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EXISTENCE THEOREM FOR NON-ABELIAN VORTICES IN THE AHARONY–BERGMAN–JAFFERIS–MALDACENA THEORY

Ruifeng Zhang and Meili Zhu

Abstract. In this paper, we discuss the existence theorem for multiple vortex solutions in the non-Abelian Chern–Simons–Higgs field theory developed by Aharony, Bergman, Jafferis, and Maldacena, on a doubly periodic domain. The governing equations are of the BPS type and derived by Auzzi and Kumar in the mass-deformed framework labeled by a continuous parameter. Our method is based on fixed point method.

1. Introduction

Vortices in non-Abelian gauge field theory play important roles in confinement mechanism and are governed by systems of nonlinear elliptic equations of complicated structures [2, 4, 7, 8, 10, 11, 12, 13, 14, 16, 22, 28, 30]. In this paper, we will focus on the vortex equations in the non-Abelian Chern–Simons–Higgs field theory developed by Aharony, Bergman, Jafferis, and Maldacena [1], known as the ABJM model, on a doubly periodic domain. The governing equations are of the BPS type and derived by Auzzi and Kumar [5] in the mass-deformed framework labeled by a continuous parameter. Developing and extending the methods of [6, 15, 17, 18, 19, 20, 21, 24, 27], we obtain the existence of a multiple vortex solution.

Recall that the ABJM model [1] is a Chern–Simons–Higgs theory within which the matter fields are four complex scalars,

$$C^I = (Q^1, Q^2, R^1, R^2), \quad I = 1, 2, 3, 4, \tag{1.1}$$

in the bifundamental matter field $(\mathbf{N}, \overline{\mathbf{N}})$ representation of the gauge group $U(N) \times U(N)$, which hosts two gauge fields, A_μ and B_μ. The Chern–Simons action associated to the two gauge group A_μ and B_μ of levels $+k$ and $-k$ is

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given by the Lagrangian density

$$\mathcal{L}_{\rm CS} = \frac{k}{4\pi}\epsilon^{\mu\nu\gamma}\mathrm{Tr}\bigg(A_\mu\partial_\nu A_\gamma + \frac{2\mathrm{i}}{3}A_\mu A_\nu A_\gamma - B_\mu\partial_\nu B_\gamma - \frac{2\mathrm{i}}{3}B_\mu B_\nu B_\gamma\bigg), \tag{1.2}$$

where the gauge-covariant derivatives on the bifundamental fields are defined as

$$D_\mu C^I = \partial_\mu C^I + \mathrm{i}A_\mu C^I - \mathrm{i}C^I B_\mu, \quad I = 1,2,3,4. \tag{1.3}$$

The scalar potential of the mass deformed theory can be written in a compact way as [9]

$$V = \mathrm{Tr}(M^{\alpha\dagger}M^\alpha + N^{\alpha\dagger}N^\alpha), \tag{1.4}$$

where

$$\begin{aligned} M^\alpha = {} & \rho Q^\alpha + \frac{2\pi}{k}(2Q^{[\alpha}Q^\dagger_\beta Q^{\beta]} + R^\beta R^\dagger_\beta Q^\alpha - Q^\alpha R^\dagger_\beta R^\beta \\ & + 2Q^\beta R^\dagger_\beta R^\alpha - 2R^\alpha R^\dagger_\beta Q^\beta), \end{aligned} \tag{1.5}$$

$$\begin{aligned} N^\alpha = {} & -\rho R^\alpha + \frac{2\pi}{k}(2R^{[\alpha}R^\dagger_\beta R^{\beta]} + Q^\beta Q^\dagger_\beta R^\alpha - R^\alpha Q^\dagger_\beta Q^\beta \\ & + 2R^\beta Q^\dagger_\beta Q^\alpha - 2Q^\alpha Q^\dagger_\beta R^\beta), \end{aligned} \tag{1.6}$$

where the Kronecker symbol $\epsilon^{\alpha\beta}$ $(\alpha,\beta = 1,2)$ is used to lower or raise indices, and $\rho > 0$ a massive parameter. Thus, when the spacetime metric is of the signature $(+--)$, the total (bosonic) Lagrangian density of ABJM model can be written as

$$\mathcal{L} = -\mathcal{L}_{\rm CS} + \mathrm{Tr}([D_\mu C^I]^\dagger[D^\mu C^I]) - V, \tag{1.7}$$

