n) = E_\Theta\left(e^{-\Lambda(\Theta)}\right),$$

which is a desired claim in (55). □

The following lemma will be applied in the proof of Theorem 2.1(ii).

Lemma 4.4. *Under the assumptions of Theorem 2.1*(ii) *on* $\{X_i\}$*,* Ψ*,* $\{u_n\}$*,* Θ *and* G*, we have for* $m < n$

$$E\left|I\left(M_n^{(2)} \leq u_n\right) - I\left(M_{m,n}^{(2)} \leq u_n\right)\right| \ll m/n, \tag{59}$$

where $M_{m,n}^{(2)}$ *denotes the second largest maximum among* $X_{m+1},\ldots,X_n$*.*

Proof. Obviously, we have

$$\begin{aligned} & E\left|I\left(M_n^{(2)} \leq u_n\right) - I\left(M_{m,n}^{(2)} \leq u_n\right)\right| \\ & = P\left(M_{m,n}^{(2)} \leq u_n\right) - P\left(M_n^{(2)} \leq u_n\right) \\ & = E_\Theta P\left(M_{m,n}^{(2)} \leq u_n \middle| \Theta\right) - E_\Theta P\left(M_n^{(2)} \leq u_n \middle| \Theta\right), \end{aligned} \tag{60}$$

where Θ is a r.v. satisfying (4)-(5).

By the definition of the second largest maxima, as well as the condition in (5) and the fact that $X_1, \ldots, X_n$ are conditionally independent given Θ, we obtain:

$$\begin{aligned} & E_\Theta P\left(M_{m,n}^{(2)} \leq u_n \middle| \Theta\right) \\ & = \sum_{s=0}^{1} \binom{n-m}{s} E_\Theta\left[\left\{1 - \left(G\left(u_n\right)\right)^\Theta\right\}^s \left\{\left(G\left(u_n\right)\right)^\Theta\right\}^{n-m-s}\right], \end{aligned} \tag{61}$$

$$\begin{aligned} & E_\Theta P\left(M_n^{(2)} \leq u_n \middle| \Theta\right) \\ & = \sum_{s=0}^{1} \binom{n}{s} E_\Theta\left[\left\{1 - \left(G\left(u_n\right)\right)^\Theta\right\}^s \left\{\left(G\left(u_n\right)\right)^\Theta\right\}^{n-s}\right]. \end{aligned} \tag{62}$$

It follows from (61)-(62) that

$$\begin{aligned} C &:= E_\Theta P\left(M_{m,n}^{(2)} \leq u_n \middle| \Theta\right) - E_\Theta P\left(M_n^{(2)} \leq u_n \middle| \Theta\right) \\ &\leq \sum_{s=0}^{1} \binom{n-m}{s} E_\Theta\left[\left\{1 - \left(G\left(u_n\right)\right)^\Theta\right\}^s \left(\left\{\left(G\left(u_n\right)\right)^\Theta\right\}^{n-m-s} - \left\{\left(G\left(u_n\right)\right)^\Theta\right\}^{n-s}\right)\right] \\ &= E_\Theta\left[\left\{\left(G\left(u_n\right)\right)^\Theta\right\}^{n-m} - \left\{\left(G\left(u_n\right)\right)^\Theta\right\}^{n}\right] \\ &\quad + (n-m-1) E_\Theta\left[\left\{1 - \left(G\left(u_n\right)\right)^\Theta\right\} \left(\left\{\left(G\left(u_n\right)\right)^\Theta\right\}^{n-m-1} - \left\{\left(G\left(u_n\right)\right)^\Theta\right\}^{n-1}\right)\right] \\ &=: C_1 + C_2. \end{aligned} \tag{63}$$

Our purpose now is to estimate the terms C_1-C_2 in (63). From the facts that: $z^{n-m} - z^n < m/n$, if $0 \leq z \leq 1$ and $1 \leq m < n$, and $0 \leq G\left(u_n\right)^\Theta \leq 1$, we immediately conclude

$$C_1 < m/n. \tag{64}$$

In order to find the bound for C_2 in (63), recall that, by the conditions of Theorem 2.1(ii), the family of r.v.'s $(X_1, \ldots, X_n)$ has the Clayton copula (i.e., the copula with a generator of the form $\Psi(t) = \frac{1}{\alpha}\left(t^{-\alpha} - 1\right)$ for some $\alpha > 0$) and (11) is satisfied.

Observe that, since: $z^{n-m-1} - z^{n-1} = z^{n-1-m} - z^{n-1} < m/(n-1)$, provided $0 \leq z \leq 1$ and $1 \leq m < n-1$, and $0 \leq G(u_n)^{\Theta} \leq 1$, we have

$$C_2 < nE_{\Theta}\left\{1 - G(u_n)^{\Theta}\right\} \frac{m}{n-1} = nE_{\Theta}\left\{1 - G(u_n)^{\Theta}\right\}\left(\frac{m-1}{n-1} + \frac{1}{n-1}\right)$$
$$\ll nE_{\Theta}\left\{1 - G(u_n)^{\Theta}\right\}\left(\frac{m}{n} + \frac{1}{n}\right) \ll nE_{\Theta}\left\{1 - G(u_n)^{\Theta}\right\}\frac{m}{n}. \tag{65}$$

Moreover, by the relations on Ψ^{-1} and G in (4) and (5)-(6), respectively, we get

$$E_{\Theta}\left\{1 - G(u_n)^{\Theta}\right\} = 1 - E_{\Theta}\left\{G(u_n)^{\Theta}\right\} = 1 - E_{\Theta}\left[\exp\left\{-\Theta\Psi(F(u_n))\right\}\right]$$
$$= 1 - \Psi^{-1}(\Psi(F(u_n))) = 1 - F(u_n). \tag{66}$$

Derivation (66) and assumption (11) yield

$$nE_{\Theta}\left\{1 - G(u_n)^{\Theta}\right\} = n(1 - F(u_n)) \ll 1/n^{\varepsilon} \quad \text{for some } \varepsilon > 1. \tag{67}$$

Therefore, due to (65) and (67), we may write that

$$C_2 \ll m/n. \tag{68}$$

By virtue of (60), (63), (64) and (68), we obtain

$$E\left|I\left(M_n^{(2)} \leq u_n\right) - I\left(M_{m,n}^{(2)} \leq u_n\right)\right|$$
$$= E_{\Theta}P\left(M_{m,n}^{(2)} \leq u_n \middle| \Theta\right) - E_{\Theta}P\left(M_n^{(2)} \leq u_n \middle| \Theta\right)$$
$$= C \leq C_1 + C_2 \ll m/n, \tag{69}$$

which is a desired claim in (59). □

The following lemma will also be employed in the proof of Theorem 2.1(ii).

Lemma 4.5. *Under the assumptions of Theorem 2.1*(ii) *on* $\{X_i\}$, Ψ, $\{u_n\}$, Θ *and* G, *we have for* $m < n$

$$\left|Cov\left(I\left(M_m^{(2)} \leq u_m\right), I\left(M_{m,n}^{(2)} \leq u_n\right)\right)\right| \ll m/n + 1/n^{\delta} \quad \textit{for some } \delta > 0. \tag{70}$$

Proof. It is clear that

$$\left|Cov\left(I\left(M_m^{(2)} \leq u_m\right), I\left(M_{m,n}^{(2)} \leq u_n\right)\right)\right|$$
$$= \left|P\left(M_m^{(2)} \leq u_m, M_{m,n}^{(2)} \leq u_n\right) - P\left(M_m^{(2)} \leq u_m\right)P\left(M_{m,n}^{(2)} \leq u_n\right)\right|$$
$$= \left|E_{\Theta}P\left(M_m^{(2)} \leq u_m, M_{m,n}^{(2)} \leq u_n \middle| \Theta\right)\right.$$
$$\left. -E_{\Theta}P\left(M_m^{(2)} \leq u_m \middle| \Theta\right)E_{\Theta}P\left(M_{m,n}^{(2)} \leq u_n \middle| \Theta\right)\right|, \tag{71}$$

where Θ is a r.v. satisfying (4)-(5).

By the definition of the second largest maxima, as well as the condition in (5) and the fact that $X_1, \ldots, X_n$ are conditionally independent given Θ, we have, in view of (71),

$$\begin{aligned} &\left|Cov\left(I\left(M_m^{(2)} \le u_m\right), I\left(M_{m,n}^{(2)} \le u_n\right)\right)\right| \\ &= \sum_{s_1=0}^{1}\sum_{s_2=0}^{1}\binom{m}{s_1}\binom{n-m}{s_2} D\left(s_1, s_2, m, n\right), \end{aligned} \tag{72}$$

where

$$\begin{aligned} D\left(s_1, s_2, m, n\right) := \Big| & E_\Theta\left[\left\{1-G\left(u_m\right)^\Theta\right\}^{s_1}\left\{G\left(u_m\right)^\Theta\right\}^{m-s_1}\left\{1-G\left(u_n\right)^\Theta\right\}^{s_2}\left\{G\left(u_n\right)^\Theta\right\}^{n-m-s_2}\right] \\ & -E_\Theta\left[\left\{1-G\left(u_m\right)^\Theta\right\}^{s_1}\left\{G\left(u_m\right)^\Theta\right\}^{m-s_1}\right] E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}^{s_2}\left\{G\left(u_n\right)^\Theta\right\}^{n-m-s_2}\right]\Big|. \end{aligned}$$

