Bull. Korean Math. Soc. **54** (2017), No. 3, pp. 817–824
https://doi.org/10.4134/BKMS.b160319
pISSN: 1015-8634 / eISSN: 2234-3016

RIGIDITY OF GRADIENT SHRINKING AND EXPANDING RICCI SOLITONS

Fei Yang and Liangdi Zhang

Abstract. In this paper, we prove that a gradient shrinking Ricci soliton is rigid if the radial curvature vanishes and the second order divergence of Bach tensor is non-positive. Moreover, we show that a complete non-compact gradient expanding Ricci soliton is rigid if the radial curvature vanishes, the Ricci curvature is nonnegative and the second order divergence of Bach tensor is nonnegative.

1. Introduction

A complete Riemannian manifold (M^n, g) is called a gradient Ricci soliton if there exists a smooth function $f : M^n \to \mathbb{R}$ such that the Ricci tensor Ric of the metric g satisfies the equation

$$Ric + \nabla^2 f = \lambda g \tag{1.1}$$

for some constant $\lambda \in \mathbb{R}$. The soliton is shrinking, steady or expanding Ricci soliton if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

The classification of gradient shrinking Ricci solitons under some conditions on the Weyl tensor and its derivatives has been a subject of interest for many people in recent years. M. Eminenti, G. La Nave and C. Mantegazza [9] proved that an n-dimensional compact shrinking Ricci soliton with vanishing Weyl tensor is a finite quotient of $\mathbb{S}^n$. The work of P. Petersen and W. Wylie [13] implied that a gradient shrinking Ricci soliton is a finite quotient of $\mathbb{R}^n$, $\mathbb{S}^{n-1} \times \mathbb{R}$, or $\mathbb{S}^n$ if the Weyl tensor vanishes and $\int_M |Ric|^2 e^{-f} dvol_g < \infty$. This integral assumption was proven to be true for gradient shrinking Ricci solitons (see [11, Theorem 1.1]). Moreover, H. D. Cao and Q. Chen [4] proved that an n-dimensional complete non-compact locally conformally flat gradient steady Ricci soliton is either flat or isometric to the Bryant soliton.

M. Fernández-López and E. García-Río [10] proved that a compact Ricci soliton is rigid if and only if it has harmonic Weyl tensor. In [11], O. Munteanu

Received April 13, 2016; Revised August 10, 2016.
2010 *Mathematics Subject Classification.* 53C25, 53C24.
Key words and phrases. rigidity, gradient Ricci soliton, Bach tensor.
This work is partially supported by National Natural Science Foundation of China (No. 11601495).

and N. Sesum [11] proved that a complete non-compact gradient shrinking Ricci soliton is rigid if it has harmonic Weyl tensor.

H. D. Cao and Q. Chen [3] studied the classification of Bach-flat gradient shrinking Ricci solitons. They proved that a 4-dimensional complete Bach-flat ($B_{ij} = 0$) gradient shrinking Ricci soliton is either Einstein or a finite quotient of $\mathbb{R}^4$ or $\mathbb{S}^3 \times \mathbb{R}$. More generally, they proved that an n-dimensional ($n \geq 5$) complete Bach-flat ($B_{ij} = 0$) gradient shrinking Ricci soliton is either Einstein, a finite quotient of Gaussian shrinking soliton $\mathbb{R}^n$ or $N^{n-1} \times \mathbb{R}$ with N being an Einstein manifold of positive scalar curvature. Moreover, H. D. Cao, G. Catino, Q. Chen, C. Mantegazza and L. Mazzieri [2] proved that a complete Bach-flat gradient steady Ricci soliton with positive Ricci curvature such that the scalar curvature attains its maximum at some interior point is isometric to the Bryant soliton. They also proved that a 3-dimensional steady gradient Ricci soliton with divergence-free Bach tensor is either flat or isometric to the Bryant soliton up to a scaling factor.