which is of a pure Chern–Simons type for the gauge field sector. The equations of motion of the Lagrangian (1.7) are rather complicated. As in [5] and [6], we concentrate on a reduced situation where (say) $R^\alpha = 0, N = 3$. In the static limit, Auzzi and Kumar [5] showed that these equations may be reduced into the first-order BPS vortex equations without assuming radial symmetry

$$(\partial_1 + \mathrm{i}\partial_2)\kappa = \mathrm{i}(a_1 + \mathrm{i}a_2)\kappa, \tag{1.8}$$

$$(\partial_1 + \mathrm{i}\partial_2)\phi = -\mathrm{i}([a_1 + \mathrm{i}a_2] - [b_1 + \mathrm{i}b_2])\phi, \tag{1.9}$$

$$a_{12} = -\frac{\lambda}{2}(2\kappa^2 - |\phi|^2 - 1), \tag{1.10}$$

$$b_{12} = -\lambda(|\phi|^2 - 1), \tag{1.11}$$

where κ is a real-valued scalar field, ϕ a complex-valued scalar field, and a_j and b_j are two real-valued gauge potential vector fields, $a_{jk} = \partial_j a_k - \partial_k a_j$ and $\lambda = 4\rho^2$.

We shall look for solutions of these equations so that κ never vanishes but ϕ vanishes exactly at the finite set of points

$$Z = \{p_1, p_2, \ldots, p_n\}. \tag{1.12}$$

Set $u = \ln \kappa^2$ and $w = \ln |\phi|^2$ and note that $|\phi|$ behaves like $|x - p_s|$ for x near p_s $(s = 1, \ldots, n)$. We see that u and w satisfy the equations [6]

$$\Delta u = \lambda(2e^u - e^w - 1), \tag{1.13}$$

$$\Delta u + \Delta w = 2\lambda(e^w - 1) + 4\pi \sum_{s=1}^{n} \delta_{p_s}(x), \tag{1.14}$$

where we have included our consideration of the zero set Z of ϕ as given in (1.12).

Chen, Zhang and Zhu [6] studied vortex equations in a supersymmetric Chern–Simons–Higgs theory in the ABJM model. They obtained a series of existence and uniqueness theorems for multiple vortex solutions of the ABJM model, over $\mathbb{R}^2$ and on a doubly periodic domain using the methods of calculus of variations.

In the present paper, we are going to discuss the non-Abelian BPS vortex equations of the ABJM model on a doubly periodic domain. We shall show how to approach the existence problem by a fixed point method via the Leray–Schauder theorem. Our approach is of independent interest because the *a priori* estimates obtained in the process may provide additional information on the governing equations. It's interesting that, our method is completely applicable to the self-dual equations governing multiple vortices in a product Abelian Higgs model may be regarded as a generalized Ginzburg–Landau theory [25, 26, 29].

2. Fixed point method

In this section, we approach the existence problem of the multiple vortex solutions in a doubly periodic domain Ω by a fixed point method where we apple the maximum principle and the Poincaré inequality to derive suitable *a priori* estimates. We introduce a background function w_0 satisfying

$$\Delta w_0 = -\frac{4\pi n}{|\Omega|} + 4\pi \sum_{s=1}^{n} \delta_{p_s}(x), \tag{2.1}$$

where δ_p is the Dirac distribution concentrated at the point p. Using the new variable v so that $w = w_0 + v$, we can modify (1.13) and (1.14) into

$$\Delta u = \lambda(2e^u - e^{w_0+v} - 1), \tag{2.2}$$

$$\Delta v = \lambda(3e^{w_0+v} - 2e^u - 1) + \frac{4\pi n}{|\Omega|}, \tag{2.3}$$

which are now in a regular (singularity-free) form. Note that, since the singularity of w_0 at p_s is of the type $\ln |x-p_s|^2$, the weight function e^{w_0} is everywhere smooth.

Let (u, v) be a solution of (2.2) and (2.3). Then (u, w) solves (1.13) and (1.14). We first derive a necessary condition for the solvability of (2.2) and (2.3). Integrating (2.2) and (2.3), we have

$$\int_\Omega e^{w_0+v}dx = |\Omega| - \frac{2\pi n}{\lambda} \equiv C_1 > 0, \tag{2.4}$$

$$\int_\Omega e^u dx = \frac{1}{2}\int_\Omega e^{w_0+v}dx + \frac{1}{2}|\Omega| = \frac{1}{2}(C_1 + |\Omega|) \equiv C_2 > 0. \tag{2.5}$$

Of course, the conditions (2.4) and (2.5) imply that the existence of an n-vortex solution requires that $C_1 > 0$ and $C_2 > 0$, which is simply

$$|\Omega| - \frac{2\pi n}{\lambda} \equiv C_1 > 0, \tag{2.6}$$

since $C_1 > 0$ contains $C_2 > 0$.