Therefore, we may write that

$$H := \left|Cov\left(I\left(M_m^{(2)} \le u_m\right), I\left(M_{m,n}^{(2)} \le u_n\right)\right)\right| \le H_1 + H_2 + H_3 + H_4, \tag{73}$$

where:

$$\begin{aligned} H_1 := \Big| & E_\Theta\left[\left\{G\left(u_m\right)^\Theta\right\}^{m}\left\{G\left(u_n\right)^\Theta\right\}^{n-m}\right] \\ & -E_\Theta\left\{G\left(u_m\right)^\Theta\right\}^{m} E_\Theta\left\{G\left(u_n\right)^\Theta\right\}^{n-m}\Big|, \end{aligned}$$

$$\begin{aligned} H_2 := m\Big| & E_\Theta\left[\left\{1-G\left(u_m\right)^\Theta\right\}\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\left\{G\left(u_n\right)^\Theta\right\}^{n-m}\right] \\ & -E_\Theta\left[\left\{1-G\left(u_m\right)^\Theta\right\}\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\right] E_\Theta\left\{G\left(u_n\right)^\Theta\right\}^{n-m}\Big|, \end{aligned}$$

$$\begin{aligned} H_3 := (n-m)\Big| & E_\Theta\left[\left\{G\left(u_m\right)^\Theta\right\}^{m}\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^{n-m-1}\right] \\ & -E_\Theta\left\{G\left(u_m\right)^\Theta\right\}^{m} E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^{n-m-1}\right]\Big|, \end{aligned}$$

$$\begin{aligned} H_4 := m\left(n-m\right)\Big| & E_\Theta\left[\left\{1-G\left(u_m\right)^\Theta\right\}\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^{n-m-1}\right] \\ & -E_\Theta\left[\left\{1-G\left(u_m\right)^\Theta\right\}\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\right] E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^{n-m-1}\right]\Big|. \end{aligned}$$

Thus, we need to give the bounds for H_1–H_4. Recall that, due to our assumptions, the family of r.v.'s $(X_1, \ldots, X_n)$ has the Clayton copula and condition (11) is satisfied. Reasoning as in the estimation of the term

$$B = \left|E_\Theta\left[\left\{G\left(u_m\right)^\Theta\right\}^{m}\left\{G\left(u_n\right)^\Theta\right\}^{n-m}\right] - E_\Theta\left\{G\left(u_m\right)^\Theta\right\}^{m} E_\Theta\left\{G\left(u_n\right)^\Theta\right\}^{n-m}\right|$$

in the proof of Lemma 4.1, for the case when assumption (7) - containing the constraint in (11) - is fulfilled (see the notations and relations in (31)-(47)), we immediately obtain

$$H_1 = B \le B_1 + B_2 + B_3 \ll m/n + 1/n^{\delta_1} \quad \text{for some } \delta_1 > 0, \tag{74}$$

where B_1-B_3 are defined in the same manner as in the mentioned proof of Lemma 4.1.

Our aim now is to find the estimate for H_2 in (73). We have

$$\begin{aligned} H_2 \le m \Bigg| & E_\Theta \left[\left\{1 - G\left(u_m\right)^\Theta\right\}\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\left\{G\left(u_n\right)^\Theta\right\}^{n-m}\right] \\ & - E_\Theta \left[\left\{1 - G\left(u_m\right)^\Theta\right\}\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\left\{G\left(u_n\right)^\Theta\right\}^{n}\right]\Bigg| \\ & + m\Bigg|E_\Theta \left[\left\{1 - G\left(u_m\right)^\Theta\right\}\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\left\{G\left(u_n\right)^\Theta\right\}^{n}\right] \\ & - E_\Theta \left[\left\{1 - G\left(u_m\right)^\Theta\right\}\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\right] E_\Theta \left\{G\left(u_n\right)^\Theta\right\}^{n}\Bigg| \\ & + m\Bigg|E_\Theta \left[\left\{1 - G\left(u_m\right)^\Theta\right\}\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\right] E_\Theta \left\{G\left(u_n\right)^\Theta\right\}^{n} \\ & - E_\Theta \left[\left\{1 - G\left(u_m\right)^\Theta\right\}\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\right] E_\Theta \left\{G\left(u_n\right)^\Theta\right\}^{n-m}\Bigg| \end{aligned}$$

$$=: H_{21} + H_{22} + H_{23}. \tag{75}$$

As $0 \le \left\{G\left(u_m\right)^\Theta\right\}^{m-1} \le 1$, it is clear that

$$\begin{aligned} H_{21} + H_{23} \le\ & m E_\Theta \left[\left\{1 - G\left(u_m\right)^\Theta\right\}\left(\left\{G\left(u_n\right)^\Theta\right\}^{n-m} - \left\{G\left(u_n\right)^\Theta\right\}^{n}\right)\right] \\ & + m E_\Theta \left\{1 - G\left(u_m\right)^\Theta\right\} E_\Theta \left(\left\{G\left(u_n\right)^\Theta\right\}^{n-m} - \left\{G\left(u_n\right)^\Theta\right\}^{n}\right) \end{aligned}$$

$$< 2m E_\Theta \left\{1 - G\left(u_m\right)^\Theta\right\} \frac{m}{n}, \tag{76}$$

where the last relation follows from the property that $z^{n-m} - z^n < m/n$, if $0 \le z \le 1$ and $1 \le m < n$.

Following the derivation in (66) and using assumption (11), we straightforwardly get

$$E_\Theta \left\{1 - G\left(u_m\right)^\Theta\right\} = 1 - F\left(u_m\right) \sim 1/m^{1+\varepsilon} \quad \text{for some } \varepsilon > 1. \tag{77}$$

The relations in (76)-(77) imply

$$H_{21} + H_{23} \ll m \frac{1}{m^{1+\varepsilon}} \frac{m}{n} \le \frac{m}{n}. \tag{78}$$

Thus, in order to complete the estimation of H_2, we only need to give the bound for H_{22} in (75). Since $0 \le \left\{G(u_m)^\Theta\right\}^{m-1} \le 1$, we may write that

$$H_{22} \le m E_\Theta \left[\left\{1 - G(u_m)^\Theta\right\} \left|\left\{G(u_n)^\Theta\right\}^n - E_\Theta \left\{G(u_n)^\Theta\right\}^n\right|\right].$$

Since in addition, the r.v.'s $1 - G(u_m)^\Theta$ and $\left|\left\{G(u_n)^\Theta\right\}^n - E_\Theta \left\{G(u_n)^\Theta\right\}^n\right|$ are bounded and have finite moments of any finite orders, we obtain, in view of the Schwarz inequality,

$$H_{22} \le m \sqrt{E_\Theta \left\{1 - G(u_m)^\Theta\right\}^2} \sqrt{E_\Theta \left[\left\{G(u_n)^\Theta\right\}^n - E_\Theta \left\{G(u_n)^\Theta\right\}^n\right]^2}. \tag{79}$$

Since $0 \le 1 - G(u_m)^\Theta \le 1$ and (77) holds, we have

$$\begin{aligned} \sqrt{E_\Theta \left\{1 - G(u_m)^\Theta\right\}^2} &\le \sqrt{E_\Theta \left\{1 - G(u_m)^\Theta\right\}} \\ &= \sqrt{1 - F(u_m)} \\ &\ll 1/m^{(1+\varepsilon)/2} \quad \text{for some } \varepsilon > 1. \end{aligned} \tag{80}$$

Furthermore, it follows from (38)-(39) and condition (11) that

$$\begin{aligned} \sqrt{E_\Theta \left[\left\{G(u_n)^\Theta\right\}^n - E_\Theta \left\{G(u_n)^\Theta\right\}^n\right]^2} &\le \sqrt{2n\Psi(F(u_n))} \\ &\ll \sqrt{n(1 - F(u_n))} \\ &\ll 1/n^{\varepsilon/2} \quad \text{for some } \varepsilon > 1. \end{aligned} \tag{81}$$

By virtue of (79)-(81) and the fact that $\varepsilon > 1$, we obtain

$$H_{22} \ll m \frac{1}{m^{(1+\varepsilon)/2}} \frac{1}{n^{\varepsilon/2}} = \frac{m}{m^{(1+\varepsilon)/2}} \frac{1}{n^{\varepsilon/2}} \le \frac{1}{n^{\varepsilon/2}} \quad \text{for some } \varepsilon > 1,$$

which, by putting $\delta_2 := \varepsilon/2$, yields

$$H_{22} \ll 1/n^{\delta_2} \quad \text{for some } \delta_2 > 0. \tag{82}$$

Thus, due to (75), (78) and (82), we have

$$H_2 \ll m/n + 1/n^{\delta_2} \quad \text{for some } \delta_2 > 0. \tag{83}$$

Our purpose now is to give the bound for H_3 in (73). We may write that

$$\begin{aligned} H_3 \le n \Big| & E_\Theta \left[\left\{G(u_m)^\Theta\right\}^m \left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n-m-1}\right] \\ & - E_\Theta \left[\left\{G(u_m)^\Theta\right\}^m \left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^n\right]\Big| \\ & + n \Big| E_\Theta \left[\left\{G(u_m)^\Theta\right\}^m \left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^n\right] \\ & - E_\Theta \left\{G(u_m)^\Theta\right\}^m E_\Theta \left[\left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^n\right]\Big| \end{aligned}$$

$$
\begin{aligned}
&+n\left|E_\Theta\left\{G\left(u_m\right)^\Theta\right\}^m E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^n\right]\right.\\
&\left.-E_\Theta\left\{G\left(u_m\right)^\Theta\right\}^m E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^{n-m-1}\right]\right|\\
&=: H_{31}+H_{32}+H_{33}. \qquad (84)
\end{aligned}
$$

