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UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH THEIR HOMOGENEOUS AND LINEAR DIFFERENTIAL POLYNOMIALS SHARING A SMALL FUNCTION

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Abstract. In this paper we study the uniqueness question of meromorphic functions whose certain differential polynomials share a small function.

1. Introduction, definitions and results

Let f be a meromorphic function in the open complex plane $\mathbb{C}$. We use the standard notations of Nevanlinna's value distribution theory such as $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $T(r,f)$ etc. as available in [2]. We denote by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ possibly outside a set of finite linear measure.

A meromorphic function $a=a(z)$ is called a small function of f if $T(r,a)=S(r,f)$. We denote by $S(f)$ the collection of all small functions of f. Clearly $\mathbb{C}\subset S(f)$.

Let f and g be two meromorphic functions in $\mathbb{C}$ and $a\in S(f)\cap S(g)$. We say that f and g share the function $a=a(z)$ CM (counting multiplicities) or IM (ignoring multiplicities) if $f-a$ and $g-a$ have the same set of zeros counting multiplicities or ignoring multiplicities respectively.

For $a\in\mathbb{C}\cap\{\infty\}$ the quantities

$$\delta(a;f)=1-\limsup_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}\qquad\text{and}\qquad\Theta(a;f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}$$

are respectively called the deficiency and ramification index of a for the function f, where $N(r,a;f)=N(r,\frac{1}{f-a})$, $\overline{N}(r,a;f)=\overline{N}(r,\frac{1}{f-a})$, $N(r,\infty;f)=N(r,f)$ and $\overline{N}(r,\infty;f)=\overline{N}(r,f)$.

Also $\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}$ and $\tau(f)=\limsup_{r\to\infty}\frac{T(r,f)}{r^{\rho(f)}}$ $(0<\rho(f)<\infty)$ are respectively called the order and type of f. A meromorphic function f

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is said to be of minimal type if $\tau(f) = 0$, which can be found, for example, in [2, pp. 16–17].

In 1976 Yang [10] asked to investigate the relationship between two nonconstant entire functions f and g if f and g share the value 0 CM and $f^{(1)}$ and $g^{(1)}$ share the value 1 CM. Many authors, including Shibazaki [9], Yi [13, 14], Yang and Yi [11], Hua [4], Mues and Reinders [8], Lahiri [5, 6], studied the question. Further, Yi [16], Chen, Wang and Zhang [1], Li and Li [7] and others also worked on this question and its extensions.

In 1990 Yi [13] proved the following result.

Theorem A ([13]). *Let f and g be two nonconstant entire functions such that f, g share the value 0 CM and $f^{(n)}$, $g^{(n)}$ share the value 1 CM. If $\delta(0; f) > \frac{1}{2}$, then either $f \equiv g$ or $f^{(n)} \cdot g^{(n)} \equiv 1$.*

Shibazaki [9] did not consider the sharing of zeros and proved the following theorem.

Theorem B ([9]). *Let f and g be two nonconstant entire functions of finite order such that $f^{(1)}$, $g^{(1)}$ share the value 1 CM. If $\delta(0; f) > 0$ and 0 is a Picard exceptional value of g, then either $f \equiv g$ or $f^{(1)} \cdot g^{(1)} \equiv 1$.*

Yi and Yang [17], Hua [4] and many others improved Theorem B in different manners. Yi and Yang [17] proved the following result.

Theorem C ([17]). *Let f and g be two nonconstant meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share the value 1 CM. If $\Theta(\infty; f) = \Theta(\infty; g) = 1$ and $\delta(0; f) + \delta(0; g) > 1$, then either $f \equiv g$ or $f^{(n)} \cdot g^{(n)} \equiv 1$.*

Also Yi [16] proved the following improvement of Theorem B.

Theorem D ([16]). *Let f and g be two nonconstant meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share the values 1 and ∞ CM. If*

$$\delta(0; f) + \delta(0; g) + (n + 2)\Theta(\infty; f) > n + 3,$$

then either $f \equiv g$ or $f^{(n)} \cdot g^{(n)} \equiv 1$.

In [16] Yi proved some others results which improve previous ones.

Theorem E ([16]). *Let f and g be two nonconstant meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share the value 1 CM. If*

$$2\delta(0; f) + (n + 4)\Theta(\infty; f) > n + 5 \text{ and}$$
$$2\delta(0; g) + (n + 4)\Theta(\infty; g) > n + 5,$$

then either $f \equiv g$ or $f^{(n)} \cdot g^{(n)} \equiv 1$.

Theorem F ([16]). *Let f and g be two nonconstant meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share the value 1 IM. If*

$$5\delta(0; f) + (4n + 7)\Theta(\infty; f) > 4n + 11 \text{ and}$$

$$5\delta(0; g) + (4n + 7)\Theta(\infty; g) > 4n + 11,$$

then either $f \equiv g$ *or* $f^{(n)} \cdot g^{(n)} \equiv 1$.

