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MULTIPLICITY OF SOLUTIONS FOR A CLASS OF NON-LOCAL ELLIPTIC OPERATORS SYSTEMS

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Abstract. In this paper, we investigate the existence and multiplicity of solutions for systems driven by two non-local integrodifferential operators with homogeneous Dirichlet boundary conditions. The main tools are the Saddle point theorem, Ekeland's variational principle and the Mountain pass theorem.

1. Introduction

This paper is concerned with the following problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u + F_u(x,u,v) & \text{in } \Omega, \\ -\mathcal{L}_G v = \mu v + F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, and λ, μ are two positive parameters. $F \in C^1(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$ satisfies some conditions which will be stated later on, $\mathcal{L}_K$ and $\mathcal{L}_G$ are the non-local operators defined by:

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^n,$$

and

$$\mathcal{L}_G v(x) = \int_{\mathbb{R}^n} (v(x+y) + v(x-y) - 2v(x))G(y)dy, \quad x \in \mathbb{R}^n,$$

respectively, here $K, G : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ are two functions such that

$$mK, mG \in L^1(\mathbb{R}^n), \quad \text{where } m(x) = \min\{|x|^2, 1\} \tag{1.2}$$

there exist $\theta_1, \theta_2 > 0$ and $s_1, s_2 \in (0,1)$ such that

$$K(x) \geq \theta_1 |x|^{-(n+2s_1)}, \quad G(x) \geq \theta_2 |x|^{-(n+2s_2)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\} \tag{1.3}$$

$$K(x) = K(-x), \quad G(x) = G(-x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \tag{1.4}$$

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A typical example for K and G is given by $K(x) = |x|^{-(n+2s_1)}$ and $G(x) = |x|^{-(n+2s_2)}$. In this case $\mathcal{L}_K$ and $\mathcal{L}_G$ are the fractional Laplace operators $-(-\Delta)^{s_1}$ and $-(-\Delta)^{s_2}$, where $-(-\Delta)^s$ is defined by

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \quad x \in \mathbb{R}^n,$$

here $s \in (0,1)$ and $n > 2s$. The fractional Laplacian $-(-\Delta)^s$ is a classical linear integro-differential operator of order $2s$ which gives the standard Laplacian when $s = 1$.

Let X_K be the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X_K belongs to $L^2(\Omega)$ and

$$\text{the map } (x,y) \to (g(x) - g(y))\sqrt{K(x-y)} \text{ is in } L^2(\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy),$$

where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. Moreover,

$$X_{0,K} = \{g \in X_K : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

Similarly, we can define the space $X_{0,G}$. Let $E_0 = X_{0,K} \times X_{0,G}$. We say that $(u,v) \in E_0$ is a weak solution of problem (1.1) if for every $(\varphi, \psi) \in E_0$, one has

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)dxdy \\
&+ \int_{\mathbb{R}^{2n}} (v(x) - v(y))(\psi(x) - \psi(y))G(x-y)dxdy \\
&- \lambda \int_\Omega u(x)\varphi(x)dx - \mu \int_\Omega v(x)\psi(x)dx - \int_\Omega F_u(x, u(x), v(x))\varphi(x)dx \\
&- \int_\Omega F_v(x, u(x), v(x))\psi(x)dx = 0.
\end{aligned}$$

The fractional Laplacian and non-local operators of elliptic type arises in both pure mathematical research and concrete applications, since these operators occur in a quite natural way in many different contexts. For an elementary introduction to this topic, see [10] and the references therein. Recently, some elliptic boundary problems driven by the non-local integrodifferential operator $\mathcal{L}_K$ have been studied in the works [3, 4, 6, 7, 8, 12, 13, 14].

In this paper, inspired by the ideas introduced in [1, 3, 12], we will show how the multiplicity of solutions of problem (1.1) changes as λ and μ vary. To the best of our knowledge, this is an interesting and new research topic for non-local operators of elliptic type.

Denote by $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \cdots$ the eigenvalues of the following non-local eigenvalue problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.5}$$

Similarly, denote by $0 < \mu_1 < \mu_2 \leq \cdots \leq \mu_k \leq \cdots$ the eigenvalues of the following non-local eigenvalue problem

$$\begin{cases} -\mathcal{L}_G v = \mu v & \text{in } \Omega \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.6}$$

Our main results are given by the following theorems.

Theorem 1.1. *Let $F(x,0,0)$ be bounded for each $x \in \Omega$. If F satisfies*

$$\lim_{|u|\to+\infty} \frac{|F_u(x,u,v)|}{|u|} = 0, \quad \lim_{|v|\to+\infty} \frac{|F_v(x,u,v)|}{|v|} = 0 \tag{H1}$$

uniformly in $x \in \bar{\Omega}$. Then for $\lambda_1 < \lambda < \lambda_2$ and $\mu_1 < \mu < \mu_2$, problem (1.1) *has at least one solution.*

Theorem 1.2. *Let $F(x,0,0)$ be bounded for each $x \in \Omega$. Assume that the nonlinearity F satisfies $(H1)$ and*

$$\lim_{|t_1|,|t_2|\to+\infty} F(x, t_1 e_1, t_2 \omega_1) = +\infty \tag{H2}$$

uniformly in $x \in \bar{\Omega}$, where e_1 is a normalized eigenfunction corresponding to λ_1 and ω_1 is a normalized eigenfunction corresponding to μ_1. Then for $\lambda < \lambda_1$ and $\mu < \mu_1$ sufficiently close to λ_1 and μ_1, problem (1.1) *has at least three solutions.*