By the property $z^{n-m-1}-z^n=z^{n-(m+1)}-z^n<(m+1)/n$, if $0\le z\le 1$ and $1\le m+1<n$, and the facts that: $0\le G\left(u_n\right)^\Theta\le 1$, $0\le\left\{G\left(u_m\right)^\Theta\right\}^m\le 1$, we have

$$
\begin{aligned}
H_{31}+H_{33}&\le 2nE_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}\left(\left\{G\left(u_n\right)^\Theta\right\}^{n-m-1}-\left\{G\left(u_n\right)^\Theta\right\}^n\right)\right]\\
&<2nE_\Theta\left\{1-G\left(u_n\right)^\Theta\right\}\left(\frac{m+1}{n}\right)\\
&=2nE_\Theta\left\{1-G\left(u_n\right)^\Theta\right\}\left(\frac{m}{n}+\frac{1}{n}\right).
\end{aligned}
$$

This and the fact that

$$
E_\Theta\left\{1-G\left(u_n\right)^\Theta\right\}=1-F\left(u_n\right)\sim 1/n^{1+\varepsilon}\quad\text{for some }\varepsilon>1, \tag{85}
$$

where the relations in (85) follow from (66) and assumption (11), imply

$$
H_{31}+H_{33}<2n\left(1-F\left(u_n\right)\right)\left(\frac{m}{n}+\frac{1}{n}\right)\ll\frac{1}{n^\varepsilon}\frac{m}{n}<\frac{m}{n}. \tag{86}
$$

Thus, in order to complete the estimation of H_3, it remains to give the bound for the term H_{32} in (84). It is clear that

$$
\begin{aligned}
H_{32}\le nE_\Theta&\left[\left\{G\left(u_m\right)^\Theta\right\}^m\right.\\
&\left.\times\left|\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^n-E_\Theta\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^n\right|\right].
\end{aligned}
$$

Therefore, in view of the Schwarz inequality,

$$
H_{32}\le n\sqrt{E_\Theta\left\{G\left(u_m\right)^\Theta\right\}^{2m}}\sqrt{D^2_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^n\right]},
$$

where $D^2_\Theta[\cdot]$ stands for the variance of the corresponding r.v.

Since: $0\le\left\{G\left(u_m\right)^\Theta\right\}^{2m}\le 1$, $0\le\left\{G\left(u_n\right)^\Theta\right\}^n\le 1$, and

$$
D^2_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^n\right]\le E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^n\right]^2,
$$

we obtain

$$
H_{32}\le n\sqrt{E_\Theta\left\{1-G\left(u_n\right)^\Theta\right\}^2}\le n\sqrt{E_\Theta\left\{1-G\left(u_n\right)^\Theta\right\}}=n\sqrt{\left(1-F\left(u_n\right)\right)},
$$

which, due to (11), implies that

$$H_{32} \ll n \frac{1}{n^{(1+\varepsilon)/2}} = \frac{1}{n^{(\varepsilon-1)/2}} \quad \text{for some } \varepsilon > 1.$$

Consequently, putting $\delta_3 := (\varepsilon - 1)/2$, we get

$$H_{32} \ll 1/n^{\delta_3} \quad \text{for some } \delta_3 > 0. \tag{87}$$

By virtue of (84), (86) and (87), we conclude

$$H_3 \ll m/n + 1/n^{\delta_3} \quad \text{for some } \delta_3 > 0. \tag{88}$$

Thus, it remains to estimate the component H_4 in (73). We may write that

$$\begin{aligned} H_4 \le mn \Big| & E_\Theta \left[\left\{1 - G(u_m)^\Theta\right\} \left\{G(u_m)^\Theta\right\}^{m-1} \left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n-m-1} \right] \\ & - E_\Theta \left\{1 - G(u_m)^\Theta\right\} E_\Theta \left[\left\{G(u_m)^\Theta\right\}^{m-1} \left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n-m-1} \right] \Big| \\ & + mn \Big| E_\Theta \left\{1 - G(u_m)^\Theta\right\} E_\Theta \left[\left\{G(u_m)^\Theta\right\}^{m-1} \left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n-m-1} \right] \\ & - E_\Theta \left\{1 - G(u_m)^\Theta\right\} E_\Theta \left[\left\{G(u_m)^\Theta\right\}^{m-1} \left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n} \right] \Big| \\ & + mn \Big| E_\Theta \left\{1 - G(u_m)^\Theta\right\} E_\Theta \left[\left\{G(u_m)^\Theta\right\}^{m-1} \left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n} \right] \\ & - E_\Theta \left\{1 - G(u_m)^\Theta\right\} E_\Theta \left\{G(u_m)^\Theta\right\}^{m-1} E_\Theta \left[\left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n} \right] \Big| \\ & + mn \Big| E_\Theta \left\{1 - G(u_m)^\Theta\right\} E_\Theta \left\{G(u_m)^\Theta\right\}^{m-1} E_\Theta \left[\left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n} \right] \\ & - E_\Theta \left[\left\{1 - G(u_m)^\Theta\right\} \left\{G(u_m)^\Theta\right\}^{m-1} \right] E_\Theta \left[\left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n} \right] \Big| \\ & + mn \Big| E_\Theta \left[\left\{1 - G(u_m)^\Theta\right\} \left\{G(u_m)^\Theta\right\}^{m-1} \right] E_\Theta \left[\left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n} \right] \\ & - E_\Theta \left[\left\{1 - G(u_m)^\Theta\right\} \left\{G(u_m)^\Theta\right\}^{m-1} \right] E_\Theta \left[\left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n-m-1} \right] \Big| \end{aligned}$$

$$=: H_{41} + H_{42} + H_{43} + H_{44} + H_{45}. \tag{89}$$

It is obvious that

$$\begin{aligned} H_{41} = mn \Big| E_\Theta \Big[\left\{1 - G(u_m)^\Theta\right\} \Big(& \left\{G(u_m)^\Theta\right\}^{m-1} \left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n-m-1} \\ & - E_\Theta \left\{G(u_m)^\Theta\right\}^{m-1} \left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n-m-1} \Big) \Big] \Big|. \end{aligned}$$

Therefore, in view of the Schwarz inequality, we get

$$H_{41} \le mn \sqrt{E_\Theta \left\{1 - G(u_m)^\Theta\right\}^2} \sqrt{D^2_\Theta \left[\left\{G(u_m)^\Theta\right\}^{m-1} \left\{1 - G(u_n)^\Theta\right\} \left\{G(u_n)^\Theta\right\}^{n-m-1} \right]}.$$

Hence, we may write as follows

$$H_{41} \leq mn\sqrt{E_\Theta\left\{1-G\left(u_m\right)^\Theta\right\}^2}\sqrt{E_\Theta\left[\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^{n-m-1}\right]^2}.$$

Since $G\left(u_m\right)^\Theta \leq G\left(u_n\right)^\Theta$ (as $m<n$, $\Theta>0$ and $\left(u_n\right)$, G are nondecreasing), we have $\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\left\{G\left(u_n\right)^\Theta\right\}^{n-m-1} \leq \left\{G\left(u_n\right)^\Theta\right\}^{m-1}\left\{G\left(u_n\right)^\Theta\right\}^{n-m-1} = \left\{G\left(u_n\right)^\Theta\right\}^{n-2}$. This, the fact that $0 \leq \left\{G\left(u_m\right)^\Theta\right\}^{m-1} \leq 1$ and the last relation for H_{41} imply

$$H_{41} \leq mn\sqrt{E_\Theta\left\{1-G\left(u_m\right)^\Theta\right\}^2}\sqrt{E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}^2\left\{G\left(u_n\right)^\Theta\right\}^{2(n-2)}\right]}.$$

Thus, we obtain

$$H_{41} \ll mn\sqrt{E_\Theta\left\{1-G\left(u_m\right)^\Theta\right\}}\sqrt{E_\Theta\left\{1-G\left(u_n\right)^\Theta\right\}}.$$

This, the relation in (80) and the fact that $\varepsilon>1$ yield

$$H_{41} \ll mn\frac{1}{m^{(1+\varepsilon)/2}}\frac{1}{n^{(1+\varepsilon)/2}} = \frac{m}{m^{(1+\varepsilon)/2}}\frac{n}{n^{(1+\varepsilon)/2}} \leq \frac{n}{n^{(1+\varepsilon)/2}} = \frac{1}{n^{(\varepsilon-1)/2}}$$

for some $\varepsilon>1$.