In 1990 Yi [14] considered the uniqueness of entire functions when they share the value 0 CM and that their derivatives share the value 1 CM. The following result of H. X. Yi [14] is an answer to the question of C. C. Yang under a general setting.

Theorem G ([14])**.** *Let* f *and* g *be two nonconstant entire functions and let* k *be a positive integer. If* f *and* g *share the value* 0 *CM,* $f^{(k)}$ *and* $g^{(k)}$ *share the value* 1 *CM and* $\delta(0; f) > \frac{1}{2}$, *then either* $f \equiv g$ *or* $f^{(k)} \cdot g^{(k)} \equiv 1$.

Recently Li and Li [7] considered the problem of replacing the derivatives by linear differential polynomials generated by entire functions.

Let h be a nonconstant meromorphic function. An expression of the form

$$P(h) = h^{(k)} + a_{k-1}h^{(k-1)} + \cdots + a_1h^{(1)} + a_0h, \tag{1.1}$$

where $a_0, a_1, \ldots, a_{k-1}$ are complex constants and k is a positive integer, is called a linear differential polynomial generated by h.

Considering following example Li and Li [7] exhibited that it is not possible to replace $f^{(k)}$ and $g^{(k)}$ in Theorem G respectively by $P(f)$ and $P(g)$.

Example 1.1 ([7])**.** Let $f = \frac{1}{2}e^{-2z}$ and $g = e^{-2z}$. If $P(h) = h^{(2)} + 2h^{(1)}$, then f, g share the value 0 CM, $P(f)$, $P(g)$ share the value 1 CM and $\delta(0; f) = 1$ but $f \not\equiv g$ and $P(f) \cdot P(g) \not\equiv 1$.

We recall the following results from Li and Li[7].

Theorem H ([7])**.** *Let* f *and* g *be two nonconstant entire functions. Suppose that* f *and* g *share the value* 0 *CM,* $P(f)$ *and* $P(g)$ *share the value* 1 *CM and* $\delta(0; f) > \frac{1}{2}$. *If* $\rho(f) \neq 1$, *then* $f \equiv g$ *unless* $P(f) \cdot P(g) \equiv 1$.

Theorem I ([7])**.** *Let* f *and* g *be two nonconstant entire functions. Suppose that* f *and* g *share the value* 0 *CM,* $P(f)$ *and* $P(g)$ *share the value* 1 *IM and* $\delta(0; f) > \frac{4}{5}$. *If* $\rho(f) \neq 1$, *then* $f \equiv g$ *unless* $P(f) \cdot P(g) \equiv 1$.

We can easily note that in Example 1.1, $P(f) \equiv 0$ and $P(g) \equiv 0$. On the other hand, in the following example we see that if $P(f)$ and $P(g)$ are nonconstant, then for an entire function of order 1 the conclusion of Theorem H may hold.

Example 1.2. Let $f = e^z$ and $g = e^{-z}$ and $P(h) = h^{(3)} - h^{(2)} - h^{(1)}$. Then f and g share the value 0 CM, $P(f) = -e^z$ and $P(g) = -e^{-z}$ share the value 1 CM and $\delta(0; f) = 1$. Also $P(f) \cdot P(g) \equiv 1$.

In the present paper we extend the results of Li and Li [7] by including the class of entire functions of order 1. We also extend some previous results to homogeneous differential polynomials.

Let h be a nonconstant meromorphic function. An expression of the form

$$P(h)=\sum_{k=1}^{n}a_k\prod_{j=0}^{p}(h^{(j)})^{l_{kj}}, \tag{1.2}$$

where $a_k\in S(h)$ for $k=1,2,\ldots,n$ and l_{kj} are nonnegative integers for $k=1,2,\ldots,n$; $j=0,1,2,\ldots,p$ and $d=\sum_{j=0}^{p}l_{kj}$ for $k=1,2,\ldots,n$, is called a homogeneous differential polynomial of degree d generated by h. Also we denote by Q the quantity $Q=\max_{1\leq k\leq n}\sum_{j=0}^{p}jl_{kj}$.

Let f and g be two nonconstant meromorphic functions. When we consider $P(f)$ and $P(g)$, as defined by (1.2), and generated by f and g respectively, then we understand that the coefficients a_k $(k=1,2,\ldots,n)$ belong to $S(f)\cap S(g)$.

We now state the results of the paper.