Remark 1.1. The case of $\lambda_1 < \frac{\lambda_2}{2}$ is attainable. In fact, if $K(x) = |x|^{-(n+2s_1)}$ ($s_1 \in (0,1)$), then $-\mathcal{L}_K = (-\Delta)^{s_1}$. In [11], Kwaśnicki studied the asymptotic behavior of the eigenvalues of the spectral problem for the one-dimensional fractional Laplace operator $(-\Delta)^{\alpha/2}$ ($\alpha \in (0,2)$) in the interval $D = (-1,1)$, from [11, Table 2], we know that eigenvalues $\lambda_1 < \frac{\lambda_2}{2}$ for $\alpha > 1$. If $\alpha \to 2$, then the fractional Laplace operator $(-\Delta)^{\alpha/2}$ reduces to the Laplace operator $-\Delta$. The eigenvalues λ_1 and λ_2 of the spectral problem for the two-dimensional Laplace operator $-\Delta$ in the rectangle $D = (0,a) \times (0,b)$ ($a > b > 0$) had been given as follows ([5], page 83):

$$\lambda_1 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}, \quad \lambda_2 = \frac{4\pi^2}{a^2} + \frac{\pi^2}{b^2}.$$

Let $a = 5$ and $b = 4$, then

$$\lambda_1 = \frac{41}{400}\pi^2 < \frac{1}{2} \cdot \frac{89}{400}\pi^2 = \frac{1}{2}\lambda_2.$$

2. Preliminaries

The space X_K is endowed with the norm defined as

$$\|g\|_K = \|g\|_{L^2(\Omega)} + \left(\int_Q |g(x) - g(y)|^2 K(x-y)\,dxdy \right)^{1/2}, \tag{2.1}$$

where $Q = \mathbb{R}^{2n} \setminus \mathcal{O}$. Here $\mathcal{O} = (\mathcal{C}\Omega \times \mathcal{C}\Omega) \subset \mathbb{R}^{2n}$ and $\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$. It is easily seen that $\|\cdot\|_K$ is a norm on X_K (see, for instance, [12] for a proof).

By [12], a sort of Poincaré-Sobolev inequality for functions in $X_{0,K}$ is given as follows.

Lemma 2.1 ([12]). *Suppose that* $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ *satisfies assumptions* (1.2)-(1.4). *Then*

(1) *there exists a positive constant* c_1*, depending only on* n *and* s_1*, such that for any* $u \in X_{0,K}$

$$\|u\|^2_{L^{2^*_{s_1}}(\Omega)} = \|u\|^2_{L^{2^*_{s_1}}(\mathbb{R}^n)} \le c_1 \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s_1}} dxdy,$$

where $2^*_{s_1} = 2n/(n-2s_1)$;

(2) *there exits a constant* $C > 1$*, depending only on* n, s_1, θ_1 *and* Ω*, such that for any* $u \in X_{0,K}$

$$\int_Q |u(x)-u(y)|^2 K(x-y)dxdy \le \|u\|^2_K \le C \int_Q |u(x)-u(y)|^2 K(x-y)dxdy,$$

that is

$$\|u\|_{X_{0,K}} = \left(\int_Q |u(x)-u(y)|^2 K(x-y)dxdy \right)^{1/2} \tag{2.2}$$

is a norm on $X_{0,K}$ *equivalent to the usual one defined in* (2.1).

Lemma 2.2 ([12]). $(X_{0,K}, \|\cdot\|_{X_K})$ *is a Hilbert space, with the scalar product*

$$\langle u, v\rangle_{X_{0,K}} = \int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)dxdy. \tag{2.3}$$

Since $v \in X_{0,K}$, we have $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$. Thus the integrals in (2.2) and in (2.3) can be extended to all $\mathbb{R}^{2n}$.

Remark 2.1. Similarly, we can define $\|u\|_{X_{0,G}}$ and $\langle u, v\rangle_{X_{0,G}}$ if only replaced K by G in Lemma 2.1 and Lemma 2.2 respectively. Moreover, there exists a positive constant c_2, depending only on n and s_2, such that for any $v \in X_{0,G}$

$$\|v\|^2_{L^{2^*_{s_2}}(\Omega)} = \|v\|^2_{L^{2^*_{s_2}}(\mathbb{R}^n)} \le c_2 \int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s_2}} dxdy,$$

where $2^*_{s_2} = 2n/(n-2s_2)$.

Space $E_0 = X_{0,K} \times X_{0,G}$ is the Cartesian product of two Hilbert spaces, which is a reflexive Banach space endowed with the norm

$$\begin{aligned}\|(u,v)\| &= \|u\|_{0,K} + \|v\|_{0,G} \\ &= \left(\int_Q |u(x)-u(y)|^2 K(x-y)dxdy \right)^{1/2} \\ &\quad + \left(\int_Q |v(x)-v(y)|^2 G(x-y)dxdy \right)^{1/2}.\end{aligned}$$

From [13, Proposition 9], we have:

Lemma 2.3 (Eigenvalues and eigenfunctions of $-\mathcal{L}_K$)**.** *Let* $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ *be a function satisfying assumptions* (1.2)-(1.4). *Then*

a)

$$\begin{aligned}\lambda_1 &= \min_{u\in X_{0,K}\setminus\{0\}} \frac{\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dxdy}{\int_\Omega |u(x)|^2 dx} \\ &= \min_{u\in X_{0,K},\ \|u\|_{L^2(\Omega)}=1} \int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dxdy; \qquad (2.4)\end{aligned}$$

b) *there exists a non-negative function* $e_1 \in X_{0,K}$*, which is an eigenfunction corresponding to* λ_1*, attaining the minimum in* (2.4)*, that is* $\|e_1\|_{L^2(\Omega)} = 1$ *and*

$$\lambda_1 = \int_{\mathbb{R}^{2n}} |e_1(x)-e_1(y)|^2 K(x-y)dxdy; \qquad (2.5)$$

c)

$$\begin{aligned}\lambda_2 &= \min_{u\in\langle e_1\rangle^\perp} \frac{\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dxdy}{\int_\Omega |u(x)|^2 dx} \\ &= \min_{u\in\langle e_1\rangle^\perp,\ \|u\|_{L^2(\Omega)}=1} \int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dxdy. \qquad (2.6)\end{aligned}$$

3. Main results

By [13] we know that $(u, v) \in E_0$ is a weak solution of problem (1.1) is equivalent to being a critical point of the functional

$$\begin{aligned}\mathcal{J}_{\lambda,\mu}(u,v) &= \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dxdy \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 G(x-y)dxdy \\ &\quad - \frac{\lambda}{2}\int_\Omega |u(x)|^2dx - \frac{\mu}{2}\int_\Omega |v(x)|^2dx - \int_\Omega F(x,u,v)dx. \qquad (3.1)\end{aligned}$$