Consequently, putting $\delta_3 := (\varepsilon-1)/2$ again, we conclude

$$H_{41} \ll 1/n^{\delta_3} \quad \text{for some } \delta_3>0. \tag{90}$$

In order to give the bound for the component H_{42} in (89), observe that, as $0 \leq \left\{G\left(u_m\right)^\Theta\right\}^{m-1} \leq 1$, we get

$$H_{42} \leq mnE_\Theta\left\{1-G\left(u_m\right)^\Theta\right\}E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}\left(\left\{G\left(u_n\right)^\Theta\right\}^{n-m-1}-\left\{G\left(u_n\right)^\Theta\right\}^n\right)\right].$$

This, the relations in (77) and (85), as well as the facts that: $z^{n-m-1}-z^n = z^{n-(m+1)}-z^n < (m+1)/n$, if $0 \leq z \leq 1$ and $1 \leq m+1 < n$, and $0 \leq G\left(u_n\right)^\Theta \leq 1$, yield

$$H_{42} \ll mn\frac{1}{m^{1+\varepsilon}}\frac{1}{n^{1+\varepsilon}}\frac{m+1}{n} \ll \frac{m}{n}. \tag{91}$$

Our aim now is to find the estimate for the component H_{43} in (89). We have

$$\begin{aligned} H_{43} \leq{} & mnE_\Theta\left\{1-G\left(u_m\right)^\Theta\right\} \\ & \times E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}\left\{G\left(u_n\right)^\Theta\right\}^n\left|\left\{G\left(u_m\right)^\Theta\right\}^{m-1}-E_\Theta\left\{G\left(u_m\right)^\Theta\right\}^{m-1}\right|\right]. \end{aligned}$$

By (77) and the property that $\left|\left\{G\left(u_m\right)^{\Theta}\right\}^{m-1}-E_{\Theta}\left\{G\left(u_m\right)^{\Theta}\right\}^{m-1}\right| \leq 2$, we obtain

$$\begin{aligned} H_{43} &\ll mn\frac{1}{m^{1+\varepsilon}}E_{\Theta}\left[\left\{1-G\left(u_n\right)^{\Theta}\right\}\left\{G\left(u_n\right)^{\Theta}\right\}^{n}\right] \\ &\leq nE_{\Theta}\left[\left\{1-G\left(u_n\right)^{\Theta}\right\}\left\{G\left(u_n\right)^{\Theta}\right\}^{n}\right]. \end{aligned}$$

Furthermore, as $0 \leq \left\{G\left(u_n\right)^{\Theta}\right\}^{n} \leq 1$ and (85) holds, we conclude

$$H_{43} \ll nE_{\Theta}\left\{1-G\left(u_n\right)^{\Theta}\right\} \ll n\frac{1}{n^{1+\varepsilon}} = \frac{1}{n^{\varepsilon}} \quad \text{for some } \varepsilon > 1. \tag{92}$$

We now wish to estimate the term H_{44} in (89). It is clear that

$$\begin{aligned} H_{44} \leq\; & mnE_{\Theta}\left[\left\{1-G\left(u_m\right)^{\Theta}\right\}\left|\left\{G\left(u_m\right)^{\Theta}\right\}^{m-1}-E_{\Theta}\left\{G\left(u_m\right)^{\Theta}\right\}^{m-1}\right|\right] \\ & \times E_{\Theta}\left[\left\{1-G\left(u_n\right)^{\Theta}\right\}\left\{G\left(u_n\right)^{\Theta}\right\}^{n}\right]. \end{aligned}$$

Since, as has already been mentioned, $\left|\left\{G\left(u_m\right)^{\Theta}\right\}^{m-1}-E_{\Theta}\left\{G\left(u_m\right)^{\Theta}\right\}^{m-1}\right|$ is bounded above by 2, we obtain

$$H_{44} \ll mnE_{\Theta}\left\{1-G\left(u_m\right)^{\Theta}\right\}E_{\Theta}\left[\left\{1-G\left(u_n\right)^{\Theta}\right\}\left\{G\left(u_n\right)^{\Theta}\right\}^{n}\right].$$

By the Schwarz inequality and the fact that $0 \leq \left\{G\left(u_n\right)^{\Theta}\right\}^{2n} \leq 1$, we get

$$\begin{aligned} H_{44} &\ll mnE_{\Theta}\left\{1-G\left(u_m\right)^{\Theta}\right\}\sqrt{E_{\Theta}\left\{1-G\left(u_n\right)^{\Theta}\right\}^{2}}\sqrt{E_{\Theta}\left\{G\left(u_n\right)^{\Theta}\right\}^{2n}} \\ &\leq mnE_{\Theta}\left\{1-G\left(u_m\right)^{\Theta}\right\}\sqrt{E_{\Theta}\left\{1-G\left(u_n\right)^{\Theta}\right\}^{2}}. \end{aligned}$$

This, the fact that $E_{\Theta}\left\{1-G\left(u_n\right)^{\Theta}\right\}^{2} \leq E_{\Theta}\left\{1-G\left(u_n\right)^{\Theta}\right\}$ together with the relations in (77) and (85) yield

$$H_{44} \ll mn\frac{1}{m^{1+\varepsilon}}\frac{1}{n^{(1+\varepsilon)/2}} \leq \frac{1}{n^{(\varepsilon-1)/2}} \quad \text{for some } \varepsilon > 1.$$

Hence, setting, as previously, $\delta_3 := (\varepsilon-1)/2$, we obtain

$$H_{44} \ll 1/n^{\delta_3} \quad \text{for some } \delta_3 > 0. \tag{93}$$

Therefore, in order to complete the estimation of H_4, we only need to find the bound for H_{45} in (89). We have

$$\begin{aligned} H_{45} \leq\; & mnE_{\Theta}\left[\left\{1-G\left(u_m\right)^{\Theta}\right\}\left\{G\left(u_m\right)^{\Theta}\right\}^{m-1}\right] \\ & \times E_{\Theta}\left[\left\{1-G\left(u_n\right)^{\Theta}\right\}\left(\left\{G\left(u_n\right)^{\Theta}\right\}^{n-m-1}-\left\{G\left(u_n\right)^{\Theta}\right\}^{n}\right)\right]. \end{aligned}$$

Hence, due to the earlier cited property that $z^{n-m-1} - z^n < (m+1)/n$, provided $0 \leq z \leq 1$ and $1 \leq m+1 < n$, and the facts that: $0 < \left\{ G(u_m)^{\Theta} \right\}^{m-1} \leq 1$, $0 \leq G(u_n)^{\Theta} \leq 1$, we get

$$\begin{aligned} H_{45} &< mnE_{\Theta} \left\{ 1 - G(u_m)^{\Theta} \right\} E_{\Theta} \left\{ 1 - G(u_n)^{\Theta} \right\} \frac{m+1}{n} \\ &\ll mnE_{\Theta} \left\{ 1 - G(u_m)^{\Theta} \right\} E_{\Theta} \left\{ 1 - G(u_n)^{\Theta} \right\} \frac{m}{n} \\ &= m^2 E_{\Theta} \left\{ 1 - G(u_m)^{\Theta} \right\} E_{\Theta} \left\{ 1 - G(u_n)^{\Theta} \right\}. \end{aligned}$$

This together with the relations in (77) and (85) imply

$$H_{45} \ll m^2 \frac{1}{m^{1+\varepsilon}} \frac{1}{n^{1+\varepsilon}} < \frac{1}{(mn)^{\varepsilon}} \quad \text{for some } \varepsilon > 1. \tag{94}$$

It stems from (89) and (90)-(94) that

$$H_4 \ll m/n + 1/n^{\delta} \quad \text{for some } \delta > 0. \tag{95}$$

Finally, combining (73), (74), (83), (88) and (95), we obtain that (70) is satisfied. Since it is the result we wished to establish, the proof of Lemma 4.5 is complete. □

The following assertion will also be needed for the proof of Theorem 2.1(ii).

Lemma 4.6. *Under the assumptions of Theorem 2.1*(ii) *on* $\{X_i\}$, Ψ, $\{u_n\}$, Θ *and* G, *we get*

$$\lim_{n\to\infty} P\left(M_n^{(2)} \leq u_n\right) = E_{\Theta}\left\{ e^{-\Lambda(\Theta)} \left(1 + \Lambda(\Theta)\right) \right\}. \tag{96}$$

Proof. In view of formula (62) from the proof of Lemma 4.4 and the fact that $P\left(M_n^{(2)} \leq u_n\right) = E_{\Theta} P\left(M_n^{(2)} \leq u_n \middle| \Theta\right)$, we immediately obtain

$$\begin{aligned} &\lim_{n\to\infty} P\left(M_n^{(2)} \leq u_n\right) \\ &\quad = \lim_{n\to\infty} \left(E_{\Theta} \left\{ (G(u_n))^{\Theta} \right\}^n + nE_{\Theta} \left[\left\{ 1 - (G(u_n))^{\Theta} \right\} \left\{ (G(u_n))^{\Theta} \right\}^{n-1} \right] \right). \end{aligned} \tag{97}$$

In addition, since $0 \leq \left\{ (G(u_n))^{\Theta} \right\}^n \leq 1$, it follows from the Lebesgue theorem on passing to the limit under the integral sign that

$$\begin{aligned} &\lim_{n\to\infty} \left(E_{\Theta} \left\{ (G(u_n))^{\Theta} \right\}^n + nE_{\Theta} \left[\left\{ 1 - (G(u_n))^{\Theta} \right\} \left\{ (G(u_n))^{\Theta} \right\}^{n-1} \right] \right) \\ &\quad = E_{\Theta} \left(\lim_{n\to\infty} \left[\left\{ (G(u_n))^{\Theta} \right\}^n \right] + \lim_{n\to\infty} \left[n \left\{ 1 - (G(u_n))^{\Theta} \right\} \left\{ (G(u_n))^{\Theta} \right\}^{n-1} \right] \right). \end{aligned} \tag{98}$$

Furthermore, the fact that $(G(u_n))^{\Theta} \to 1$ as $n \to \infty$, as well as a Poisson approximation to the binomial distribution with $np_n = nP\left(X_1 > u_n \middle| \Theta\right) =$

$n\left\{1-(G(u_n))^{\Theta}\right\}$ and the property that, due to the assumption from (9), $n\left\{1-(G(u_n))^{\Theta}\right\}\overset{a.s.}{\to}\Lambda(\Theta)$, as $n\to\infty$, yield

$$\lim_{n\to\infty}\left\{(G(u_n))^{\Theta}\right\}^{n-1}=\lim_{n\to\infty}\left\{(G(u_n))^{\Theta}\right\}^{n}=e^{-\Lambda(\Theta)}\text{ a.s.} \tag{99}$$

By virtue of (97)-(99), we have

$$\begin{aligned}\lim_{n\to\infty}P\left(M_n^{(2)}\le u_n\right)&=E_{\Theta}\left\{e^{-\Lambda(\Theta)}+\Lambda(\Theta)\,e^{-\Lambda(\Theta)}\right\}\\&=E_{\Theta}\left\{e^{-\Lambda(\Theta)}\left(1+\Lambda(\Theta)\right)\right\},\end{aligned}$$

which is a desired claim in (96). □

5. Proofs of the ASCLTs for ordinary maxima and the second largest maxima

We are now in a position to prove Theorems 2.1-2.2.