Theorem 1.1. *Let f and g be two nonconstant meromorphic functions and $a=a(z)\in S(f)\cap S(g)$ and $a\not\equiv 0,\infty$. Suppose that $P(f)$ and $P(g)$, as defined by* (1.2), *are nonconstant. If $P(f)$ and $P(g)$ share $a=a(z)$ IM and*

$$\min\left\{5\delta(0;f)+\frac{4Q+7}{d}\Theta(\infty;f),\ 5\delta(0;g)+\frac{4Q+7}{d}\Theta(\infty;g)\right\}>\frac{4Q+4d+7}{d}, \tag{1.3}$$

then either $P(f)\equiv P(g)$ or $P(f)\cdot P(g)\equiv a^2$.

Remark 1. If $P(f)$ and $P(g)$ share $a=a(z)$ CM, then the condition (1.3) of Theorem 1.1 can be replaced by the following

$$\min\left\{2\delta(0;f)+\frac{Q+4}{d}\Theta(\infty;f),\ 2\delta(0;g)+\frac{Q+4}{d}\Theta(\infty;g)\right\}>\frac{Q+d+4}{d}.$$

Theorem 1.2. *Let f and g be two nonconstant meromorphic functions and $a=a(z)(\not\equiv 0,\infty)\in S(f)\cap S(g)$. Suppose that $P(f)$ and $P(g)$, as defined by* (1.2), *are nonconstant. If f and g share the values* 0 *CM and ∞ IM and $P(f)$, $P(g)$ share $a=a(z)$ IM and*

$$5\delta(0;f)+\frac{4Q+7}{d}\Theta(\infty;f)>\frac{4Q+4d+7}{d},$$

then either $P(f)\equiv P(g)$ or $P(f)\cdot P(g)\equiv a^2$.

Theorem 1.3. *Let f and g be two nonconstant entire functions and $a=a(z)(\not\equiv 0,\infty)\in S(f)\cap S(g)$. Suppose that $P(f)$ and $P(g)$, as defined by* (1.2), *are nonconstant. If f and g share the value* 0 *CM and $P(f)$, $P(g)$ share $a=a(z)$ CM and $\delta(0;f)>\frac{1}{2}$, then either $P(f)\equiv P(g)$ or $P(f)\cdot P(g)\equiv a^2$.*

Remark 2. If $P(f)$ and $P(g)$ share $a=a(z)$ IM, then the condition $\delta(0;f)>\frac{1}{2}$ of Theorem 1.3 has to be replaced by $\delta(0;f)>\frac{4}{5}$.

As the consequences of the main results we obtain the following corollaries.

Corollary 1.1. *Let f and g be two nonconstant meromorphic functions. Suppose that $\alpha(f^{(k)})^n$ and $\alpha(g^{(k)})^n$ are nonconstant and share the value 1 IM, where $\alpha(\neq 0)$ is a constant and k, n are positive integers. If*

$$\min\left\{5\delta(0;f)+\frac{4kn+7}{n}\Theta(\infty;f),\ 5\delta(0;g)+\frac{4kn+7}{n}\Theta(\infty;g)\right\}>\frac{4kn+4n+7}{n},$$

then either $\alpha^2(f^{(k)}g^{(k)})^n\equiv 1$ or $f\equiv\omega g$, where $\omega^n=1$.

If, in addition, $f(z_0)=g(z_0)\neq 0$ for some $z_0\in\mathbb{C}$, then $\omega=1$.

Corollary 1.2. *Let f and g be two nonconstant entire functions such that $P(f)$ and $P(g)$, as defined by* (1.1)*, are nonconstant. Suppose that f and g share the value* 0 *CM and $P(f)$, $P(g)$ share the value* 1 *CM. If $\delta(0;f)>\frac{1}{2}$, then either $f\equiv g$ or $P(f)\cdot P(g)\equiv 1$ under any one of the following hypotheses*:

(i) *$\rho(f)\neq 1$,*

(ii) *$\rho(f)=1$ and*

  (a) *f has at most a finite number of zeros, or*

  (b) *f has infinitely many zeros and f is of minimal type.*

We now recall some well known notations of the value distribution theory. Let F and G be two nonconstant meromorphic functions, which share the value 1 IM. We denote by $\overline{N}_L(r,1;F)$ the reduced counting function of those zeros of $F-1$ in $\{z :\mid z\mid< r\}$, which have larger multiplicities than those of the corresponding zeros of $G-1$. Also we denote by $N_E^{1)}(r,1;F)$ the reduced counting function of common simple zeros of $F-1$ and $G-1$ in $\{z :\mid z\mid< 1\}$, and denote by $\overline{N}_E^{(2}(r,1;F)$ the counting function of those common multiple zeros of $F-1$ and $G-1$ in $\{z :\mid z\mid< r\}$, where each such common multiple zero of $F-1$ and $G-1$ has the same multiplicity related to $F-1$ and $G-1$. Likewise we define $\overline{N}_L(r,1;G)$, $N_E^{1)}(r,1;G)$ and $\overline{N}_E^{(2}(r,1;G)$.