We now proceed to prove that (2.4) and (2.5) are also sufficient for the existence of a solution to the equations (2.2) and (2.3).

We use $W^{1,2}(\Omega)$ to denote the usual Sobolev space of scalar-valued or vector-valued Ω- periodic L^2-functions whose derivatives are also in $L^2(\Omega)$. For this purpose, we rewrite each $f \in W^{1,2}(\Omega)$ as follows

$$f = \underline{f} + f',$$

where $\underline{f}$ denotes the integral mean of f, $\underline{f} = \frac{1}{|\Omega|}\int_\Omega f dx$ and $\int_\Omega f' dx = 0$. We can derive from (2.4) and (2.5) the expressions

$$\underline{v} = \ln C_1 - \ln\bigg(\int_\Omega e^{w_0+v'}dx\bigg), \tag{2.7}$$

$$\underline{u} = \ln C_2 - \ln\bigg(\int_\Omega e^{u'}dx\bigg). \tag{2.8}$$

For $X = \left\{ f' \in W^{1,2}(\Omega) \middle| \int_\Omega f' dx = 0 \right\}$ and $Y = X \times X$ define a operator $T : Y \longrightarrow Y$ be setting

$$(U', V') = T(u', v'), \quad (u', v') \in Y, \tag{2.9}$$

where $(U', V') \in Y$ is the unique solution of the system of the equations

$$\Delta U' = \lambda\bigg(\frac{2C_2 e^{u'}}{\int_\Omega e^{u'}dx} - \frac{C_1 e^{w_0+v'}}{\int_\Omega e^{w_0+v'}dx} - 1\bigg), \tag{2.10}$$

$$\Delta V' = \lambda\bigg(\frac{3C_1 e^{w_0+v'}}{\int_\Omega e^{w_0+v'}dx} - \frac{2C_2 e^{u'}}{\int_\Omega e^{u'}dx} - 1\bigg) + \frac{4\pi n}{|\Omega|}. \tag{2.11}$$

The existence and uniqueness of a solution of the system of equations (2.10) and (2.11) may easily be seen since the right-hand sides of (2.10) and (2.11) have zero average value on Ω as a consequence of the definitions of (2.7) and (2.8). By the Poincaré inequality [23], we may define the norm of Y as follow

$$\|(u', v')\|_Y = \|\nabla u'\|_{L^2(\Omega)} + \|\nabla v'\|_{L^2(\Omega)}. \tag{2.12}$$

Theorem 2.1. *The system of equation* (1.13) *and* (1.14) *has a solution if and only if the conditions* (2.4) *and* (2.5) *are valid.*

We will prove Theorem 2.1 in terms of two lemmas as follows.

Lemma 2.1. *The operator $T : Y \longmapsto Y$ is completely continuous.*

Proof. Let $(u_n', v_n') \to (u_0', v_0')$ weakly in Y as $n \to \infty$. Then $(u_n', v_n') \to (u_0', v_0')$ strongly in $L^p(\Omega) \times L^p(\Omega)\,(p \geq 1)$. The Egorov theorem imply that for any $\varepsilon > 0$ there is a sufficiently large number $K_\varepsilon > 0$ and a subset $\Omega_\varepsilon \subset \Omega$ such that $|u_n'|, |v_n'| \leq K_\varepsilon, x \in \Omega - \Omega_\varepsilon, |\Omega_\varepsilon| < \varepsilon$.

Set $(U_n', V_n') = T(u_n', v_n')$ and $(U_0', V_0') = T(u_0', v_0')$. Then

(2.13)
$$\Delta(U_n' - U_0') = \lambda\left(\frac{2C_2 e^{u_n'}}{\int_\Omega e^{u_n'} dx} - \frac{C_1 e^{w_0 + v_n'}}{\int_\Omega e^{w_0 + v_n'} dx} - \frac{2C_2 e^{u_0'}}{\int_\Omega e^{u_0'} dx} + \frac{C_1 e^{w_0 + v_0'}}{\int_\Omega e^{w_0 + v_0'} dx}\right),$$