Proof of Theorems 2.1-2.2. First, we will show that the following property holds true

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(I(M_n\le u_n)-P(M_n\le u_n)\right)=0\text{ a.s.} \tag{100}$$

By virtue of Lemma 3.1 in Csaki and Gonchigdanzan [5], in order to prove (100), it is enough to establish that

$$Var\left(\sum_{n=1}^{N}\frac{1}{n}I(M_n\le u_n)\right)\ll\frac{(\log N)^2}{(\log\log N)^{1+\varepsilon}}\quad\text{for some }\varepsilon>0. \tag{101}$$

Put

$$\xi_n:=I(M_n\le u_n). \tag{102}$$

Then,

$$\begin{aligned}Var\left(\sum_{n=1}^{N}\frac{1}{n}I(M_n\le u_n)\right)&=Var\left(\sum_{n=1}^{N}\frac{1}{n}\xi_n\right)\\&\le\sum_{n=1}^{N}\frac{1}{n^2}Var(\xi_n)+2\sum_{1\le m<n\le N}\frac{1}{mn}\left|Cov(\xi_m,\xi_n)\right|\\&=:\sum\nolimits_1+\sum\nolimits_2.\end{aligned} \tag{103}$$

It is clear that

$$\sum\nolimits_1\le\sum_{n=1}^{N}\frac{1}{n^2}<\infty. \tag{104}$$

Thus, it remains to estimate the second component $\sum_2$ in (104). It follows from Lemmas 4.1-4.2 that, under the assumptions of Theorems 2.1(i) and 2.2, respectively,

$$\begin{aligned} |Cov\,(\xi_m, \xi_n)| &= |Cov\,(I\,(M_m \leq u_m)\,, I\,(M_n \leq u_n))| \\ &\ll m/n + 1/n^{\delta} \quad \text{for some } \delta > 0. \end{aligned} \tag{105}$$

Consequently, we may write that

$$\begin{aligned} \sum\nolimits_2 &\ll \sum_{m=1}^{N-1} \sum_{n=m+1}^{N} \frac{1}{mn}\frac{m}{n} + \sum_{m=1}^{N-1} \sum_{n=m+1}^{N} \frac{1}{mn}\frac{1}{n^{\delta}} \\ &=: \sum\nolimits_{21} + \sum\nolimits_{22}\,. \end{aligned} \tag{106}$$

The estimation of the component $\sum_{21}$ in (106) is straightforward. We clearly have

$$\sum\nolimits_{21} \ll \sum_{m=1}^{N-1} \sum_{n=m+1}^{N} \frac{1}{n^2} \leq \sum_{m=1}^{N-1} \frac{1}{m} \ll \log N. \tag{107}$$

Furthermore, in order to give the bound for $\sum_{22}$ in (106), observe that

$$\begin{aligned} \sum\nolimits_{22} &\ll \sum_{m=1}^{N-1} \sum_{n=m+1}^{N} \frac{1}{mn}\frac{1}{n^{\delta}} = \sum_{m=1}^{N-1} \frac{1}{m} \sum_{n=m+1}^{N} \frac{1}{n^{\delta+1}} \\ &\leq \frac{1}{\delta} \sum_{m=1}^{N-1} \frac{1}{m^{1+\delta}} \quad \text{for some } \delta > 0. \end{aligned}$$

Consequently,

$$\sum\nolimits_{22} \ll \sum_{m=1}^{N-1} \frac{1}{m^{1+\delta}} < \infty. \tag{108}$$

In view of (106)-(108), we obtain

$$\sum\nolimits_2 \ll \log N. \tag{109}$$

Due to (103), (104) and (109), we get

$$Var \left(\sum_{n=1}^{N} \frac{1}{n} I\,(M_n \leq u_n) \right) \ll \log N.$$

Thus, (101) is fulfilled and (100) holds true.

The convergence in (100), Lemma 4.3 and the regularity property of logarithmic mean imply (10) and hence, the proofs of Theorems 2.1(i) and 2.2 are complete.

Thus, we need to prove Theorem 2.1(ii). The approach leading to the proof of Theorem 2.1(ii) is identical as the method applied in the proof of Theorems 2.1(i) and 2.2, with the only exception that Lemmas 4.4-4.6 are implied here instead of Lemmas 4.1-4.3. First, we will show that the following property is satisfied

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(I\left(M_n^{(2)}\le u_n\right)-P\left(M_n^{(2)}\le u_n\right)\right)=0 \text{ a.s.} \tag{110}$$

In view of Lemma 3.1 from Csaki and Gonchigdanzan [5], in order to prove (110), it suffices to show that

$$Var\left(\sum_{n=1}^{N}\frac{1}{n}I\left(M_n^{(2)}\le u_n\right)\right)\ll\frac{(\log N)^2}{(\log\log N)^{1+\varepsilon}}\quad\text{for some }\varepsilon>0. \tag{111}$$

Set

$$\zeta_n:=I\left(M_n^{(2)}\le u_n\right). \tag{112}$$

Then,

$$\begin{aligned}Var\left(\sum_{n=1}^{N}\frac{1}{n}I\left(M_n^{(2)}\le u_n\right)\right)&=Var\left(\sum_{n=1}^{N}\frac{1}{n}\zeta_n\right)\\ &\le\sum_{n=1}^{N}\frac{1}{n^2}Var\left(\zeta_n\right)+2\sum_{1\le m<n\le N}\frac{1}{mn}\left|Cov\left(\zeta_m,\zeta_n\right)\right|\\ &=:\tilde{\sum}_1+\tilde{\sum}_2.\end{aligned} \tag{113}$$

Obviously, we get

$$\tilde{\sum}_1\le\sum_{n=1}^{N}\frac{1}{n^2}<\infty. \tag{114}$$

Thus, it remains to estimate the second component $\tilde{\sum}_2$ in (113). By virtue of Lemmas 4.4-4.5, we have that, under the assumptions of Theorem 2.1(ii),

$$\begin{aligned}\left|Cov\left(\zeta_m,\zeta_n\right)\right|&=\left|Cov\left(I\left(M_m^{(2)}\le u_m\right),I\left(M_n^{(2)}\le u_n\right)\right)\right|\\ &\ll m/n+1/n^{\delta}\quad\text{for some }\delta>0.\end{aligned} \tag{115}$$

Using (115) and reasoning as in the estimation of $\sum_2$ in the previous proof (see the derivations in (106)-(109)), we immediately obtain

$$\tilde{\sum}_2\ll\sum_{m=1}^{N-1}\sum_{n=m+1}^{N}\frac{1}{mn}\frac{m}{n}+\sum_{m=1}^{N-1}\sum_{n=m+1}^{N}\frac{1}{mn}\frac{1}{n^{\delta}}\ll\log N. \tag{116}$$

In view of (113), (114) and (116), we get

$$Var\left(\sum_{n=1}^{N}\frac{1}{n}I\left(M_n^{(2)}\le u_n\right)\right)\ll\log N.$$

Thus, (111) is fulfilled and (110) holds true.

Finally, the convergence in (110), Lemma 4.6 and the regularity property of logarithmic mean imply (12), and, as a consequence, the proof of Theorem 2.1 (ii) is complete. □

Furthermore, we also prove the validity of Corollaries 2.1-2.2 in this part of our paper. As has already been mentioned, both of these claims are the straightforward consequences of (the previously proved) Theorems 2.1 and 2.2, as well as the results stated by Wüthrich [28].

Proof of Corollary 2.1(i). By Assumption 5.1 and Corollary 5.2 in Wüthrich [28] (see Example 5.3 for the case of the Clayton copula there), we have

$$\lim_{n\to\infty} P\left(M_n \le 1 - x/n\right) = (1+\alpha x)^{-1/\alpha}. \tag{117}$$

Put $u_n := 1 - x_n/n$, where $x_n \sim 1/n^{\varepsilon}$ for some $\varepsilon > 0$. Thus, by virtue of the relation in (117),

$$\lim_{n\to\infty} P\left(M_n \le u_n\right) = \lim_{n\to\infty} P\left(M_n \le 1 - x_n/n\right) = \lim_{n\to\infty} (1+\alpha x_n)^{-1/\alpha} = 1. \tag{118}$$

Since F is a cdf of the $U(0,1)$ distribution, we immediately get

$$n\left(1 - F\left(u_n\right)\right) = n \cdot x_n/n = x_n \sim 1/n^{\varepsilon} \quad \text{for some } \varepsilon > 0,$$

and assumption (7) is satisfied. Therefore, relation (118) and Theorem 2.1 imply the a.s. convergence in (14). □

Proof of Corollary 2.1(ii). Put $u_n := 1 - x_n/n$, where $x_n \sim n^{\varepsilon}$ for some $\varepsilon \in (1 - 1/(1+\alpha);1)$. Hence, in view of (117),

$$\lim_{n\to\infty} P\left(M_n \le u_n\right) = \lim_{n\to\infty} P\left(M_n \le 1 - x_n/n\right) = \lim_{n\to\infty} (1+\alpha x_n)^{-1/\alpha} = 0. \tag{119}$$