Also we denote by $N_{1)}(r,0;F)$ the counting function of simple zeros of F and by $\overline{N}_{(2}(r,0;F)$ the reduced counting function of multiple zeros of F in $\{z :\mid z\mid< r\}$.

2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1. *Let f be a nonconstant meromorphic function and $P(f)$ be defined by* (1.2)*. Then*

$$T(r,P)\leq dT(r,f)+Q\overline{N}(r,\infty;f)+S(r,f)$$

and

$$\begin{aligned}N(r,0;P)&\leq T(r,P)-dT(r,f)+dN(r,0;f)+S(r,f)\\&\leq Q\overline{N}(r,\infty;f)+dN(r,0;f)+S(r,f).\end{aligned}$$

Proof. Since

$$N(r,P) \le dN(r,f) + Q\overline{N}(r,\infty;f) + S(r,f) \text{ and}$$
$$m(r,f) \le m(r,\frac{P}{f^d}) + m(r,f^d) = dm(r,f) + S(r,f),$$

we get

$$T(r,P) \le dT(r,f) + Q\overline{N}(r,\infty;f) + S(r,f). \tag{2.1}$$

Now

$$m(r,0;f^d) \le m(r,0;P) + m(r,\frac{P}{f^d}) = m(r,0;P) + S(r,f)$$

and so

$$T(r,f^d) - N(r,0;f^d) \le T(r,P) - N(r,0;P) + S(r,f)$$

i.e.,

$$N(r,0;P) \le T(r,P) - dT(r,f) + dN(r,0;f) + S(r,f). \tag{2.2}$$

The lemma follows from (2.1) and (2.2). □

Lemma 2.2 ([16])**.** *Let F and G be two nonconstant meromorphic functions such that F and G share 1 IM. Then*

$$T(r,F) \le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + N_E^{1)}(r,1;F)$$
$$+ \overline{N}_L(r,1;F) - N_0(r,\infty;F^{(1)}) - N_0(r,0;G^{(1)}) + S(r,F) + S(r,G),$$

where $N_0(r,0;F^{(1)})$ denotes the counting function corresponding to the zeros of $F^{(1)}$ that are not zeros of F and $F-1$, $N_0(r,0;G^{(1)})$ denotes the counting function corresponding to the zeros of $G^{(1)}$ that are not zeros of G and $G-1$.

Lemma 2.3 ([2, p. 47])**.** *Let f be a nonconstant meromorphic function and a_1, a_2, a_3 be three distinct members of $S(f)$. Then*

$$T(r,f) \le \overline{N}(r,0;f-a_1) + \overline{N}(r,0;f-a_2) + \overline{N}(r,0;f-a_3) + S(r,f).$$

Lemma 2.4 ([3])**.** *Let f be a transcendental meromorphic function and $P(f)$, defined by (1.2), be nonconstant and $d \ge 1$. Then*

$$dT(r,f) \le \overline{N}(r,\infty;f) + \overline{N}(r,1;P(f)) + dN(r,0;f) - N_0(r,0;(P(f))^{(1)}) + S(r,f),$$

where $N_0(r,0;(P(f))^{(1)})$ denotes the counting function corresponding to the zeros of $(P(f))^{(1)}$ which are not the zeros of $P(f)$ and $P(f)-1$.

Remark 3. In fact Lemma 2.4 is a special case of Lemma 1 [3].

Lemma 2.5 ([12, p. 92])**.** *Suppose that f_1, f_2,…,f_n $(n \ge 3)$ are meromorphic functions which are not constants except for f_n. Furthermore, let $\sum_{j=1}^n f_j \equiv 1$. If $f_n \not\equiv 0$ and*

$$\sum_{j=1}^n N(r,0;f_j) + (n-1)\sum_{j=1}^n \overline{N}(r,\infty;f_j) < \{\lambda + o(1)\}T(r,f_k),$$

where $r \in I$, *a set of infinite linear measure,* $k = 1, 2, \ldots, n-1$ *and* $0 < \lambda < 1$, *then* $f_n \equiv 1$.