(2.14)
$$\Delta(V_n' - V_0') = \lambda\left(\frac{-2C_2 e^{u_n'}}{\int_\Omega e^{u_n'} dx} + \frac{3C_1 e^{w_0 + v_n'}}{\int_\Omega e^{w_0 + v_n'} dx} + \frac{2C_2 e^{u_0'}}{\int_\Omega e^{u_0'} dx} - \frac{3C_1 e^{w_0 + v_0'}}{\int_\Omega e^{w_0 + v_0'} dx}\right).$$

Multiplying (2.13) and (2.14) by $U_n' - U_0'$ and $V_n' - V_0'$, and integrating by parts, respectively, we obtain

$$\begin{aligned}\int_\Omega |\nabla(U_n' - U_0')|^2 dx = \int_\Omega \lambda\Bigg\{ & \frac{2C_2 e^{u_0'}}{\int_\Omega e^{u_0'} dx} - \frac{2C_2 e^{u_n'}}{\int_\Omega e^{u_n'} dx} \\ & + \frac{C_1 e^{w_0 + v_n'}}{\int_\Omega e^{w_0 + v_n'} dx} - \frac{C_1 e^{w_0 + v_0'}}{\int_\Omega e^{w_0 + v_0'} dx}\Bigg\}(U_n' - U_0') dx, \end{aligned} \tag{2.15}$$

$$\begin{aligned}\int_\Omega |\nabla(V_n' - V_0')|^2 dx = \int_\Omega \lambda\Bigg\{ & \frac{2C_2 e^{u_n'}}{\int_\Omega e^{u_n'} dx} - \frac{2C_2 e^{u_0'}}{\int_\Omega e^{u_0'} dx} \\ & - \frac{3C_1 e^{w_0 + v_n'}}{\int_\Omega e^{w_0 + v_n'} dx} + \frac{3C_1 e^{w_0 + v_0'}}{\int_\Omega e^{w_0 + v_0'} dx}\Bigg\}(V_n' - V_0') dx. \end{aligned} \tag{2.16}$$

Note that the boundedness of $\{(u_n', v_n')\}$ in Y and the Trudinger-Moser inequality [3] imply that

$$\sup_n \int_\Omega e^{u_n'} dx \leq C < \infty, \tag{2.17}$$

$$\sup_n \int_\Omega e^{v_n'} dx \leq C < \infty. \tag{2.18}$$

For any $\varepsilon > 0$, let Ω_ε be a neighborhood of the points $p_1, p_2, \ldots, p_n$ so that $p_s \in \Omega_\varepsilon (\forall \varepsilon)$ and $|\Omega_\varepsilon| < \varepsilon$. On the other hand, since there is a constant $\varepsilon_0 > 0$ such that $e^{w_0(x)} \geq \varepsilon_0$ for all $x \in \Omega - \Omega_\varepsilon$.

Therefore, from (2.15), we obtain

$$\int_\Omega |\nabla(U_n' - U_0')|^2 dx \leq \lambda\Bigg\{\frac{4C_2}{\int_\Omega e^{u_n'} dx} \int_\Omega e^{\tilde{u}_n'} |u_n' - u_0'||U_n' - U_0'| dx$$

$$
\begin{aligned}
&\quad + \frac{2C_1}{\int_\Omega e^{w_0+v_n'}dx}\int_\Omega e^{w_0+\tilde{v}_n'}|v_n'-v_0'||U_n'-U_0'|dx\Big\} \\
&\le \lambda\Big\{\frac{4C_2}{|\Omega|}\int_\Omega e^{\tilde{u}_n'}|u_n'-u_0'||U_n'-U_0'|dx \\
&\quad + \frac{2C_1}{K_{\Omega,\varepsilon}}\int_\Omega e^{w_0+\tilde{v}_n'}|v_n'-v_0'||U_n'-U_0'|dx\Big\}, \qquad (2.19)
\end{aligned}
$$

where $\tilde{u}_n'$ and $\tilde{v}_n'$ lie between u_n', v_n' and u_0', v_0', respectively. In (2.19), we have used the inequalities

$$\int_\Omega e^{u_n'}dx \ge |\Omega|\exp\Big(\frac{1}{|\Omega|}\int_\Omega u_n'dx\Big) = |\Omega|,$$

and

$$\int_\Omega e^{w_0+v_n'}dx \ge \int_{\Omega-\Omega_\varepsilon} e^{w_0+v_n'}dx \ge \varepsilon_0|\Omega-\Omega_\varepsilon|\exp(-K_\varepsilon) \equiv K_{\Omega,\varepsilon}.$$