G. Catino, P. Mastrolia and D.D. Monticelli [6] proved that a gradient shrinking Ricci soliton is rigid if $div^4 W = 0$ ($div^4 W := \nabla_k \nabla_j \nabla_i \nabla_l W_{ijkl}$). In particular, they showed that a 3-dimensional gradient steady Ricci soliton with $div^3 C = 0$ ($div^3 C := \nabla_k \nabla_j \nabla_i C_{ijk}$) is isometric to a finite quotient of $\mathbb{R}^3$ or the Bryant soliton up to scaling. Moreover, an expanding Ricci soliton with $div^3 C = 0$ and $Ric \geq 0$ is rotationally symmetric. They showed that $div^4 W = 0$ is equivalent to $div^3 C = 0$ if $n \geq 4$ and $div^2 B = 0$ is equivalent to $div^3 C = 0$ if $n = 3$. We will see that last equivalence does not always hold for $n \geq 4$ in Section 2.

For a Ricci soliton, we say that the *radial curvature vanishes* if $Rm(\cdot, \nabla f)\nabla f = 0$ (see [12]). A Ricci soliton is called *radially flat* if $sec(E, \nabla f) = 0$ (see [12]). A gradient soliton is *rigid* if it is of the type $N^{n-k} \times_\Gamma \mathbb{R}^k$, where Γ acts freely on N and by orthogonal transformations on $\mathbb{R}^k$ with N being Einstein with Einstein constant λ and $\mathbb{R}^k$ the Gaussian soliton with $f = \frac{\lambda}{2}|x|^2$.

Our aim in this paper is to prove that a gradient shrinking Ricci soliton is rigid if $div^2 B \leq 0$ and $Rm(\cdot, \nabla f)\nabla f = 0$. Moreover, a complete non-compact gradient expanding Ricci soliton is rigid if $Ric \geq 0$, $div^2 B \geq 0$ and $Rm(\cdot, \nabla f)\nabla f = 0$. These results are generalizations of the classification of Bach-flat shrinking gradient Ricci solitons (see [3]) and the classification of 3-dimensional expanding gradient Ricci soliton with $div^3 C = 0$ (see [6]), respectively.

The purpose of this article is to prove the following rigid theorems.

Theorem 1.1. *Let (M^n, f, g) ($n \geq 5$) be a complete gradient shrinking Ricci soliton. If the radial curvature vanishes and $div^2 B \leq 0$, then the soliton is a finite quotient of $N^{n-k} \times \mathbb{R}^k$ ($0 \leq k \leq n$), the product of an Einstein manifold N with positive scalar curvature and the Gaussian shrinking soliton $\mathbb{R}^k$.*

Theorem 1.2. *Let (M^n, f, g) ($n \geq 5$) be a complete non-compact gradient expanding Ricci soliton. If the radial curvature vanishes, $Ric \geq 0$ and $div^2 B \geq$*

0, *then the soliton is a finite quotient of* $N^{n-k} \times \mathbb{R}^k$ $(0 \leq k \leq n)$, *the product of an Einstein manifold* N *and the Gaussian expanding soliton* $\mathbb{R}^k$.

We arrange this paper as follows. In Section 2, we give the notations needed in this paper. In Section 3, we prove Theorems 1.1-1.2.

2. Preliminaries

On an n-dimensional Riemannian manifold (M^n, g) $(n \geq 4)$, the Weyl tensor is given by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}),$$

the Cotton tensor by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R).$$

In fact,

$$C_{ijk} = -C_{jik}, \quad g^{ij}C_{ijk} = g^{ik}C_{ijk} = 0, \tag{2.2}$$

$$C_{ijk} = -\frac{n-2}{n-3}\nabla_l W_{ijkl}. \tag{2.3}$$

The covariant 3-tensor D_{ijk} is defined as

$$D_{ijk} = \frac{1}{n-2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) + \frac{1}{2(n-1)(n-2)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) - \frac{R}{(n-1)(n-2)}(g_{jk}\nabla_i f - g_{ik}\nabla_j f),$$

and the Bach tensor is given by

$$B_{ij} = \frac{1}{n-2}(\nabla_k C_{kij} + R_{kl}W_{ikjl}).$$

Proposition 2.1 (H. D. Cao and Q. Chen [3]). *If* (M^n, f, g) $(n \geq 4)$ *is a complete gradient Ricci soliton satisfying* (1.1), *we have*

$$D_{ijk} = C_{ijk} + W_{ijkl}\nabla_l f. \tag{2.4}$$

$$|D|^2 = \frac{1}{(n-2)^2}(|R_{jk}\nabla_i f - R_{ik}\nabla_j f|^2 - \frac{2}{n-1}|\frac{1}{2}\nabla R - R\nabla f|^2). \tag{2.5}$$