Since the potential F satisfies (H1), it follows that $\mathcal{J}_{\lambda,\mu} \in C^1(E, \mathbb{R})$. Thanks to the fact that $L^{2^*_{s_1}}(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, we get

$$\|u\|^2_{L^2(\Omega)} \le |\Omega|^{(2^*_{s_1}-2)/2^*_{s_1}} \|u\|^2_{L^{2^*_{s_1}}(\Omega)}. \qquad (3.2)$$

Using (1.3) and Lemma 2.1(1), we have

$$\begin{aligned}\|u\|_{L^{2^*_{s_1}}(\Omega)} &\le \sqrt{c_1}\left(\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s_1}}dxdy\right)^{1/2} \\ &\le \sqrt{\frac{c_1}{\theta_1}}\left(\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dxdy\right)^{1/2} \\ &= \sqrt{\frac{c_1}{\theta_1}}\|u\|_{0,K}. \qquad (3.3)\end{aligned}$$

Substituting (3.3) into (3.2), we get

$$\|u\|_{L^2(\Omega)} \leq |\Omega|^{(2^*_{s_1}-2)/2\cdot 2^*_{s_1}} \sqrt{\frac{c_1}{\theta_1}} \|u\|_{0,K}. \tag{3.4}$$

Similarly, we have

$$\|v\|_{L^2(\Omega)} \leq |\Omega|^{(2^*_{s_2}-2)/2\cdot 2^*_{s_2}} \sqrt{\frac{c_2}{\theta_2}} \|v\|_{0,G}. \tag{3.5}$$

The main results of Theorem 1.1 are proved by the saddle point theorem [9] and those of Theorem 1.2 are based on Ekeland's variational principle and the Mountain pass theorem [2].

Proof Theorem 1.1. Let $\{z_n\} = \{(u_n, v_n)\} \subset E_0$ satisfy

$$\mathcal{J}_{\lambda,\mu}(z_n) \to c \in \mathbb{R}, \quad \|\mathcal{J}'_{\lambda,\mu}(z_n)\|_{E_0^*} \to 0 \tag{3.6}$$

as $n \to \infty$. Firstly, we prove that $\{z_n\}$ in bounded in E_0. From (H1) and the continuity of the potential F, for any $\varepsilon > 0$, there exists a positive constant W_ε such that

$$\left|\frac{\partial F}{\partial u}(x,u,v)\right| \leq \varepsilon|u| + W_\varepsilon, \quad \left|\frac{\partial F}{\partial v}(x,u,v)\right| \leq \varepsilon|v| + W_\varepsilon \tag{3.7}$$

for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R}^2$. Putting $Z = \langle e_1 \rangle \times \langle \omega_1 \rangle$, and

$$Z' = \{(u,v) \in E_0 : u \in \langle e_1 \rangle^\perp,\ v \in \langle \omega_1 \rangle^\perp\}.$$

We can easily know that Z' is a complementary subspace of Z. Hence we have the following direct sum

$$E_0 = Z \bigoplus Z'.$$

Let $z_n = z_n^- + z_n^+ \in E_0$, where $z_n^- = (u_n^-, v_n^-) \in Z$, $z_n^+ = (u_n^+, v_n^+) \in Z'$. For large n, we obtain by (3.6) that

$$\begin{aligned}
&\left|\mathcal{J}'_{\lambda,\mu}(u_n,v_n)\left(\frac{u_n^+}{2}, \frac{v_n^+}{2}\right)\right| \\
&= \left|\frac{1}{2}\int_{\mathbb{R}^{2n}} (u_n(x)-u_n(y))(u_n^+(x)-u_n^+(y))K(x-y)dxdy\right. \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^{2n}} (v_n(x)-v_n(y))(v_n^+(x)-v_n^+(y))G(x-y)dxdy \\
&\quad - \frac{\lambda}{2}\int_\Omega u_n(x)u_n^+(x)dx - \frac{\mu}{2}\int_\Omega v_n(x)v_n^+(x)dx \\
&\quad \left. -\frac{1}{2}\int_\Omega F_u(x,u_n,v_n)u_n^+(x)dx - \frac{1}{2}\int_\Omega F_v(x,u_n,v_n)v_n^+(x)dx\right| \\
&\leq \left\|\left(\frac{u_n^+}{2}, \frac{v_n^+}{2}\right)\right\|.
\end{aligned} \tag{3.8}$$

On the other hand, by Claim 2 in Appendix A of [13], by (2.6), (3.4), (3.5) and (3.7), we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^{2n}}(u_n(x)-u_n(y))(u_n^+(x)-u_n^+(y))K(x-y)dxdy \\
&+\frac{1}{2}\int_{\mathbb{R}^{2n}}(v_n(x)-v_n(y))(v_n^+(x)-v_n^+(y))G(x-y)dxdy \\
&-\frac{\lambda}{2}\int_{\Omega}u_n(x)u_n^+(x)dx-\frac{\mu}{2}\int_{\Omega}v_n(x)v_n^+(x)dx \\
&=\frac{1}{2}\|u_n^+\|_{0,K}^2+\frac{1}{2}\|v_n^+\|_{0,G}^2-\frac{\lambda}{2}\int_{\Omega}|u_n^+(x)|^2dx-\frac{\mu}{2}\int_{\Omega}|v_n^+(x)|^2dx, \qquad (3.9)
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega}F_u(x,u_n,v_n)u_n^+dx\right| &\le \int_{\Omega}(\epsilon|u_n|+W_\epsilon)|u_n^+|dx \\
&\le \frac{3\epsilon}{2}\|u_n^+\|_{L^2(\Omega)}^2+\frac{\epsilon}{2}\|u_n^-\|_{L^2(\Omega)}^2+W_\epsilon|\Omega|^{1/2}\|u_n^+\|_{L^2(\Omega)} \\
&\le \frac{\epsilon}{2}|\Omega|^{(2^*_{s_1}-2)/2^*_{s_1}}\frac{c_1}{\theta_1}(3\|u_n^+\|_{0,K}^2+\|u_n^-\|_{0,K}^2) \\
&\quad + W_\epsilon|\Omega|^{(2^*_{s_1}-1)/2^*_{s_1}}\sqrt{\frac{c_1}{\theta_1}}\|u_n^+\|_{0,K}, \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_{\Omega}F_v(x,u_n,v_n)v_n^+dx\right| &\le \frac{\epsilon}{2}|\Omega|^{(2^*_{s_2}-2)/2^*_{s_2}}\frac{c_2}{\theta_2}(3\|v_n^+\|_{0,G}^2+\|v_n^-\|_{0,G}^2) \\
&\quad + W_\epsilon|\Omega|^{(2^*_{s_2}-1)/2^*_{s_2}}\sqrt{\frac{c_2}{\theta_2}}\|v_n^+\|_{0,K}. \qquad (3.11)
\end{aligned}$$