3. Proof of theorems and corollaries

Proof of Theorem 1.1. Let $F = \frac{P(f)}{a}$ and $G = \frac{P(g)}{a}$. Then F and G share 1 IM and so by Lemma 2.2 we get

$$\begin{aligned} T(r,F) \leq{}& \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + N_E^{1)}(r,1;F) \\ &+ \overline{N}_L(r,1;F) - N_0(r,0;F^{(1)}) - N_0(r,0;G^{(1)}) + S(r,F) + S(r,G). \end{aligned} \tag{3.1}$$

Let

$$H = \left(\frac{F^{(2)}}{F^{(1)}} - \frac{2F^{(1)}}{F-1}\right) - \left(\frac{G^{(2)}}{G^{(1)}} - \frac{2G^{(1)}}{G-1}\right).$$

We suppose that $H \not\equiv 0$. Then by a simple calculation we see that

$$\begin{aligned} N_E^{1)}(r,1;F) &\leq N(r,0;H) \\ &\leq T(r,H) \\ &\leq N(r,\infty;H) + S(r,F) + S(r,G) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} N(r,\infty;H) \leq{}& \overline{N}_{(2}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}_{(2}(r,0;G) + \overline{N}(r,\infty;G) \\ &+ \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + N_0(r,0;F^{(1)}) + N_0(r,0;G^{(1)}). \end{aligned} \tag{3.3}$$

Noting that $\overline{N}(r,0;F) + \overline{N}_{(2}(r,0;F) \leq N(r,0;F)$ and combining (3.1), (3.2) and (3.3) we get

$$\begin{aligned} T(r,F) \leq{}& N(r,0;F) + 2\overline{N}(r,\infty;F) + N(r,0;G) + 2\overline{N}(r,\infty;G) \\ &+ 2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + S(r,F) + S(r,G). \end{aligned} \tag{3.4}$$

Now by Lemma 2.1 and (3.4) we get

$$\begin{aligned} N(r,0;F) &\leq T(r,F) - dT(r,F) + dN(r,0;f) + S(r,f) \\ &\leq N(r,0;F) + 2\overline{N}(r,\infty;F) + Q\overline{N}(r,\infty;g) + dN(r,0;g) \\ &\quad + 2\overline{N}(r,\infty;g) + 2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) - dT(r,f) \\ &\quad + dN(r,0;f) + S(r,f) + S(r,g) \end{aligned}$$

and so

$$\begin{aligned} dT(r,f) \leq{}& dN(r,0;f) + 2\overline{N}(r,\infty;f) + dN(r,0;g) + (Q+2)\overline{N}(r,\infty;g) \\ &+ 2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + S(r,f) + S(r,g). \end{aligned} \tag{3.5}$$

Again using Lemma 2.1 we obtain

$$\begin{aligned} \overline{N}_L(r,1;F) &\leq N(r,1;F) - \overline{N}(r,1;F) \\ &\leq N(r,0;F^{(1)}) \\ &\leq N(r,0;F) + \overline{N}(r,\infty;F) + S(r,F) \end{aligned}$$

$$\leq dN(r,0;f) + (Q+1)\overline{N}(r,\infty;f) + S(r,f). \tag{3.6}$$

Similarly

$$\overline{N}_L(r,1;G) \leq dN(r,0;g) + (Q+1)\overline{N}(r,\infty;g) + S(r,g). \tag{3.7}$$

Combining (3.5), (3.6) and (3.7) we obtain

$$\begin{aligned} T(r,f) \leq\ & 3N(r,0;f) + \frac{2Q+4}{d}\overline{N}(r,\infty;f) + 2N(r,0;g) \\ & + \frac{2Q+3}{d}\overline{N}(r,\infty;g) + S(r,f) + S(r,g). \end{aligned} \tag{3.8}$$

Likewise we have

$$\begin{aligned} T(r,g) \leq\ & 3N(r,0;g) + \frac{2Q+4}{d}\overline{N}(r,\infty;g) + 2N(r,0;f) \\ & + \frac{2Q+3}{d}\overline{N}(r,\infty;f) + S(r,f) + S(r,g). \end{aligned} \tag{3.9}$$

Adding (3.8) and (3.9) we obtain

$$\begin{aligned} T(r,f) + T(r,g) \leq\ & 5N(r,0;f) + \frac{4Q+7}{d}\overline{N}(r,\infty;f) + 5N(r,0;g) \\ & + \frac{4Q+7}{d}\overline{N}(r,\infty;g) + S(r,f) + S(r,g), \end{aligned}$$

which implies a contradiction to the hypothesis. Therefore $H \equiv 0$ and so on integration we get

$$\frac{1}{G-1} = \frac{A}{F-1} + B,$$

where $A(\neq 0)$ and B are constants. This gives

$$G = \frac{(B+1)F + (A-B-1)}{BF + A - B} \tag{3.10}$$

and

$$F = \frac{(B-A)G + (A-B-1)}{BG - (B+1)}. \tag{3.11}$$

We now consider the following three cases.