$$\nabla_j B_{ij} = \frac{n-4}{(n-2)^2}C_{ijk}R_{jk}. \tag{2.6}$$

Remark 2.1. We study the relation between $div^2 B$ and $div^3 C$ here. Calculating directly, we have

$$\nabla_j\nabla_i B_{ij} = \frac{1}{n-2}\nabla_j\nabla_i(\nabla_k C_{kij} + R_{kl}W_{ikjl})$$

$$
\begin{aligned}
&= \frac{1}{n-2}(\nabla_j\nabla_i\nabla_k C_{kij} + \nabla_i R_{kl}\nabla_j W_{ikjl} \\
&\quad + \nabla_j R_{kl}\nabla_i W_{ikjl} + \nabla_j\nabla_i R_{kl} W_{ikjl} + R_{kl}\nabla_j\nabla_i W_{ikjl}).
\end{aligned}
$$

Note that

$$\nabla_i R_{kl}\nabla_j W_{ikjl} = -\frac{n-3}{n-2}\nabla_i R_{kl} C_{kil},$$

and

$$\nabla_j R_{kl}\nabla_i W_{ikjl} = -\frac{n-3}{n-2}\nabla_j R_{kl} C_{ljk},$$

we obtain

$$
\begin{aligned}
&\nabla_i R_{kl}\nabla_j W_{ikjl} + \nabla_j R_{kl}\nabla_i W_{ikjl} \\
= &-\frac{2(n-3)}{n-2}\nabla_i R_{kl} C_{kil} = \frac{n-3}{n-2}|C|^2.
\end{aligned}
$$

Moreover, we have

$$\nabla_j\nabla_i R_{kl} W_{ikjl} = \frac{1}{2}\nabla_j(\nabla_i R_{kl} - \nabla_k R_{il})W_{ikjl} = \frac{1}{2}\nabla_j C_{ikl} W_{ikjl} = -\frac{1}{2}\nabla_l C_{ijk} W_{ijkl},$$

and

$$R_{kl}\nabla_j\nabla_i W_{ikjl} = -\frac{n-3}{n-2}R_{kl}\nabla_j C_{ljk} = \frac{n-3}{n-2}R_{jk}\nabla_i C_{ijk}.$$

Therefore, the relation between $div^3 C := \nabla_j\nabla_i\nabla_k C_{kij}$ and $div^2 B := \nabla_i\nabla_j B_{ij}$ is

$$(n-2)div^2 B = div^3 C + \frac{n-3}{n-2}|C|^2 - \frac{1}{2}\nabla_l C_{ijk} W_{ijkl} + \frac{n-3}{n-2}R_{jk}\nabla_i C_{ijk}.$$

We can see that $div^2 B = 0$ is equivalent to $div^3 C = 0$ in dimension 3 and it does not always hold for $n \geq 4$.

3. Proof of main results

Before we prove Theorems 1.1 and 1.2, we present a useful formula.

Lemma 3.1. *Let (M^n, f, g) $(n \geq 4)$ be a gradient Ricci soliton satisfying* (1.1). *Then we have*

$$\nabla_j B_{ij}\nabla_i f = \frac{n-4}{2(n-2)^2}\left(\frac{|\nabla R|^2}{2n-2} - \frac{R\langle\nabla R, \nabla f\rangle}{n-1} - 2R_{ijkl}\nabla_i f R_{jk}\nabla_l f\right). \tag{3.7}$$

Proof. By direct computations, we have

$$
\begin{aligned}
\nabla_j B_{ij}\nabla_i f &= \frac{n-4}{(n-2)^2}C_{ijk}R_{jk}\nabla_i f \\
&= \frac{n-4}{2(n-2)^2}C_{ijk}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) \\
&= \frac{n-4}{2(n-2)}C_{ijk}D_{ijk} \\
&= \frac{n-4}{2(n-2)}(|D|^2 - D_{ijk}W_{ijkl}\nabla_l f), \qquad (3.8)
\end{aligned}
$$

where we used (2.6) in the first equality. In the second and third equalities, we used (2.2). Moreover, we used (2.5) in the last equality.