Substituting (3.9), (3.10) and (3.11) into (3.8), we obtain

$$\begin{aligned}
\left\|\left(\frac{u_n^+}{2},\frac{v_n^+}{2}\right)\right\| &\ge \frac{1}{2}\|u_n^+\|_{0,K}^2+\frac{1}{2}\|v_n^+\|_{0,G}^2-\frac{\lambda}{2}\int_{\Omega}|u_n^+(x)|^2dx-\frac{\mu}{2}\int_{\Omega}|v_n^+(x)|^2dx \\
&\quad -\frac{1}{2}\left[\epsilon|\Omega|^{(2^*_{s_1}-2)/2^*_{s_1}}\frac{c_1}{\theta_1}(3\|u_n^+\|_{0,K}^2+\|u_n^-\|_{0,K}^2)\right. \\
&\qquad \left.+W_\epsilon|\Omega|^{(2^*_{s_1}-1)/2^*_{s_1}}\sqrt{\frac{c_1}{\theta_1}}\|u_n^+\|_{0,K}\right] \\
&\quad -\frac{1}{2}\left[\epsilon|\Omega|^{(2^*_{s_2}-2)/2^*_{s_2}}\frac{c_2}{\theta_2}(3\|v_n^+\|_{0,G}^2+\|v_n^-\|_{0,G}^2)\right. \\
&\qquad \left.+W_\epsilon|\Omega|^{(2^*_{s_2}-1)/2^*_{s_2}}\sqrt{\frac{c_2}{\theta_2}}\|v_n^+\|_{0,G}\right]. \qquad (3.12)
\end{aligned}$$

From (3.6), we have for large n that

$$\left|\mathcal{J}'_{\lambda,\mu}(u_n,v_n)\left(\frac{u_n^-}{2},\frac{v_n^-}{2}\right)\right|$$

$$
\begin{aligned}
&= \Bigg| \frac{1}{2}\int_{\mathbb{R}^{2n}} (u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y))K(x-y)dxdy \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^{2n}} (v_n(x)-v_n(y))(v_n^-(x)-v_n^-(y))G(x-y)dxdy \\
&\quad - \frac{\lambda}{2}\int_\Omega u_n(x)u_n^-(x)dx - \frac{\mu}{2}\int_\Omega v_n(x)v_n^-(x)dx \\
&\quad - \frac{1}{2}\int_\Omega F_u(x,u_n,v_n)u_n^-(x)dx - \frac{1}{2}\int_\Omega F_v(x,u_n,v_n)v_n^-(x)dx \Bigg| \\
&\leq \left\| \left(\frac{u_n^-}{2}, \frac{v_n^-}{2} \right) \right\|. \qquad (3.13)
\end{aligned}
$$

Similar to the proof of (3.9)-(3.11), we get by (3.13) that

$$
\begin{aligned}
\left\| \left(\frac{u_n^-}{2}, \frac{v_n^-}{2} \right) \right\| &\geq \frac{\lambda}{2}\int_\Omega |u_n^-(x)|^2 dx + \frac{\mu}{2}\int_\Omega |v_n^-(x)|^2 dx - \frac{1}{2}\|u_n^-\|_{0,K}^2 - \frac{1}{2}\|v_n^-\|_{0,G}^2 \\
&\quad - \frac{1}{2}\Bigg[\epsilon|\Omega|^{(2_{s_1}^*-2)/2_{s_1}^*} \frac{c_1}{\theta_1}(3\|u_n^-\|_{0,K}^2 + \|u_n^+\|_{0,K}^2) \\
&\qquad + W_\epsilon|\Omega|^{(2_{s_1}^*-1)/2_{s_1}^*} \sqrt{\frac{c_1}{\theta_1}} \|u_n^-\|_{0,K} \Bigg] \\
&\quad - \frac{1}{2}\Bigg[\epsilon|\Omega|^{(2_{s_2}^*-2)/2_{s_2}^*} \frac{c_2}{\theta_2}(3\|v_n^-\|_{0,G}^2 + \|v_n^+\|_{0,G}^2) \\
&\qquad + W_\epsilon|\Omega|^{(2_{s_2}^*-1)/2_{s_2}^*} \sqrt{\frac{c_2}{\theta_2}} \|v_n^-\|_{0,G} \Bigg]. \qquad (3.14)
\end{aligned}
$$

Since

$$
\begin{aligned}
(\|u_n^+\|_{0,K} + \|u_n^-\|_{0,K})^2 &\leq 2(\|u_n^+\|_{0,K}^2 + \|u_n^-\|_{0,K}^2) \\
&= 2\|u_n\|_{0,K}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
(\|v_n^+\|_{0,G} + \|v_n^-\|_{0,G})^2 &\leq 2(\|v_n^+\|_{0,G}^2 + \|v_n^-\|_{0,G}^2) \\
&= 2\|v_n\|_{0,G}^2,
\end{aligned}
$$

we have

$$
\begin{aligned}
\|(u_n^+, v_n^+)\| + \|(u_n^-, v_n^-)\| &= (\|u_n^+\|_{0,K} + \|u_n^-\|_{0,K}) + (\|v_n^+\|_{0,G} + \|v_n^-\|_{0,G}) \\
&\leq \sqrt{2}(\|u_n\|_{0,K} + \|v_n\|_{0,G}) \\
&= \sqrt{2}\|(u_n, v_n)\|. \qquad (3.15)
\end{aligned}
$$