Case 1: Let $B \neq 0, -1$. From (3.11) we have $\overline{N}(r, \frac{B+1}{B}; G) = \overline{N}(r,\infty;F)$. Now by the second fundamental theorem and Lemma 2.2 we get

$$\begin{aligned} T(r,G) &\leq N(r,0;G) + \overline{N}(r, \frac{B+1}{B}; G) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq T(r,G) - dT(r,g) + dN(r,0;g) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,g) \end{aligned}$$

i.e.,

$$dT(r,g) \leq dN(r,0;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,g). \tag{3.12}$$

If $A-B-1\neq 0$, from (3.10) we have $N(r,\frac{B+1-A}{B+1};F)=N(r,0;G)$. Hence by the second fundamental theorem and Lemma 2.2 we get

$$\begin{aligned} T(r,F) &\le N(r,0;F)+\overline{N}(r,\frac{B+1-A}{B+1};F)+\overline{N}(r,\infty;F)+S(r,F)\\ &\le T(r,F)-dT(r,f)+dN(r,0;f)+N(r,0;G)+\overline{N}(r,\infty;f)+S(r,f) \end{aligned}$$

i.e.,

$$\begin{aligned} dT(r,f) \le{}& dN(r,0;f)+dN(r,0;g)+\overline{N}(r,\infty;f)\\ &+Q\overline{N}(r,\infty;g)+S(r,f)+S(r,g). \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13) we obtain

$$\begin{aligned} T(r,f)+T(r,g) \le{}& N(r,0;f)+\frac{2}{d}\overline{N}(r,\infty;f)+2N(r,0;g)\\ &\frac{Q+1}{d}\overline{N}(r,\infty;g)+S(r,f)+S(r,g), \end{aligned}$$

a contradiction.

Hence $A-B-1=0$ and from (3.10) we get

$$G=\frac{(B+1)F}{BF+1}.$$

Therefore $\overline{N}(r,0;F+\frac{1}{B})=\overline{N}(r,\infty;G)$. Again by the second fundamental theorem and Lemma 2.2 we obtain

$$\begin{aligned} T(r,F) &\le N(r,0;F)+\overline{N}(r,0;F+\frac{1}{B})+\overline{N}(r,\infty;F)+S(r,F)\\ &\le T(r,F)-dT(r,f)+dN(r,0;f)+\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)+S(r,f) \end{aligned}$$

i.e.,

$$dT(r,f)\le dN(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,f). \tag{3.14}$$

Combining (3.12) and (3.14) we have

$$\begin{aligned} T(r,f)+T(r,g) \le{}& N(r,0;f)+N(r,0;g)+\frac{2}{d}\overline{N}(r,\infty;f)\\ &+\frac{2}{d}\overline{N}(r,\infty;g)+S(r,f)+S(r,g), \end{aligned}$$

a contradiction.

Case 2: We suppose that $B=0$. From (3.10) and (3.11) we have

$$G=\frac{F+A-1}{A} \quad\text{and}\quad F=AG+1-A.$$

If $A-1\neq 0$, then it follows that

$$N(r,1-A;F)=N(r,0;G) \quad\text{and}\quad N(r,\frac{A-1}{A};G)=N(r,0;F).$$

Using the similar argument of Case 1 we arrive at a contradiction. Therefore $A-1=0$ and so $P(f)\equiv P(g)$.

Case 3: We suppose that $B = -1$. From (3.10) and (3.11) we get

$$G = \frac{A}{A+1-F} \quad \text{and} \quad F = \frac{(A+1)G - A}{G}.$$

If $A + 1 \neq 0$, we obtain

$$\overline{N}(r, A+1; F) = \overline{N}(r, \infty; G) \quad \text{and} \quad N(r, \frac{A}{A+1}; G) = N(r, 0; F).$$

Using the similar argument of Case 1 we arrive at a contradiction. Therefore $A + 1 = 0$ and so $P(f)P(g) \equiv a^2$. This proves the theorem. □

Proof of Theorem 1.2. Let $F = \frac{P(f)}{a}$ and $G = \frac{P(g)}{a}$. Then F and G share 1 IM and so by Lemma 2.2 and Lemma 2.5 we get

$$\begin{aligned} dT(r, f) &\leq \overline{N}(r, \infty; f) + \overline{N}(r, 1; F) + dN(r, 0; f) + S(r, f) \\ &= \overline{N}(r, \infty; g) + \overline{N}(r, 1; G) + dN(r, 0; g) + S(r, f) \\ &\leq (1 + 2d + Q)T(r, g) + S(r, f) + S(r, g). \end{aligned} \tag{3.15}$$

Similarly

$$dT(r, g) \leq (1 + 2d + Q)T(r, f) + S(r, f) + S(r, g). \tag{3.16}$$

From (3.15) and (3.16) we get $S(r, f) = S(r, g)$. The rest of the proof is similar to that of Theorem 1.1. This proves the theorem. □