Applying the Cauchy inequality and Hölder inequality, and (2.17), we have

$$
\begin{aligned}
\int_\Omega e^{\tilde{u}_n'}|u_n'-u_0'||U_n'-U_0'|dx &\le \frac{1}{2\varepsilon}\int_\Omega e^{2\tilde{u}_n'}|u_n'-u_0'|^2dx + \frac{\varepsilon}{2}\int_\Omega |U_n'-U_0'|^2dx \\
&\le \frac{1}{2\varepsilon}\Big(\int_\Omega e^{4\tilde{u}_n'}dx\Big)^{\frac12}\Big(\int_\Omega |u_n'-u_0'|^4x\Big)^{\frac12} \\
&\quad + \frac{C_3\varepsilon}{2}\|\nabla(U_n'-U_0')\|^2_{L^2(\Omega)} \\
&\le C_\varepsilon\|u_n'-u_0'\|^2_{L^4(\Omega)} + \frac{C_3\varepsilon}{2}\|\nabla(U_n'-U_0')\|^2_{L^2(\Omega)}. \qquad (2.20)
\end{aligned}
$$

Similarly,

(2.21)

$$\int_\Omega e^{w_0+\tilde{v}_n'}|v_n'-v_0'||U_n'-U_0'|dx \le C_\varepsilon\|v_n'-v_0'\|^2_{L^4(\Omega)} + \frac{C_4\varepsilon}{2}\|\nabla(U_n'-U_0')\|^2_{L^2(\Omega)}.$$

Inserting (2.20) and (2.21) into (2.19), and letting $\varepsilon > 0$ be small enough, we have

$$\|\nabla(U_n'-U_0')\|^2_{L^2(\Omega)} \le C\Big(\|u_n'-u_0'\|^2_{L^4(\Omega)} + \|v_n'-v_0'\|^2_{L^4(\Omega)}\Big), \qquad (2.22)$$

where $C > 0$ is a constant.

For (2.16), we have

$$\|\nabla(V_n'-V_0')\|^2_{L^2(\Omega)} \le C\Big(\|u_n'-u_0'\|^2_{L^4(\Omega)} + \|v_n'-v_0'\|^2_{L^4(\Omega)}\Big). \qquad (2.23)$$

From (2.22) and (2.23), we arrive at

$$\|(U_n'-U_0', V_n'-V_0')\|_Y \le C\Big(\|u_n'-u_0'\|^2_{L^4(\Omega)} + \|v_n'-v_0'\|^2_{L^4(\Omega)}\Big), \qquad (2.24)$$

where $C > 0$ is a constant. This proves that $(U'_n, V'_n) \to (U'_0, V'_0)$ strongly in Y and the lemma follows. $\square$

We now study the fixed point equation labeled by a parameter t,

$$(u'_t, v'_t) = tT(u'_t, v'_t), \quad 0 \le t \le 1. \tag{2.25}$$

Lemma 2.2. *There is a constant $C > 0$ independent of $t \in [0,1]$ so that*

$$\|(u'_t, v'_t)\|_Y \le C, \quad 0 < t \le 1. \tag{2.26}$$

Consequently, T has a fixed point in Y.

Proof. When $t > 0$, it is straightforward to check that (u'_t, v'_t) satisfies the equations

$$\Delta u'_t = \lambda t(\frac{2C_2 e^{u'_t}}{\int_\Omega e^{u'_t} dx} - \frac{C_1 e^{w_0 + v'_t}}{\int_\Omega e^{w_0 + v'_t} dx} - 1), \tag{2.27}$$

$$\Delta v'_t = \lambda t(\frac{-2C_2 e^{u'_t}}{\int_\Omega e^{u'_t} dx} + \frac{3C_1 e^{w_0 + v'_t}}{\int_\Omega e^{w_0 + v'_t} dx} - 1) + \frac{4\pi n}{|\Omega|} t. \tag{2.28}$$

Set $w'_t = w_0 + v'_t$. Then the equations (2.27) and (2.28) are modified into

$$\Delta u'_t = \lambda t(\frac{2C_2 e^{u'_t}}{\int_\Omega e^{u'_t} dx} - \frac{C_1 e^{w'_t}}{\int_\Omega e^{w'_t} dx} - 1), \tag{2.29}$$

$$\Delta w'_t = \lambda t(\frac{-2C_2 e^{u'_t}}{\int_\Omega e^{u'_t} dx} + \frac{3C_1 e^{w'_t}}{\int_\Omega e^{w'_t} dx} - 1) + \frac{4\pi n}{|\Omega|}(t-1) + 4\pi \sum_{s=1}^n \delta_{p_s}(x), \tag{2.30}$$

where $\Delta w_0 = -\frac{4\pi n}{|\Omega|} + 4\pi \sum_{s=1}^n \delta_{p_s}(x)$.