Thus, from (3.13)-(3.15), we obtain

$$
\begin{aligned}
&\frac{\sqrt{2}}{2}\|(u_n, v_n)\| \\
&\geq \left\| \left(\frac{u_n^+}{2}, \frac{v_n^+}{2} \right) \right\| + \left\| \left(\frac{u_n^-}{2}, \frac{v_n^-}{2} \right) \right\|
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{1}{2}\|u_n^+\|_{0,K}^2 - \frac{\lambda}{2}\int_\Omega |u_n^+(x)|^2 dx + \frac{1}{2}\|v_n^+\|_{0,G}^2 - \frac{\mu}{2}\int_\Omega |v_n^+(x)|^2 dx \\
&\quad + \frac{\lambda}{2}\int_\Omega |u_n^-(x)|^2 dx + \frac{\mu}{2}\int_\Omega |v_n^-(x)|^2 dx - \frac{1}{2}\|u_n^-\|_{0,K}^2 - \frac{1}{2}\|v_n^-\|_{0,G}^2 \\
&\quad - 2\epsilon|\Omega|^{(2^*_{s_1}-2)/2^*_{s_1}} \frac{c_1}{\theta_1}(\|u_n^+\|_{0,K}^2 + \|u_n^-\|_{0,K}^2) \\
&\quad - \frac{1}{2}W_\epsilon|\Omega|^{(2^*_{s_1}-1)/2^*_{s_1}} \sqrt{\frac{c_1}{\theta_1}}(\|u_n^+\|_{0,K} + \|u_n^-\|_{0,K}) \\
&\quad - 2\epsilon|\Omega|^{(2^*_{s_2}-2)/2^*_{s_2}} \frac{c_2}{\theta_2}(\|v_n^+\|_{0,G}^2 + \|v_n^-\|_{0,G}^2) \\
&\quad - \frac{1}{2}W_\epsilon|\Omega|^{(2^*_{s_2}-1)/2^*_{s_2}} \sqrt{\frac{c_2}{\theta_2}}(\|v_n^+\|_{0,G} + \|v_n^-\|_{0,G}) \\
&\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_2}\right)\|u_n^+\|_{0,K}^2 + \frac{1}{2}\left(1 - \frac{\mu}{\mu_2}\right)\|v_n^+\|_{0,G}^2 \\
&\quad + \frac{1}{2}\left(\frac{\lambda}{\lambda_1} - 1\right)\|u_n^-\|_{0,K}^2 + \frac{1}{2}\left(\frac{\mu}{\mu_1} - 1\right)\|v_n^-\|_{0,G}^2 \\
&\quad - 2\epsilon|\Omega|^{(2^*_{s_1}-2)/2^*_{s_1}} \frac{c_1}{\theta_1}\|u_n\|_{0,K}^2 - 2\epsilon|\Omega|^{(2^*_{s_2}-2)/2^*_{s_2}} \frac{c_2}{\theta_2}\|v_n\|_{0,G}^2 \\
&\quad - \frac{1}{2}W_\epsilon \max\left\{|\Omega|^{(2^*_{s_1}-1)/2^*_{s_1}}\sqrt{\frac{c_1}{\theta_1}}, |\Omega|^{(2^*_{s_2}-1)/2^*_{s_2}}\sqrt{\frac{c_2}{\theta_2}}\right\}\sqrt{2}\|(u_n, v_n)\| \\
&\geq \frac{1}{2}\min\left\{\left(1 - \frac{\lambda}{\lambda_2}\right), \left(\frac{\lambda}{\lambda_1} - 1\right), \left(1 - \frac{\mu}{\mu_2}\right), \left(\frac{\mu}{\mu_1} - 1\right)\right\} \\
&\quad (\|u_n\|_{0,K}^2 + \|v_n\|_{0,G}^2) \\
&\quad - 2\epsilon \max\left\{|\Omega|^{(2^*_{s_1}-2)/2^*_{s_1}}\frac{c_1}{\theta_1}, |\Omega|^{(2^*_{s_2}-2)/2^*_{s_2}}\frac{c_2}{\theta_2}\right\}(\|u_n\|_{0,K}^2 + \|v_n\|_{0,G}^2) \\
&\quad - \frac{\sqrt{2}}{2}W_\epsilon \max\left\{|\Omega|^{(2^*_{s_1}-1)/2^*_{s_1}}\sqrt{\frac{c_1}{\theta_1}}, |\Omega|^{(2^*_{s_2}-1)/2^*_{s_2}}\sqrt{\frac{c_2}{\theta_2}}\right\}\|(u_n, v_n)\| \\
&\geq \frac{1}{4}\min\left\{\left(1 - \frac{\lambda}{\lambda_2}\right), \left(\frac{\lambda}{\lambda_1} - 1\right), \left(1 - \frac{\mu}{\mu_2}\right), \left(\frac{\mu}{\mu_1} - 1\right)\right\}\|(u_n, v_n)\|^2 \\
&\quad - 2\epsilon \max\left\{|\Omega|^{(2^*_{s_1}-2)/2^*_{s_1}}\frac{c_1}{\theta_1}, |\Omega|^{(2^*_{s_2}-2)/2^*_{s_2}}\frac{c_2}{\theta_2}\right\}\|(u_n, v_n)\|^2 \\
&\quad - \frac{\sqrt{2}}{2}W_\epsilon \max\left\{|\Omega|^{(2^*_{s_1}-1)/2^*_{s_1}}\sqrt{\frac{c_1}{\theta_1}}, |\Omega|^{(2^*_{s_2}-1)/2^*_{s_2}}\sqrt{\frac{c_2}{\theta_2}}\right\}\|(u_n, v_n)\|. \qquad (3.16)
\end{aligned}
$$

So $\{z_n\}$ is bounded in E_0 by $\lambda_1 < \lambda < \lambda_2$, $\mu_1 < \mu < \mu_2$, and ϵ sufficiently small. Similar to the proof of Step 2 of Proposition 2 in [15], we can obtain that $\{z_n\}$ has a convergent subsequence. So, the functional $\mathcal{J}_{\lambda,\mu}$ satisfies the (PS) condition. In the following, we will show that the functional $\mathcal{J}_{\lambda,\mu}$ has the geometry of the saddle point theorem.