Proof of Corollary 1.1. By Theorem 1.1 we get either $\alpha^2(f^{(k)}g^{(k)})^n \equiv 1$ or $(f^{(k)})^n \equiv (g^{(k)})^n$. We suppose that $(f^{(k)})^n \equiv (g^{(k)})^n$. Then $f^{(k)} = \omega g^{(k)}$, where ω is a constant satisfying $\omega^n = 1$. Integrating k times we obtain $f = \omega g + p$, where p is a polynomial of degree at most $k - 1$. From the hypothesis it is clear that f and g are transcendental meromorphic functions. If $p \not\equiv 0$, by Lemma 2.3 we get

$$\begin{aligned} T(r, f) &\leq N(r, 0; f) + N(r, 0; f - p) + \overline{N}(r, \infty; f) + S(r, f) \\ &= N(r, 0; f) + N(r, 0; g) + \overline{N}(r, \infty; f) + S(r, f) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} T(r, g) &\leq N(r, 0; g) + N(r, 0; g + \frac{p}{\omega}) + \overline{N}(r, \infty; g) + S(r, g) \\ &= N(r, 0; g) + N(r, 0; f) + \overline{N}(r, \infty; g) + S(r, g). \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18) we obtain

$$\begin{aligned} T(r, f) + T(r, g) \leq\ & 2N(r, 0; f) + 2N(r, 0; g) + \overline{N}(r, \infty; f) \\ & + \overline{N}(r, \infty; g) + S(r, f) + S(r, g), \end{aligned}$$

which contradicts the hypothesis. Therefore $p \equiv 0$ and so $f \equiv \omega g$.

If, further, $f(z_0) = g(z_0) \neq 0$ for some $z_0 \in \mathbb{C}$, then clearly $\omega = 1$ and so $f \equiv g$. This proves the corollary. □

Proof of Corollary 1.2. By Theorem 1.3 we get either $P(f) \equiv P(g)$ or $P(f) \cdot P(g) \equiv 1$. Let $P(f) \equiv P(g)$ so that $P(g-f) \equiv 0$. Then

$$g - f = \sum_{j=1}^{m} p_j(z) e^{\alpha_j z}, \tag{3.19}$$

where $m(\leq k)$ is a positive integer, α_j's are distinct complex constants and $p_j(z)$'s are nonzero polynomials.

Since f and g share 0 CM, we can put $g = f \cdot e^h$, where h is an entire function.

Let $e^h \not\equiv 1$, otherwise we are done. So from (3.19) we get

$$f = \frac{\sum_{j=1}^{m} p_j(z) e^{\alpha_j z}}{e^h - 1}.$$

Since f is entire, we see that $N(r, 0; e^h - 1) \leq N(r, 0; \sum_{j=1}^{m} p_j(z) e^{\alpha_j z})$ and by the second fundamental theorem we get

$$\begin{aligned}
T(r, e^h) &\leq \overline{N}(r, \infty; e^h) + \overline{N}(r, 0; e^h) + \overline{N}(r, 0; e^h - 1) + S(r, e^h) \\
&\leq N\left(r, 0; \sum_{j=1}^{m} p_j(z) e^{\alpha_j z}\right) + S(r, e^h) \\
&\leq T\left(r, \sum_{j=1}^{m} p_j(z) e^{\alpha_j z}\right) + S(r, e^h) \\
&\leq \sum_{j=1}^{m} \{T(r, p_j(z)) + T(r, e^{\alpha_j z})\} + S(r, e^h) \\
&= O(\log r) + O(r) + S(r, e^h). \qquad (3.20)
\end{aligned}$$

If h is transcendental or a polynomial of degree at least 2, then from (3.20) we see that $T(r, e^h) = S(r, e^h)$, contradiction. Hence h is a polynomial of degree at most 1.

First we assume that h is a constant. Then $P(f) \equiv P(g) \equiv e^h P(f)$ and so $e^h \equiv 1$, which contradicts our assumption.

Next we assume that $h(z) = az + b$, where $a(\neq 0)$ and b are constants. Then

$$f = \frac{\sum_{j=1}^{m} p_j(z) e^{\alpha_j z}}{e^{az+b} - 1} \quad \text{and so} \quad \rho(f) \leq 1.$$

We now consider the following cases.

Case 1: Let $\rho(f) < 1$.