Since F is a cdf of the $U(0,1)$ distribution, we instantly obtain

$$n\left(1 - F\left(u_n\right)\right) = n \cdot x_n/n = x_n \sim n^{\varepsilon} \quad \text{for some } \varepsilon \in (1 - 1/(1+\alpha);1),$$

and assumption (8) is fulfilled. Therefore, relation (119) and Theorem 2.1 imply the a.s. convergence in (15). □

Proof of Corollary 2.2. By Assumption 5.1 and Corollary 5.2 in Wüthrich [28] (see Example 5.3 for the case of the Gumbel copula there), we have

$$\lim_{n\to\infty} P\left(M_n \le 1 - x/n^{1/\alpha}\right) = \exp(-x). \tag{120}$$

Put $u_n := 1 - x_n/n^{1/\alpha}$, where $x_n \sim n^{\gamma}$ for some $\gamma \in (0; 1/\alpha)$. Therefore, by (120),

$$\lim_{n\to\infty} P\left(M_n \le u_n\right) = \lim_{n\to\infty} P\left(M_n \le 1 - x_n/n^{1/\alpha}\right) = \lim_{n\to\infty} \exp(-x_n) = 0. \tag{121}$$

Since F is a cdf of the $U(0,1)$ distribution and $0 < \gamma < 1/\alpha$, we immediately get

$$n\left(1 - F\left(u_n\right)\right) = n \cdot x_n/n^{1/\alpha} = n^{1-1/\alpha} x_n$$

$$\sim n^{1-1/\alpha+\gamma} \sim n^{\varepsilon} \quad \text{for some } \varepsilon \in (1-1/\alpha; 1)\,.$$

and condition (13) is satisfied. Consequently, a desired result in (11) directly follows from derivation (121) and Theorem 2.2. ☐

6. Proofs of the ASCLTs for the kth largest maxima

In this part of our work, we shall give the proofs of Theorem 3.1 and Corollary 3.1.

Proof of Theorem 3.1. Suppose that $n-m > k-1$. Let us estimate the expressions J and K defined as follows:

$$J := \sum_{s=0}^{k-1} \binom{n}{s} E_\Theta \left[\left\{1-G(u_n)^\Theta\right\}^s \left(\left\{G(u_n)^\Theta\right\}^{n-m-s} - \left\{G(u_n)^\Theta\right\}^{n-s} \right) \right], \tag{122}$$

$$K := K_1 + K_2 + K_3 + K_4, \tag{123}$$

where:

$$K_1 := E_\Theta \left[\left\{G\left(u_m\right)^\Theta\right\}^m \left\{G\left(u_n\right)^\Theta\right\}^{n-m} \right] - E_\Theta \left[\left\{G\left(u_m\right)^\Theta\right\}^m \right] E_\Theta \left[\left\{G\left(u_n\right)^\Theta\right\}^{n-m} \right],$$

$$K_2 := nE_\Theta \left[\left\{G\left(u_m\right)^\Theta\right\}^m \left\{1-G\left(u_n\right)^\Theta\right\} \left\{G\left(u_n\right)^\Theta\right\}^{n-m-1} \right],$$

$$K_3 := mE_\Theta \left[\left\{G\left(u_m\right)^\Theta\right\}^m \left\{1-G\left(u_m\right)^\Theta\right\} \left\{G\left(u_n\right)^\Theta\right\}^{n-m} \right],$$

$$K_4 := \sum_{s_1=1}^{k-1} \sum_{s_2=1}^{k-1} m^{s_1} n^{s_2} E_\Theta \left[\left\{1-G\left(u_m\right)^\Theta\right\}^{s_1} \left\{1-G\left(u_n\right)^\Theta\right\}^{s_2} \right].$$

In order to find the bound for J, we shall use the property that $z^{n-s-m} - z^{n-s} < m/(n-s)$, if $1 \le m < n-s$ and $0 \le z \le 1$. Thus, we have that for any $s \in \{0, 1, \ldots, k-1\}$

$$\left\{G(u_n)^\Theta\right\}^{n-m-s} - \left\{G(u_n)^\Theta\right\}^{n-s} < \frac{m}{n-s} \le \frac{m}{n-k+1}.$$

Hence, we may write that

$$J < \frac{m}{n-k+1} \sum_{s=0}^{k-1} n^s E_\Theta \left[\left\{1-G(u_n)^\Theta\right\}^s \right]. \tag{124}$$

Our purpose now is to estimate the sum $\sum_{s=0}^{k-1} n^s E_\Theta \left[\left\{1-G(u_n)^\Theta\right\}^s \right]$. It follows from assumption (17) that:

$$\mu_t := \max_{0 \le s \le t} E_\Theta \Theta^s < \infty \quad \text{for any } t \le 2\,(k-1)\,. \tag{125}$$

In view of (4), (6), (125) and the well-known fact that $1-e^{-x} \le x$ for all $x \in \mathbb{R}$, we get for any $s \in \{0, 1, \ldots, k-1\}$

$$\mathbb{E}_\Theta \left\{1-G^\Theta(u_n)\right\}^s = \mathbb{E}_\Theta \left\{1-\exp\left(-\Theta\Psi\left(F\left(u_n\right)\right)\right)\right\}^s$$

$$\leq \mathbb{E}_\Theta \{\Theta\Psi(F(u_n))\}^s = \{\Psi(F(u_n))\}^s \mathbb{E}_\Theta \Theta^s$$

$$\leq \{\Psi(F(u_n))\}^s \mu_{k-1}. \tag{126}$$

Furthermore, by assumption (18) and the facts that F is nondecreasing and Ψ^{-1} is decreasing, we obtain

$$\Psi(F(u_n)) \leq C/n^\beta \quad \text{for some generic constants } C > 0 \text{ and } \beta > 1,$$

and consequently,

$$n\Psi(F(u_n)) \ll 1/n^\gamma \quad \text{for some } \gamma > 0. \tag{127}$$

Since $C/n^\gamma \to 0$, as $n \to \infty$, the relation in (127) implies that

$$n\Psi(F(u_n)) \leq q < 1 \quad \text{for all sufficiently large } n. \tag{128}$$

By virtue of (126) and (128), we have

$$\sum_{s=0}^{k-1} n^s E_\Theta \left[\left\{1 - G(u_n)^\Theta\right\}^s\right] = 1 + \sum_{s=1}^{k-1} n^s E_\Theta \left[\left\{1 - G(u_n)^\Theta\right\}^s\right]$$

$$\leq 1 + \mu_{k-1} \sum_{s=1}^{k-1} n^s \{\Psi(F(u_n))\}^s \ll 1 + \mu_{k-1} \frac{n\Psi(F(u_n))}{1 - n\Psi(F(u_n))}$$

$$\leq 1 + \mu_{k-1} \frac{q}{1-q} \ll 1, \tag{129}$$

where the penultimate relation follows from the fact that the function $y = x/(1-x)$ is an increasing one.

Due to (124) and (129), we obtain

$$J \ll \frac{m}{n-k+1}. \tag{130}$$

Our task now is to estimate K in (123). In order to find the bound for component K_1 in (123), let us notice that K_1 is defined identically as component B in (31). Thus, we may write that

$$K_1 \leq B_1 + B_2 + B_3, \tag{131}$$

where B_1-B_3 are defined in the same manner as the terms B_1-B_3 in (32), with an exception that the conditions of Theorem 3.1 are satisfied in the current case. Thus, it follows from (33) that

$$B_1 + B_3 < 2(m/n). \tag{132}$$

Furthermore, by (34)-(37), we get

$$B_2 \leq \sqrt{\begin{array}{c} \left|\Psi^{-1}(2n\Psi(F(u_n))) - \Psi^{-1}(n\Psi(F(u_n)))\right| \Psi^{-1}(n\Psi(F(u_n))) \\ + \left|\Psi^{-1}(n\Psi(F(u_n))) - \Psi^{-1}(0)\right| \Psi^{-1}(2n\Psi(F(u_n))) \end{array}}. \tag{133}$$

Observe also that Ψ^{-1} is a Lipschitz function, as, in view of the mean value theorem, we have

$$\left|\Psi^{-1}(x) - \Psi^{-1}(y)\right| = |\mathbb{E}_\Theta \exp(-x\Theta) - \mathbb{E}_\Theta \exp(-y\Theta)| \tag{134}$$

$$\leq |x-y| \mathbb{E}_\Theta \Theta \leq L |x-y| \quad \text{for some constant } L>0 \text{ and any } x,y>0.$$

Combining (134) with (133), we obtain

$$B_2 \leq \sqrt{2n\Psi\left(F\left(u_n\right)\right)}. \tag{135}$$

Consequently, in view of (135) and (127), we get

$$B_2 \ll 1/n^{\gamma/2} \quad \text{for some } \gamma>0. \tag{136}$$

Thus, due to (131), (132) and (136),

$$K_1 \ll m/n + 1/n^{\gamma/2} \quad \text{for some } \gamma>0. \tag{137}$$

In order to estimate K_2 in (123), let us notice that

$$K_2 \leq nE_\Theta\left[1-G\left(u_n\right)^\Theta\right]. \tag{138}$$

Assumptions (17)-(18) (and relation (127), in particular) together with derivation (126) and relation (138) imply

$$K_2 \leq nE_\Theta\left[1-G\left(u_n\right)^\Theta\right] \leq n\Psi\left(F\left(u_n\right)\right)\mu_1 \ll 1/n^\gamma \quad \text{for some } \gamma>0. \tag{139}$$