In the doubly periodic domain Ω, we let $p, q \in \Omega$ so that

$$u'_t(p) = \max\{u'_t(x) | x \in \Omega\}, \quad w'_t(q) = \max\{w'_t(x) | x \in \Omega\}.$$

To facilitate our computation, we adopt the notation

$$h'_t(x) = \frac{C_2 e^{u'_t}}{\int_\Omega e^{u'_t} dx}, \qquad g'_t(x) = \frac{C_1 e^{w'_t}}{\int_\Omega e^{w'_t} dx}. \tag{2.31}$$

Then from (2.29), we have

$$0 \ge (\Delta u'_t)(p) = \lambda t(2h'_t(p) - g'_t(p) - 1).$$

Therefore

$$2h'_t(p) \le g'_t(p) + 1 \le \frac{C_1 e^{w'_t(q)}}{\int_\Omega e^{w'_t} dx} + 1 = g'_t(q) + 1.$$

Hence, for any $x \in \Omega$, we have

$$2h'_t(x) \le g'_t(q) + 1, \quad \forall x \in \Omega. \tag{2.32}$$

From (2.30), using (2.32), we obtain

$$g'_t(q) \le 1 + \frac{2\pi n}{\lambda|\Omega|} \cdot \frac{1-t}{t}, \quad 0 < t \le 1. \tag{2.33}$$

In view of (2.32) and (2.33), for any $x \in \Omega$, we have

$$g_t'(x) \leq 1, \qquad h_t'(x) \leq 1 + \frac{\pi n}{\lambda|\Omega|} \cdot \frac{1-t}{t}, \quad x \in \Omega. \tag{2.34}$$

Multiplying (2.27) and (2.28) by u_t', v_t' and integrating by parts, respectively, and using (2.34), we have

$$\begin{aligned}
&\|(\nabla u_t', \nabla v_t')\|^2_{L^2(\Omega)\times L^2(\Omega)} \\
\leq & \int_\Omega \left| \lambda t \left(\frac{2C_2 e^{u_t'}}{\int_\Omega e^{u_t'}dx} - \frac{C_1 e^{w_0+v_t'}}{\int_\Omega e^{w_0+v_t'}dx} - 1 \right) \cdot u_t' \right| dx \\
&+ \int_\Omega \left| \left\{ \lambda t \left(\frac{-2C_2 e^{u_t'}}{\int_\Omega e^{u_t'}dx} + \frac{3C_1 e^{w_0+v_t'}}{\int_\Omega e^{w_0+v_t'}dx} - 1 \right) + \frac{4\pi n}{|\Omega|} t \right\} \cdot v_t' \right| dx \\
\leq & \int_\Omega \left\{ (1+1+2)\lambda |u_t'| + \left[(1+3+2)\lambda + \frac{4\pi n}{|\Omega|} \right] |v_t'| \right\} dx \\
\leq & \tilde{C}_1 \int_\Omega |u_t'| dx + \tilde{C}_2 \int_\Omega |v_t'| dx \\
\leq & C_\varepsilon + \tilde{C}\varepsilon \|(\nabla u_t', \nabla v_t')\|^2_{L^2(\Omega)\times L^2(\Omega)}.
\end{aligned} \tag{2.35}$$

Let $\varepsilon > 0$ be small enough, we have

$$\|(u_t', v_t')\|_Y = \|(\nabla u_t', \nabla v_t')\|_{L^2(\Omega)\times L^2(\Omega)} \leq C, \tag{2.36}$$

where $C > 0$ is a constant. The existence of a fixed point is a consequence of Lemma 2.2, the apriori estimate (2.26) and the Leray–Schauder theory. In particular, the existence of a fixed point of T, say (u', v'), follows. $\square$

Set $u = \underline{u} + u'$ and $v = \underline{v} + v'$. We see that (u, v) is a solution of the system of equations (2.2) and (2.3). This completes the proof of Theorem 2.1.

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Ruifeng Zhang
Institute of Contemporary Mathematics Henan University
College of Mathematics and Statistics Henan University
Kaifeng 475001, P. R. China
E-mail address: zrf615@henu.edu.cn

Meili Zhu
Chunlai Senior High School of Chunlai Eduction Group
Shangqiu 476000, P. R. China
E-mail address: meili.xiaoyue@163.com