Since $Ric(\nabla f, \cdot) = \frac{1}{2}\nabla R$, we have

$$\begin{aligned}
W_{ijkl}\nabla_l f &= R_{ijkl}\nabla_l f - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik})\nabla_l f \\
&\quad + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk})\nabla_l f \\
&= R_{ijkl}\nabla_l f - \frac{1}{2(n-2)}(g_{ik}\nabla_j R - g_{jk}\nabla_i R) \\
&\quad + \frac{1}{n-2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) + \frac{R}{(n-1)(n-2)}(g_{ik}\nabla_j f - g_{jk}\nabla_i f).
\end{aligned}$$

Hence,

(3.9)

$$\begin{aligned}
&D_{ijk}W_{ijkl}\nabla_l f \\
&= \frac{1}{n-2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f)W_{ijkl}\nabla_l f \\
&= \frac{2}{n-2}W_{ijkl}\nabla_l f R_{jk}\nabla_i f \\
&= \frac{2}{n-2}R_{ijkl}\nabla_l f R_{jk}\nabla_i f - \frac{1}{(n-2)^2}(\frac{|\nabla R|^2}{2} - R\langle\nabla R, \nabla f\rangle) \\
&\quad + \frac{2}{(n-2)^2}(|Ric|^2|\nabla f|^2 - \frac{|\nabla R|^2}{4}) + \frac{2R}{(n-1)(n-2)^2}(\frac{\langle\nabla R, \nabla f\rangle}{2} - R|\nabla f|^2) \\
&= \frac{2}{n-2}R_{ijkl}\nabla_l f R_{jk}\nabla_i f - \frac{1}{(n-2)^2}|\nabla R|^2 + \frac{n}{(n-1)(n-2)^2}R\langle\nabla R, \nabla f\rangle \\
&\quad + \frac{2}{(n-2)^2}|Ric|^2|\nabla f|^2 - \frac{2}{(n-1)(n-2)^2}R^2|\nabla f|^2.
\end{aligned}$$

From (2.5), we have

(3.10)

$$\begin{aligned}
|D|^2 &= \frac{1}{(n-2)^2}(|R_{jk}\nabla_i f - R_{ik}\nabla_j f|^2 - \frac{2}{n-1}|\frac{1}{2}\nabla R - R\nabla f|^2) \\
&= \frac{2}{(n-2)^2}|Ric|^2|\nabla f|^2 - \frac{1}{2(n-2)^2}|\nabla R|^2 - \frac{2}{(n-1)(n-2)^2}R^2|\nabla f|^2 \\
&\quad + \frac{2}{(n-1)(n-2)^2}R\langle\nabla R, \nabla f\rangle - \frac{1}{2(n-1)(n-2)^2}|\nabla R|^2 \\
&= \frac{2}{(n-2)^2}|Ric|^2|\nabla f|^2 - \frac{n}{2(n-1)(n-2)^2}|\nabla R|^2 \\
&\quad - \frac{2}{(n-1)(n-2)^2}R^2|\nabla f|^2
\end{aligned}$$

$$+\frac{2}{(n-1)(n-2)^2}R\langle\nabla R,\nabla f\rangle.$$

Combining (3.9) and (3.10), we obtain

$$\begin{aligned}&|D|^2-D_{ijk}W_{ijkl}\nabla_l f\\&=\frac{1}{2(n-1)(n-2)}|\nabla R|^2-\frac{1}{(n-1)(n-2)}R\langle\nabla R,\nabla f\rangle\\&-\frac{2}{n-2}R_{ijkl}\nabla_i fR_{jk}\nabla_l f.\end{aligned}\tag{3.11}$$

Plugging (3.11) into (3.8), (3.7) follows. □

We are ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We divide the arguments into two cases:

• Case 1: $\nabla f=0$ on some non-empty open set. Since every complete Ricci soliton is real analytic in suitable coordinates (see [1] and [7, Theorem 2.4]), we have $\nabla f\equiv 0$ on M^n. It follows that M^n is Einstein.

• Case 2: The set $\{p\in M|\nabla f(p)\neq 0\}$ is dense in M.

Since $Rm(\cdot,\nabla f)\nabla f=0$, $\langle\nabla R,\nabla f\rangle=2Ric(\nabla f,\nabla f)=0$. By Lemma 3.1, we obtain

$$\nabla_j B_{ij}\nabla_i f=\frac{n-4}{4(n-1)(n-2)^2}|\nabla R|^2\geq 0.\tag{3.12}$$

Let $\phi(t)=\frac{s-t}{s}$ on $[0,s]$ and $\phi=0$ on $t\geq s$ for any fixed $s>0$.