Since $F(x,0,0)$ is bounded on Ω, there exists a constant $M_1 > 0$ such that $|F(x,0,0)| \le M_1$ for any $x \in \Omega$. From (3.7), we obtain

$$\begin{aligned}
|F(x,u,v)| &= \left|\int_0^u \frac{\partial F}{\partial s}(x,s,v)ds + F(x,0,v)\right| \\
&= \left|\int_0^u \frac{\partial F}{\partial s}(x,s,v)ds + \int_0^v \frac{\partial F}{\partial s}(x,0,s)ds + F(x,0,0)\right| \\
&\le \int_0^{|u|}(\varepsilon|s| + W_\varepsilon)ds + \int_0^{|v|}(\varepsilon|s| + W_\varepsilon)ds + M_1 \\
&= \frac{\varepsilon}{2}(u^2+v^2) + W_\varepsilon(|u|+|v|) + M_1.
\end{aligned}$$

Thus, By (3.4), (3.5) and Hölder's inequality, we have

$$\begin{aligned}
\left|\int_\Omega F(x,u,v)dx\right| \le\ & \int_\Omega |F(x,u,v)|dx \\
\le\ & \frac{\varepsilon}{2}\left(\int_\Omega u^2dx + \int_\Omega v^2dx\right) \\
& + W_\varepsilon|\Omega|^{1/2}\left[\left(\int_\Omega u^2dx\right)^{1/2} + \left(\int_\Omega v^2dx\right)^{1/2}\right] + M_1|\Omega| \\
\le\ & \frac{\varepsilon}{2\lambda_1}\int_Q |u(x)-u(y)|^2K(x-y)dxdy \\
& + \frac{\varepsilon}{2\mu_1}\int_Q |v(x)-v(y)|^2G(x-y)dxdy \\
& + W_\varepsilon\left[|\Omega|^{(2^*_{s_1}-1)/2^*_{s_1}}\sqrt{\frac{c_1}{\theta_1}}\|u\|_{0,K} + |\Omega|^{(2^*_{s_2}-1)/2^*_{s_2}}\right. \\
& \left. + \sqrt{\frac{c_2}{\theta_2}}\|v\|_{0,G}\right] + M_1|\Omega| \\
\le\ & \frac{\varepsilon}{2\lambda_1}\|u\|^2_{0,K} + \frac{\varepsilon}{2\mu_1}\|v\|^2_{0,G} + M_2(\|u\|_{0,K} + \|v\|_{0,G}) + M_1|\Omega|, \qquad (3.17)
\end{aligned}$$

where $M_2 = \max\{W_\varepsilon|\Omega|^{(2^*_{s_1}-1)/2^*_{s_1}}\sqrt{\frac{c_1}{\theta_1}},\ W_\varepsilon|\Omega|^{(2^*_{s_2}-1)/2^*_{s_2}}\sqrt{\frac{c_2}{\theta_2}}\}$.

For any $z = (u,v) \in Z$, we get from (3.17) and Lemma 2.3(b) that

$$\begin{aligned}
\mathcal{J}_{\lambda,\mu}(u,v) &= \frac{1}{2}(\|u\|^2_{0,K} + \|v\|^2_{0,G}) - \frac{\lambda}{2}\int_\Omega |u(x)|^2dx \\
&\quad - \frac{\mu}{2}\int_\Omega |v(x)|^2dx - \int_\Omega F(x,u,v)dx \\
&= \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^2_{0,K} + \frac{1}{2}\left(1 - \frac{\mu}{\mu_1}\right)\|v\|^2_{0,G} - \int_\Omega F(x,u,v)dx \\
&\le \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1} + \frac{\varepsilon}{\lambda_1}\right)\|u\|^2_{0,K} + \frac{1}{2}\left(1 - \frac{\mu}{\mu_1} + \frac{\varepsilon}{\mu_1}\right)\|v\|^2_{0,G}
\end{aligned}$$

$$+ M_2\|(u,v)\| + M_1|\Omega|. \tag{3.18}$$

By $\lambda_1 < \lambda$, $\mu_1 < \mu$, letting $\varepsilon = \frac{1}{2}\min\{\lambda - \lambda_1, \mu - \mu_1\}$, from (3.18), it follows that $\mathcal{J}_{\lambda,\mu}(z) \to -\infty$, as $\|z\| \to \infty$, $z \in Z$.

For any $w = (u,v) \in Z'$, from (2.6) and (3.17), we obtain

$$\begin{aligned}\mathcal{J}_{\lambda,\mu}(u,v) \geq &\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_2} - \frac{\varepsilon}{\lambda_1}\right)\|u\|_{0,K}^2 + \frac{1}{2}\left(1 - \frac{\mu}{\mu_2} - \frac{\varepsilon}{\mu_1}\right)\|v\|_{0,G}^2 \\ &- M_2\|(u,v)\| - M_1|\Omega|\end{aligned}$$

and consequently, for $\lambda < \lambda_2$, $\mu < \mu_2$, letting

$$\varepsilon = \frac{1}{2}\min\left\{\lambda_1\left(1 - \frac{\lambda}{\lambda_2}\right), \mu_1\left(1 - \frac{\mu}{\mu_2}\right)\right\},$$

it follows that $\mathcal{J}_{\lambda,\mu}$ is bounded below in Z'. By the saddle point theorem, we obtain a critical point is a solution of problem (1.1). The proof is complete. □

Proof of Theorem 1.2. The functional $\mathcal{J}_{\lambda,\mu}$ is coercive in E_0, $\mathcal{J}_{\lambda,\mu}$ is bounded from below on Z' and there is a constant b, independent of λ, μ, such that $\inf_{Z'} \mathcal{J}_{\lambda,\mu} \geq b$.