Using identical reasoning as in the estimation of component K_2, we have the following estimate for K_3 in (123)

$$K_3 \leq mE_\Theta\left[1-G\left(u_m\right)^\Theta\right] \leq m\Psi\left(F\left(u_m\right)\right)\mu_1 \ll 1/m^\gamma \quad \text{for some } \gamma>0. \tag{140}$$

Thus, in order to complete the estimation of K in (123), we only need to give the bound for component K_4 in (123). It follows from the Schwarz inequality that

$$\begin{aligned} K_4 &\leq \sum_{s_1=1}^{k-1}\sum_{s_2=1}^{k-1} m^{s_1}n^{s_2}\sqrt{E_\Theta\left[\left\{1-G\left(u_m\right)^\Theta\right\}^{2s_1}\right]}\sqrt{E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}^{2s_2}\right]} \\ &= \sum_{s_1=1}^{k-1} m^{s_1}\sqrt{E_\Theta\left[\left\{1-G\left(u_m\right)^\Theta\right\}^{2s_1}\right]}\sum_{s_2=1}^{k-1} n^{s_2}\sqrt{E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}^{2s_2}\right]}. \end{aligned} \tag{141}$$

Applying derivation (126) and assumption (17), we immediately obtain that, for $s_2 \in \{1,2,\ldots,k-1\}$,

$$E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}^{2s_2}\right] \leq \left(\Psi\left(F\left(u_n\right)\right)\right)^{2s_2}\mu_{2s_2} \leq \left(\Psi\left(F\left(u_n\right)\right)\right)^{2s_2}\mu_{2(k-1)},$$

where, in view of definition (125), $\mu_{2(k-1)} := \max_{0\leq s\leq 2(k-1)} E_\Theta\Theta^s < \infty$.

Therefore,

$$\sqrt{E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}^{2s_2}\right]} \leq \left(\Psi\left(F\left(u_n\right)\right)\right)^{s_2}\sqrt{\mu_{2(k-1)}}.$$

Thus, reasoning similarly as in the estimation of $\sum_{s=1}^{k-1} n^s E_\Theta\left[\left\{1-G(u_n)^\Theta\right\}^s\right]$ in (129), we get

$$\begin{aligned}\sum_{s_2=1}^{k-1} n^{s_2}\sqrt{E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}^{2s_2}\right]} &\leq \sqrt{\mu_{2(k-1)}}\sum_{s_2=1}^{k-1} n^{s_2}\left\{\Psi\left(F\left(u_n\right)\right)\right\}^{s_2}\\ &\leq \sqrt{\mu_{2(k-1)}}\frac{n\Psi\left(F\left(u_n\right)\right)}{1-n\Psi\left(F\left(u_n\right)\right)}.\end{aligned}$$

This, (127), (125) and the fact that $y = x/\left(1-x\right)$ is an increasing function yield

$$\sum_{s_2=1}^{k-1} n^{s_2}\sqrt{E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}^{2s_2}\right]} \ll \sqrt{\mu_{2(k-1)}}\frac{1/n^\gamma}{1-1/n^\gamma} \quad \text{for some } \gamma > 0,$$

and hence,

$$\sum_{s_2=1}^{k-1} n^{s_2}\sqrt{E_\Theta\left[\left\{1-G\left(u_n\right)^\Theta\right\}^{2s_2}\right]} \ll 1/n^\gamma \quad \text{for some } \gamma > 0. \tag{142}$$

Analogously, the following relation holds true

$$\sum_{s_1=1}^{k-1} m^{s_1}\sqrt{E_\Theta\left[\left\{1-G\left(u_m\right)^\Theta\right\}^{2s_1}\right]} \ll 1/m^\gamma \quad \text{for some } \gamma > 0. \tag{143}$$

By virtue of (141)-(143), we have

$$K_4 \ll 1/\left(mn\right)^\gamma \quad \text{for some } \gamma > 0. \tag{144}$$

In view of (123), (137), (139), (140) and (144), we get

$$K \ll m/n + 1/m^\delta \quad \text{for some } \delta > 0. \tag{145}$$

It follows from (130) and (145) that

$$J + K \ll \frac{m}{n-k+1} + \frac{1}{m^\delta} \quad \text{for some } \delta > 0. \tag{146}$$

Continuing our proof, observe that

$$\begin{aligned}&\left|Cov\left(I\left(M_m^{(k)} \leq u_m\right), I\left(M_n^{(k)} \leq u_n\right)\right)\right|\\ &\ll E\left|I\left(M_n^{(k)} \leq u_n\right) - I\left(M_{m,n}^{(k)} \leq u_n\right)\right|\\ &\quad + \left|Cov\left(I\left(M_m^{(k)} \leq u_m\right), I\left(M_{m,n}^{(k)} \leq u_n\right)\right)\right|\\ &= \left\{P\left(M_{m,n}^{(k)} \leq u_n\right) - P\left(M_n^{(k)} \leq u_n\right)\right\}\\ &\quad + \left|Cov\left(I\left(M_m^{(k)} \leq u_m\right), I\left(M_{m,n}^{(k)} \leq u_n\right)\right)\right|.\end{aligned} \tag{147}$$

On the other hand, using the assumptions in (5)-(6), it is easy to check that:

$$P\left(M_{m,n}^{(k)} \leq u_n\right) = E_\Theta P\left(M_{m,n}^{(k)} \leq u_n \middle| \Theta\right)$$

$$= \sum_{s=0}^{k-1} \binom{n-m}{s} E_{\Theta} \left[\left\{1 - G(u_n)^{\Theta}\right\}^s \left\{G(u_n)^{\Theta}\right\}^{n-m-s} \right],$$

$$P\left(M_n^{(k)} \le u_n\right) = E_{\Theta} P\left(M_n^{(k)} \le u_n \middle| \Theta\right)$$

$$= \sum_{s=0}^{k-1} \binom{n}{s} E_{\Theta} \left[\left\{1 - G(u_n)^{\Theta}\right\}^s \left\{G(u_n)^{\Theta}\right\}^{n-s} \right],$$

and consequently that

$$P\left(M_{m,n}^{(k)} \le u_n\right) - P\left(M_n^{(k)} \le u_n\right) \le J, \tag{148}$$

where J is such as in (122).

In addition, it is clear that

$$\left| Cov\left(I\left(M_m^{(k)} \le u_m\right), I\left(M_{m,n}^{(k)} \le u_n\right)\right)\right|$$

$$= \sum_{s_1=0}^{k-1} \sum_{s_2=0}^{k-1} \binom{m}{s_1} \binom{n-m}{s_2} D\left(s_1, s_2, m, n\right),$$

where $D\left(s_1, s_2, m, n\right)$ is such as in (72).

This implies that

$$\left| Cov\left(I\left(M_m^{(k)} \le u_m\right), I\left(M_{m,n}^{(k)} \le u_n\right)\right)\right|$$

$$\le K := K_1 + K_2 + K_3 + K_4, \tag{149}$$

where K and K_1-K_4 are defined as in (123).

The relations in (147)-(149) and (146) yield

$$\left| Cov\left(I\left(M_m^{(k)} \le u_m\right), I\left(M_n^{(k)} \le u_n\right)\right)\right|$$

$$\le J + K$$

$$\ll \frac{m}{n-k+1} + \frac{1}{m^{\delta}} \quad \text{for some } \delta > 0. \tag{150}$$

Next, let us find the limit $\lim_{n\to\infty} P\left(M_n^{(k)} \le u_n\right)$. Using the same ideas and theoretical background as in the proofs of Lemmas 4.3 and 4.6 (in particular, a Poisson approximation to the binomial distribution with $np_n = nP\left(X_1 > u_n \middle| \Theta\right) = n\left\{1 - \left(G\left(u_n\right)\right)^{\Theta}\right\} \overset{a.s.}{\to} \Lambda\left(\Theta\right)$, as $n \to \infty$), we have

$$\lim_{n\to\infty} P\left(M_n^{(k)} \le u_n\right) = \lim_{n\to\infty} E_{\Theta} P\left(M_n^{(k)} \le u_n \middle| \Theta\right)$$

$$= E_{\Theta} \left\{ \lim_{n\to\infty} \sum_{s=0}^{k-1} \binom{n}{s} E_{\Theta} \left[\left\{1 - G(u_n)^{\Theta}\right\}^s \left\{G(u_n)^{\Theta}\right\}^{n-s} \right] \right\}$$

$$= E_{\Theta} \left\{ \lim_{n\to\infty} \sum_{s=0}^{k-1} \frac{n^s}{s!} \left\{1 - G(u_n)^{\Theta}\right\}^s \left\{G(u_n)^{\Theta}\right\}^{n-s} \right\}$$

$$= E_\Theta \left\{ \lim_{n\to\infty} \sum_{s=0}^{k-1} \frac{\left(n\left\{1-(G(u_n))^\Theta\right\}\right)^s}{s!} e^{-\Lambda(\Theta)} \right\}$$

$$= e^{-\Lambda(\Theta)} \sum_{s=0}^{k-1} \frac{(\Lambda(\Theta))^s}{s!}. \tag{151}$$

The derivations in (150)-(151) will be applied in the final stage of our proof.