Since f is of quadratic growth (see [5]), $e^{-f}\phi(f)$ has compact support for any fixed $s>0$. Integrating by parts, we have

$$\int_M\nabla_j B_{ij}\nabla_i fe^{-f}\phi(f)=\int_M\nabla_i\nabla_j B_{ij}e^{-f}\phi(f)+\int_M\nabla_j B_{ij}\nabla_i fe^{-f}\phi'(f).\tag{3.13}$$

Note that $\phi\geq 0$, $\phi'\leq 0$ and $\nabla_i\nabla_j B_{ij}\leq 0$. Combining (3.12) with (3.13), we have

$$\int_M\nabla_j B_{ij}\nabla_i fe^{-f}\phi(f)=0.$$

From (3.12), we obtain $\nabla R=0$ on the compact set $\{x\in M:f(x)\leq s\}$. By taking $s\to+\infty$, $\nabla R=0$ on M. Therefore, R is a constant on M.

It follows from $Rm(\cdot,\nabla f)\nabla f=0$ that $sec(E,\nabla f)=0$. Note that a gradient Ricci soliton is rigid if it is radially flat and has constant scalar curvature (see [12, Theorem 1.2]). Moreover, every gradient shrinking Ricci soliton has nonnegative scalar curvature (see [8, Corollary 2.5]). In this case, we obtain that the soliton is a finite quotient of $N^{n-k}\times\mathbb{R}^k$ $(1\leq k\leq n)$, the product of an Einstein manifold N with positive scalar curvature and the Gaussian shrinking soliton $\mathbb{R}^k$.

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We divide the arguments into two cases:

• Case 1: $\nabla f = 0$ on some non-empty open set. Since every complete Ricci soliton is real analytic in suitable coordinates (see [1] and [7, Theorem 2.4]), we have $\nabla f \equiv 0$ on M^n. It follows that M is Einstein.

• Case 2: The set $\{p \in M | \nabla f(p) \neq 0\}$ is dense in M.

Recall that $\phi(t) = \frac{s-t}{s}$ on $[0, s]$ and $\phi = 0$ on $t \geq s$ for any fixed $s > 0$. Since $Ric \geq 0$, $-f$ is of quadratic growth (see [2, Lemma 5.5]). Therefore, $e^f \phi(-f)$ has compact support for any fixed $s > 0$. Integrating by parts, we obtain

$$\int_M \nabla_j B_{ij} \nabla_i f e^f \phi(-f) = -\int_M \nabla_i \nabla_j B_{ij} e^f \phi(-f) + \int_M \nabla_j B_{ij} \nabla_i f e^f \phi'(-f). \tag{3.14}$$

Note that $\phi \geq 0$, $\phi' \leq 0$ and $\nabla_i \nabla_j B_{ij} \geq 0$. Combining (3.12) with (3.14), we have

$$\int_M \nabla_j B_{ij} \nabla_i f e^{-f} \phi(f) = 0.$$

Hence, $\nabla R = 0$ on the compact set $\{x \in M | -f(x) \leq s\}$. Taking $s \to +\infty$, we have R is a constant on M.

It follows from $Rm(\cdot, \nabla f)\nabla f = 0$ that $sec(E, \nabla f) = 0$. Note that a gradient Ricci soliton is rigid if it is radially flat and has constant scalar curvature (see [12, Theorem 1.2]). In this case, we obtain that the soliton is a finite quotient of $N^{n-k} \times \mathbb{R}^k$ $(1 \leq k \leq n)$, the product of an Einstein manifold N and the Gaussian shrinking soliton $\mathbb{R}^k$.

This completes the proof of Theorem 1.2. $\square$

Acknowledgements. We would like to thank Jia-Yong Wu for useful discussions.

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Fei Yang
School of Mathematics and Physics
China University of Geosciences
Wuhan, P. R. China
E-mail address: yangfei810712@163.com

Liangdi Zhang
School of Mathematics and Physics
China University of Geosciences
Wuhan, P. R. China
E-mail address: zliangd@sina.com