For $\lambda < \lambda_1$ and $\mu < \mu_1$, by the definition of λ_1, μ_1 and (3.17), we obtain

$$\begin{aligned}\mathcal{J}_{\lambda,\mu}(u,v) &= \frac{1}{2}(\|u\|_{0,K}^2 + \|v\|_{0,G}^2) - \frac{\lambda}{2}\int_\Omega |u(x)|^2 dx - \frac{\mu}{2}\int_\Omega |v(x)|^2 dx \\ &\quad - \int_\Omega F(x,u,v)dx \\ &\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1} - \frac{\varepsilon}{\lambda_1}\right)\|u\|_{0,K}^2 + \frac{1}{2}\left(1 - \frac{\mu}{\mu_1} - \frac{\varepsilon}{\mu_1}\right)\|v\|_{0,G}^2 \\ &\quad - M_2\|(u,v)\| - M_1|\Omega|.\end{aligned} \tag{3.19}$$

Set $\varepsilon = \frac{1}{2}\min\{\lambda_1 - \lambda, \mu_1 - \mu\}$. We have by (3.19) and the inequality $2(a^2 + b^2) \geq (a+b)^2$ that

$$\mathcal{J}_{\lambda,\mu}(u,v) \geq \frac{1}{8}\min\left\{1 - \frac{\lambda}{\lambda_1}, 1 - \frac{\mu}{\mu_1}\right\}\|(u,v)\|^2 - M_2\|(u,v)\| - M_1|\Omega|,$$

which implies that $\mathcal{J}_{\lambda,\mu}$ is coercive in E_0.

For $(u,v) \in Z'$, from (2.6) and (3.17), we have

$$\begin{aligned}\mathcal{J}_{\lambda,\mu}(u,v) &\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_2} - \frac{\varepsilon}{\lambda_1}\right)\|u\|_{0,K}^2 + \frac{1}{2}\left(1 - \frac{\mu}{\mu_2} - \frac{\varepsilon}{\mu_1}\right)\|v\|_{0,G}^2 \\ &\quad - M_2\|(u,v)\| - M_1|\Omega| \\ &\geq \frac{1}{2}\left(1 - \frac{\lambda_1}{\lambda_2} - \frac{\varepsilon}{\lambda_1}\right)\|u\|_{0,K}^2 + \frac{1}{2}\left(1 - \frac{\mu_1}{\mu_2} - \frac{\varepsilon}{\mu_1}\right)\|v\|_{0,G}^2 \\ &\quad - M_2\|(u,v)\| - M_1|\Omega|.\end{aligned}$$

Putting $\varepsilon = \frac{1}{2}\min\left\{\lambda_1\left(1-\frac{\lambda_1}{\lambda_2}\right),\ \mu_1\left(1-\frac{\mu_1}{\mu_2}\right)\right\}$, thus $\mathcal{J}_{\lambda,\mu}$ is coercive in Z' and $\mathcal{J}_{\lambda,\mu}$ is bounded from below on Z', that is, there is a constant b, independent of λ, μ, such that $\inf_{Z'} \mathcal{J}_{\lambda,\mu} \geq b$.

In the following, we will show that if $\lambda < \lambda_1$, and $\mu < \mu_1$ are sufficiently close to λ_1, μ_1, there exist $t_1^- < 0 < t_1^+$, $t_2^- < 0 < t_2^+$ such that $\mathcal{J}_{\lambda,\mu}(t_1^{\pm}\varphi_1, t_2^{\pm}\psi_1) < b$. In fact, by Fatou's Lemma and condition (H2), there exist sufficiently large positive numbers t_1^+ and t_2^+ such that

$$\int_\Omega F(x, t_1^+ e_1, t_2^+ \omega_1)dx > -b+1. \tag{3.20}$$

For $\lambda_1 - \frac{\lambda_1}{(t_1^+)^2} < \lambda < \lambda_1$ and $\mu_1 - \frac{\mu_1}{(t_2^+)^2} < \mu < \mu_1$, from (3.20), we have

$$\begin{aligned}
\mathcal{J}_{\lambda,\mu}(t_1^+ e_1, t_2^+ \omega_1) &= \frac{(t_1^+)^2}{2}\|e_1\|_{0,K}^2 + \frac{(t_2^+)^2}{2}\|\omega_1\|_{0,G}^2 - \frac{\lambda(t_1^+)^2}{2}\int_\Omega |e_1|^2 dx \\
&\quad - \frac{\mu(t_2^+)^2}{2}\int_\Omega |\omega_1|^2 dx - \int_\Omega F(x, t_1^+ e_1, t_2^+ \omega_1)dx \\
&= \frac{(t_1^+)^2}{2}\|e_1\|_{0,K}^2 + \frac{(t_2^+)^2}{2}\|\omega_1\|_{0,G}^2 - \frac{\lambda(t_1^+)^2}{2\lambda_1}\|e_1\|_{0,K}^2 \\
&\quad - \frac{\mu(t_2^+)^2}{2\mu_1}\|\omega_1\|_{0,G}^2 - \int_\Omega F(x, t_1^+ e_1, t_2^+ \omega_1)dx \\
&= \frac{1}{2}\left(1-\frac{\lambda}{\lambda_1}\right)(t_1^+)^2 + \frac{1}{2}\left(1-\frac{\mu}{\mu_1}\right)(t_2^+)^2 \\
&\quad - \int_\Omega F(x, t_1^+ e_1, t_2^+ \omega_1)dx \\
&< 1 - \int_\Omega F(x, t_1^+ e_1, t_2^+ \omega_1)dx < b. \qquad (3.21)
\end{aligned}$$

A similar condition holds for $t_1^-, t_2^- < 0$.