Following Lemma 3.1 in Csaki and Gonchigdanzan [5], in order to complete the proof of Theorem 3.1, it is sufficient to show that

$$Var\left(\sum_{n=k}^{N} \frac{1}{n} I\left(M_n^{(k)} \le u_n\right)\right) \ll \frac{(\log N)^2}{(\log\log N)^{1+\varepsilon}} \quad \text{for some } \varepsilon > 0. \tag{152}$$

Clearly,

$$\begin{aligned} &Var\left(\sum_{n=k}^{N} \frac{1}{n} I\left(M_n^{(k)} \le u_n\right)\right) \\ &\le \sum_{n=k}^{N} \frac{1}{n^2} Var\left(I\left(M_n^{(k)} \le u_n\right)\right) \\ &\quad + 2 \sum_{k\le m<n\le N} \frac{1}{mn} \left|Cov\left(I\left(M_m^{(k)} \le u_m\right), I\left(M_n^{(k)} \le u_n\right)\right)\right| \\ &\le \sum_{n=k}^{N} \frac{1}{n^2} + 2 \sum_{k\le m<n\le N} \frac{1}{mn} \left|Cov\left(I\left(M_m^{(k)} \le u_m\right), I\left(M_n^{(k)} \le u_n\right)\right)\right|. \end{aligned}$$

This and (150) imply

$$\begin{aligned} &Var\left(\sum_{n=1}^{N} \frac{1}{n} I\left(M_n^{(k)} \le u_n\right)\right) \\ &\ll \sum_{n=1}^{N} \frac{1}{n^2} + \sum_{k\le m<n\le N} \frac{1}{mn} \frac{m}{n-k+1} + \sum_{k\le m<n\le N} \frac{1}{mn} \frac{1}{m^\delta} \quad \text{for some } \delta > 0. \end{aligned}$$

Therefore, we may write that

$$\begin{aligned} &Var\left(\sum_{n=1}^{N} \frac{1}{n} I\left(M_n^{(k)} \le u_n\right)\right) \\ &\ll \sum_{n=1}^{N} \frac{1}{n^2} + \sum_{m=1}^{N-1} \sum_{n=m+1}^{N} \frac{1}{mn} \frac{m}{n} + \sum_{m=1}^{N-1} \frac{1}{m^{1+\delta}} \sum_{n=1}^{N} \frac{1}{n} \\ &= \sum_{n=1}^{N} \frac{1}{n^2} + \sum_{m=1}^{N-1} \sum_{n=m+1}^{N} \frac{1}{n^2} + \sum_{m=1}^{N-1} \frac{1}{m^{1+\delta}} \sum_{n=1}^{N} \frac{1}{n} \end{aligned}$$

$$\leq \sum_{n=1}^{N} \frac{1}{n^2} + \sum_{m=1}^{N-1} \frac{1}{m} + \sum_{m=1}^{N-1} \frac{1}{m^{1+\delta}} \sum_{n=1}^{N} \frac{1}{n},$$

and hence,

$$Var\left(\sum_{n=1}^{N} \frac{1}{n} I\left(M_n^{(k)} \leq u_n\right)\right) \ll \log N.$$

Consequently, condition (152) is satisfied. This, Lemma 3.1 in Csaki and Gonchigdanzan [5] together with derivation (151) and the property of logarithmic average yield a desired claim in (19). □

We only need to prove our latest statement.

Proof of Corollary 3.1. First, let us notice that, as $\Psi(t) = (-\ln t)^{\alpha}$ for some $\alpha > 1$ and (20) holds, we have

$$\begin{aligned} n\Psi(F(u_n)) &= 2n(-\ln F(u_n))^{\alpha} \sim 2n\left(-\ln\left(1-\frac{\tau}{n}\right)\right)^{\alpha} \\ &= 2n\frac{1}{n^{\alpha}}\left(-n\ln\left(1-\frac{\tau}{n}\right)\right)^{\alpha} \sim 2n^{1-\alpha}\left(-\ln\left(1-\frac{\tau}{n}\right)^{n}\right)^{\alpha} \\ &\sim 2n^{1-\alpha}(-\ln\exp(-\tau))^{\alpha} \sim 2n^{1-\alpha}(\tau)^{\alpha} \\ &\ll 1/n^{\alpha-1} \text{ for some } \alpha > 1. \end{aligned}$$

Consequently:

$$\Psi(F(u_n)) \ll 1/n^{\alpha} \quad \text{for some } \alpha > 1,$$

and, as F in nondecreasing and Ψ^{-1} is decreasing,

$$u_n \geq F^{-1}\left(\Psi^{-1}(1/n^{\alpha})\right) \quad \text{for all sufficiently large } n \text{ and some } \alpha > 1.$$

Therefore, condition (18) of Theorem 3.1 is fulfilled with $\beta = \alpha$.

Thus, in view of Theorem 3.1, we obtain (see (19))

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(M_n^{(k)} \leq u_n\right) = E_{\Theta}\left\{e^{-\Lambda(\Theta)} \sum_{s=0}^{k-1} \frac{(\Lambda(\Theta))^s}{s!}\right\} \text{ a.s.,} \tag{153}$$

where $\Lambda(\Theta)$ is such as in (9), i.e.,

$$\lim_{n\to\infty} n\left\{1-(G(u_n))^{\Theta}\right\} = \Lambda(\Theta) \text{ a.s.}$$

The assumptions imposed in (6), (4) and (20) together with previously recollected condition (9) imply

$$\begin{aligned} E_{\Theta}(\Lambda(\Theta)) &= \lim_{n\to\infty} E_{\Theta}\left[n\left\{1-(G(u_n))^{\Theta}\right\}\right] \\ &= \lim_{n\to\infty}\left[n\left\{1-E_{\Theta}(G(u_n))^{\Theta}\right\}\right] = \lim_{n\to\infty} n\left[1-(\Psi)^{-1}(\Psi(F(u_n)))\right] \\ &= \lim_{n\to\infty} n(1-F(u_n)) = \tau. \end{aligned} \tag{154}$$

Finally, due to (153)-(154), we conclude that (19) holds, which is the result we wished to prove. □

7. Appendix

In this part of our paper, some comments on the conditions assumed in Theorems 2.1-2.2 are placed. We shall focus on the assumptions concerning the convergence of the sequence $n(1-F(u_n))$, i.e., the conditions

$$n(1-F(u_n)) \sim 1/n^{\varepsilon} \text{ or } n(1-F(u_n)) \sim n^{\varepsilon} \quad \text{for some } \varepsilon > 0. \tag{155}$$

Obviously, a natural question arises: why the assumptions of the form

$$n(1-F(u_n)) \to \tau, \text{ where } \tau \text{ stands for some non-negative constant,} \tag{156}$$

have not been considered in the statements of the mentioned propositions? The reason for the omission of the conditions as in (156) is that under the assumptions of this sort, the key estimation leading to the proofs of Lemmas 4.1, 4.2 and 4.5 (and consequently, to the proofs of Theorems 2.1-2.2, respectively), i.e., the inequality

$$2n\Psi(F(u_n)) > \Psi(1-F(u_n)) \quad \text{for all sufficiently large } n, \tag{157}$$

(see the relations between (44) and (45) and between (51) and (52)) is not satisfied.

It may be checked that the inequality in (157) is fulfilled for some class of the Clayton copulas, namely for the copulas with a generator of the form $\Psi(t) = \frac{1}{\alpha}(t^{-\alpha}-1)$ under the restriction that α is some negative number from the interval $(-1;0)$. On the other side, it follows from [17]-[18] that $\Psi(t) = \frac{1}{\alpha}(t^{-\alpha}-1)$ generates the Archimedean copula in dimension d if and only if $\alpha \geq -1/(d-1)$. Since we consider the copulas of dimension n, where $n \to \infty$, we have that the Clayton copula is the Archimedean one if and only if $\alpha \geq -1/(n-1) \to 0$, as $n \to \infty$, which contradicts the condition that $\alpha \in (-1;0)$.

It turns out that assumption (156) may be considered in the proof of the ASCLT for the k-th largest maxima, if we additionally assume that conditions (17) and (18) of Theorem 3.1 hold true. It is shown in the proof of Corollary 3.1 that (18) is satisfied for some class of the Gumbel copulas, namely for the copulas with a generator of the form $\Psi(t) = (-\ln t)^{\alpha}$ under the constraint that $\alpha > 1$.

It is not difficult to prove that assumption (17) may be replaced by condition (21), as we stated in Remark 3.1. In order to justify this fact, let us first notice that the n-th derivative of the inverse function Ψ^{-1} may be expressed as follows

$$\left(\Psi^{-1}(v)\right)^{(n)} = (-1)^n E\left\{\Theta^n \exp(-\Theta v)\right\},$$

since $\Psi^{-1}(v) = E\{\exp(-\Theta v)\}$.

Hence, we immediately get

$$E(\Theta^n) = (-1)^n \left(\Psi^{-1}(v)\right)^{(n)}\Big|_{v=0}.$$

Therefore, the condition in (21), i.e., the constraint

$$(-1)^n \left(\Psi^{-1}(v)\right)^{(2(k-1))}\Big|_{v=0} < \infty,$$

implies the property that $E\left(\Theta^{2(k-1)}\right) < \infty$, which is the mentioned assumption in (17).

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Marcin Dudziński
Faculty of Applied Informatics and Mathematics
Warsaw University of Life Sciences - SGGW w Warszawie
Nowoursynowska 159, 02-776 Warsaw, Poland
E-mail address: marcin_dudzinski@sggw.pl

Konrad Furmańczyk
Faculty of Applied Informatics and Mathematics
Warsaw University of Life Sciences - SGGW w Warszawie
Nowoursynowska 159, 02-776 Warsaw, Poland
E-mail address: konrad_furmanczyk@sggw.pl