Then by Milloux basic result [2, Theorem 3.2, p. 57] we get

$$\begin{aligned}
T(r, f) &\leq N(r, 0; f) + \overline{N}(r, 1; P(f)) + S(r, f) \\
&= N(r, 0; g) + \overline{N}(r, 1; P(g)) + S(r, f) \\
&\leq T(r, g) + T(r, P(g)) + S(r, f)
\end{aligned}$$

$$= T(r,g) + m(r, P(g)) + S(r,f)$$
$$\leq T(r,g) + m(r,g) + m\left(r, \frac{P(g)}{g}\right) + S(r,f)$$
$$= 2T(r,g) + S(r,g) + S(r,f). \tag{3.21}$$

Similarly

$$T(r,g) \leq 2T(r,f) + S(r,f) + S(r,g). \tag{3.22}$$

Since f and so g is of finite order, from (3.21) and (3.22) we see that $\rho(f) = \rho(g)$. Therefore

$$\rho(e^{az+b}) = \rho\left(\frac{g}{f}\right) \leq \max\{\rho(f),\ \rho(g)\} < 1,$$

which is impossible as $a \neq 0$.

Case 2: Let $\rho(f) = 1$.

We now consider the following subcases.

Subcase 2.1: Let f have at most a finite number of zeros.

We put $f(z) = q(z)e^{cz+d}$, where $q(z)$ is a polynomial. Then

$$g(z) = q(z)e^{(a+c)z+(b+d)}$$

and so $P(f) \equiv P(g)$ implies

$$q_1(z)e^{cz+d} = q_2(z)e^{(a+c)z+(b+d)},$$

where q_1, q_2 are polynomials. This implies $q_2(z)e^{az+b} = q_1(z)$, which is impossible as $a \neq 0$.

Subcase 2.2: Let f have infinitely many zeros and f be of minimal type.

We put

$$H_j(z) = -\frac{p_j(z)e^{\alpha_j z}}{f} \text{ for } 1 \leq j \leq m, \text{ and } H_{m+1}(z) = e^{az+b}.$$

Then $f = \dfrac{\sum_{j=1}^{m} p_j(z)e^{\alpha_j z}}{e^{az+b} - 1}$ implies

$$\sum_{j=1}^{m+1} H_j(z) \equiv 1. \tag{3.23}$$

Let one of α_j's, say α_1 be zero. Then $H_1 \not\equiv 0$ and we rewrite (3.23) as

$$\sum_{j=2}^{m+1} H_j(z) + H_1(z) \equiv 1.$$

Now

$$\sum_{j=1}^{m+1} N(r,0;H_j) + m\sum_{j=1}^{m+1} \overline{N}(r,\infty;H_j) = \sum_{j=1}^{m+1} N(r,0;p_j) + m^2\overline{N}(r,0;f)$$
$$= O(\log r) + m^2\overline{N}(r,0;f). \tag{3.24}$$

Since $e^{\alpha_j z} = -\frac{H_j(z)}{p_j(z)} f$, we get

$$T(r, e^{\alpha_j z}) \le T(r, H_j) + T(r, f) + O(\log r).$$

This implies

$$\frac{|\alpha_j|}{\pi} \le \frac{T(r, H_j)}{r} + \frac{T(r, f)}{r} + o(1)$$

and so

$$\liminf_{r\to\infty} \frac{T(r, H_j)}{r} + \limsup_{r\to\infty} \frac{T(r, f)}{r} \ge \frac{|\alpha_j|}{\pi}.$$

Since f is of minimal type, we get

$$\liminf_{r\to\infty} \frac{T(r, H_j)}{r} \ge K \text{ for } j = 2, 3, \ldots, m,$$

where $K = \min_{2\le j\le m} \frac{|\alpha_j|}{\pi} > 0$.

Hence for $j = 1, 2, \ldots, m$ we get

$$\limsup_{r\to\infty} \frac{\overline{N}(r, 0; f)}{T(r, H_j)} \le \limsup_{r\to\infty} \frac{T(r, f)}{r} \cdot \limsup_{r\to\infty} \frac{r}{T(r, H_j)} = 0.$$

Also

$$\limsup_{r\to\infty} \frac{\overline{N}(r, 0; f)}{T(r, H_{m+1})} \le \frac{\pi}{|a|} \limsup_{r\to\infty} \frac{T(r, f)}{r} = 0.$$

So from (3.24) we see that

$$\sum_{j=1}^{m+1} N(r, 0; H_j) + m \sum_{j=1}^{m+1} \overline{N}(r, \infty; H_j) < \{\lambda + o(1)\} T(r, H_k)$$

for $k = 2, 3, \ldots, m+1$, where λ $(0 < \lambda < 1)$ is a suitable constant.

Therefore by Lemma 2.5 we get $H_1(z) \equiv 1$, which is impossible as $\rho(f) = 1$.

So, $\alpha_j \ne 0$ for $j = 1, 2, \ldots, m$. Now adopting the same technique as above we get $H_{m+1}(z) \equiv 1$, which contradicts our assumption that $e^h \not\equiv 1$. This proves the corollary. $\square$

Remark 4. It is an interesting open problem to examine the validity of corollary 1.2 for entire functions f and g where f is of unit order with nonminimal type and f has infinitely many zeros.

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