If $\{z_n = (u_n, v_n)\}$ is a (PS) sequence of $\mathcal{J}_{\lambda,\mu}$, we get $\{(u_n, v_n)\}$ must be bounded, since $\mathcal{J}_{\lambda,\mu}$ is coercive. Then passing to a subsequence if necessary, there exists $z = (u, v) \in E_0$ such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E_0. Thus, there exists a strictly decreasing subsequence ϵ_n, $\lim_{n\to\infty} \epsilon_n = 0$ such that

$$|\mathcal{J}'_{\lambda,\mu}(u_n, v_n)(u_n - u, 0)| \leq \epsilon_n \|(u_n - u, 0)\|.$$

In particular

$$\begin{aligned}
&\Bigg|\int_{\mathbb{R}^{2n}} (u_n(x) - u_n(y))((u_n - u)(x) - (u_n - u)(y))K(x-y)dxdy \\
&-\lambda\int_\Omega u_n(u_n - u)dx - \int_\Omega F_u(x, u_n, v_n)(u_n - u)dx\Bigg| \leq \epsilon_n\|(u_n - u, 0)\|. \qquad (3.22)
\end{aligned}$$

By Lemma 8 of [10], we know that $u_n \to u$ in $L^2(\Omega)$, $v_n \to v$ in $L^2(\Omega)$. Thus

$$\lim_{n\to\infty}\int_\Omega u_n(u_n-u)dx \le \lim_{n\to\infty}\left(\int_\Omega u_n^2dx\right)^{1/2}\left(\int_\Omega |u_n-u|^2dx\right)^{1/2}=0. \tag{3.23}$$

Since the potential F satisfies (H1), it is easy to know that

$$\int_\Omega F_u(x,u_n,v_n)(u_n-u)dx \to 0. \tag{3.24}$$

Combining (3.22) with (3.23) and (3.24) we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^{2n}}(u_n(x)-u_n(y))((u_n-u)(x)-(u_n-u)(y))K(x-y)dxdy=0. \tag{3.25}$$

On the other hand, since

$$\begin{aligned}&\int_{\mathbb{R}^{2n}}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))K(x-y)dxdy\\ \to &\int_{\mathbb{R}^{2n}}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dxdy \quad \text{for any } \varphi\in X_0\end{aligned}$$

as $n\to+\infty$, we have

$$\int_{\mathbb{R}^{2n}}((u_n-u)(x)-(u_n-u)(y))(\varphi(x)-\varphi(y))K(x-y)dxdy\to 0, \quad n\to\infty. \tag{3.26}$$

Let $\varphi=u$, then (3.26) reduces to

$$\lim_{n\to\infty}\int_{\mathbb{R}^{2n}}(u(x)-u(y))((u_n-u)(x)-(u_n-u)(y))K(x-y)dxdy=0. \tag{3.27}$$

Adding (3.25) to (3.27), we conclude that

$$0=\lim_{n\to\infty}\left[\int_{\mathbb{R}^{2n}}(u_n(x)-u_n(y))^2K(x-y)dxdy-\int_{\mathbb{R}^{2N}}(u(x)-u(y))^2K(x-y)dxdy\right]$$

which implies $\|u_n\|_{0,K}^2\to\|u\|_{0,K}^2$. So, $\|u_n\|_{0,K}\to\|u\|_{0,K}$.

Similarly we have $\|v_n\|_{0,G}\to\|v\|_{0,G}$. The uniform convexity of E_0 yields that $\{z_n\}$ converges strongly to z in E_0. If $\lambda<\lambda_1$, $\mu<\mu_1$, the functional $\mathcal{J}_{\lambda,\mu}$ satisfies the (PS) condition. In addition, let

$$\sum_{\pm}=\{z\in E_0 : z=\pm(t_1e_1,t_2\omega_1)+w \text{ with } t_1,t_2>0 \text{ and } w\in Z'\}.$$

$\mathcal{J}_{\lambda,\mu}$ satisfies $(PC)_{c,\sum_+}$ and $(PC)_{c,\sum_-}$ for all $c<b$.

Let $\{z_n\}\subset\sum_+$ and $\mathcal{J}_{\lambda,\mu}(z_n)\to c<b$ and $\mathcal{J}'_{\lambda,\mu}(z_n)\to 0$ as $n\to\infty$. Since $\mathcal{J}_{\lambda,\mu}$ is coercive and the potential F satisfies (H1), there is $z\in E_0$ such that $z_n\to z$ strongly in E_0. If $z\in\partial\sum_+=Z'$, from $\inf_{Z'}\mathcal{J}_{\lambda,\mu}\ge b$, we get $\mathcal{J}_{\lambda,\mu}(z_n)\to c\ge b$, which is a contradiction. Thus $z\in\sum_+$ and $\mathcal{J}_{\lambda,\mu}$ satisfies the $(PC)_{c,\sum_+}$ condition. In a similar way, we get that $(PC)_{c,\sum_-}$ holds for all $c<b$.

If $\lambda < \lambda_1$, $\mu < \mu_1$ are sufficiently close to λ_1, μ_1, respectively, we obtain

$$-\infty < \inf_{\sum_\pm} \mathcal{J}_{\lambda,\mu} < b,$$

which implies that $\mathcal{J}_{\lambda,\mu}$ is bounded below in $\sum_+$. Consequently, according to Ekeland's variational principle, there exists $\{z_n\} \subset \sum_+$ such that $\mathcal{J}_{\lambda,\mu}(z_n) \to \inf_{\sum_+} \mathcal{J}_{\lambda,\mu}$ and $\mathcal{J}'_{\lambda,\mu}(z_n) \to 0$ as $n \to \infty$. Since $\mathcal{J}_{\lambda,\mu}$ satisfies $(PC)_{c,\sum_+}$ for all $c < b$, there is $z^+ \in \sum_+$ such that $\mathcal{J}_{\lambda,\mu}(z^+) = \inf_{\sum_+} \mathcal{J}_{\lambda,\mu}$, i.e., the infimum is obtained in $\sum_+$. A similar conclusion holds in $\sum_-$. So $\mathcal{J}_{\lambda,\mu}$ has two distinct critical points, denoted by z^+, z^-.

As in [10], we can obtain the third critical point z of $\mathcal{J}_{\lambda,\mu}$ by applying Mountain pass theorem such that $\mathcal{J}_{\lambda,\mu}(z) = c \geq b$. $\square$

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