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# Some limiting properties for GARCH$(p,q)$-X processes†

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**Abstract**

In this paper, we propose a modified GARCH$(p,q)$-X model which is obtained by adding the exogenous variables to the modified GARCH$(p,q)$ process. Some limiting properties are shown under various stationary and nonstationary exogenous processes which are generated by another process independent of the noise process. The proposed model extends the GARCH$(1,1)$-X model studied by Han (2015) to various GARCH$(p,q)$-type models such as GJR GARCH, asymptotic power GARCH and VGARCH combined with exogenous process. In comparison with GARCH$(1,1)$-X, we expect that many stylized facts including long memory property of the financial time series can be explained effectively by modified GARCH$(p,q)$ model combined with proper additional covariate.

*Keywords*: Conditional heteroskedasticity, exogenous variable, GARCH-X model, nonstationarity.

## 1. Introduction

After Engle (1982) and Bollerslev (1986), various modified version of GARCH models such as GJR-GARCH, EGARCH, MSGARCH etc. have been proposed and used in analyzing data from economic, finance and other various fields. The purpose of this variation is to explain many characteristic phenomena of data, such as many stylized facts and long memory. In all those models, the equation for conditional volatility is changed while keeping the same variable. In this paper we consider the generalized GARCH models, obtained by extending GARCH models with exogenous variables, so-called GARCH-X models. GARCH-X models as proposed by Hwang and Satchell (2005), Brenner *et al.* (1996) or Engle and Patton (2001) directly include the exogenous variable in the basic GARCH specification of Bollerslev (1986). The GARCH-X model is widely used by empirical researcher and practitioners (Fleming *et al.*, 2008). The idea behind this procedure for financial applications is that additional sources of information help to better understand the market's behavior and hence to improve the prediction of the market's reactions. We study some limiting properties

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for the modified GARCH($p, q$)-X model with stationary and non-stationary exogenous variables (Park, 2002; Han and Park, 2008; Han, 2015). Asymptotics for the sample variance of GARCH($p, q$)-X under various stationary or nonstationary regressors are given. The proofs of the theorems in the paper rely on the previous results given by, for example, Davidson and De Jong (2000), Park and Phillips (1999; 2001), and Han (2015).

In this paper, we consider the modified GARCH($p, q$)-X model defined as follows:

$$y_t = \sigma_t e_t, \tag{1.1}$$

$$\sigma_t^\delta = \sum_{i=1}^p c_i(e_{t-i})\sigma_{t-i}^\delta + \sum_{j=1}^q g_j(e_{t-j}) + u(x_{t-1}), \tag{1.2}$$

where $\delta > 0$ and $c_i(\cdot), g_j(\cdot)$ $(i = 1, \cdots, p,\ j = 1, \cdots, q)$, and $u(\cdot)$ are real valued nonnegative continuous function. Let $(e_t)$ be a sequence of independent and identically distributed(iid) random variables with mean zero and $E(|e_t|^\delta) < \infty$ for given $\delta > 0$. $(x_t)$ represents the exogenous process used for the improvement of the modeling behavior. We assume that the process $\{(y_t, x_t)\}$ is adapted to $\mathcal{F}_t$, where $\mathcal{F}_t$ represents the set of all information available until time $t$. We consider the case where exogenous process $(x_t)$ is generated by another process which is independent of the noise process $(e_t)$, for example, $x_t = \rho x_{t-1} + v_t$ with $|\rho| \le 1$, $(v_t) \sim iid(0, \sigma_v^2)$ and $(v_t)$ is independent on $(e_t)$.

When $u(x) = 0$, then our model includes many well known GARCH-type models, such as classical GARCH($\delta = 2, g(x) = \omega, c(x) = \beta + \alpha x^2$), GJR GARCH($\delta = 2, g(x) = \omega, c(x) = \beta + (\alpha + \gamma I_{\{x>0\}} x^2)$), asymmetric power GARCH( $c_i(e_t) = \alpha_i(|e_t| - \gamma_i e_t)^\delta + \beta_i$), VGARCH( $g_j(e_t) = \omega/p + \alpha_j(e_t + \gamma_j)^2, c_i(e_t) = \beta_i$), and EGARCH( $\delta \to 0, g_j(e_t) = \omega/p + \alpha_j e_t + \gamma_j(|e_t| - E|e_t|), c_i(e_t) = \beta_i$) etc. Asymptotics and applications for the model given by (1.1) and (1.2) with $u(x) = 0$ are studied by many authors, e.g., Atsmegiorgis *et al.* (2016), Giraitis *et al.* (2000), Jeong and Lee (2017), and Lee (2014).

## 2. Main results

We make the assumption:
(A) $\rho_0 := \sum_{i=1}^p E(c_i(e_0)) < 1$ and $G = \sum_{j=1}^q E(g_j(e_0)) < \infty$.

**Theorem 2.1** Consider the process $\sigma_t^\delta$ given by (1.1) and (1.2). Suppose that $(x_t)$ is stationary ergodic and independent of $(e_t)$ with $E(u(x_t)) < \infty$. Assume $p = q$. If the assumption (A) holds, then there is a unique strictly stationary solution $\sigma_t^\delta$ to (1.1) and (1.2). The solution is given as

$$\sigma_t^\delta = \sum_{k=1}^\infty \sum_{1 \le i_1, \cdots, i_{k-1} \le p} \Big(\sum_{i_k=1}^p g_{i_k}(e_{t-i_1-\cdots-i_k}) + u(x_{t-i_1-\cdots-i_{k-1}-1})\Big)\Pi_{j=1}^{k-1} c_{i_j}(e_{t-i_1-\cdots-i_j}). \tag{2.1}$$

Here we let $\Pi_{j=1}^0 c_{i_j}(e_{t-i_1-\cdots-i_j}) = 1$.

**Proof**: We may rewrite the equation (1.2) as

$$\sigma_t^\delta = \sum_{i=1}^p c_i(e_{t-i})\sigma_{t-i}^\delta + \omega_{t-1}, \tag{2.2}$$

where $\omega_{t-1} := \sum_{j=1}^{q} g_j(e_{t-j}) + u(x_{t-1})$. Applying the equation (2.2) recursively, after $m$ steps, we have that

$$\sigma_t^\delta = \sum_{k=1}^{m} \sum_{1\le i_1,i_2,\cdots,i_{k-1}\le p} \omega_{t-1-i_1-\cdots-i_{k-1}} \Pi_{j=1}^{k-1} c_{i_j}(e_{t-i_1-\cdots-i_j}) + \sum_{1\le i_1,i_2,\cdots,i_m\le p} \Pi_{j=1}^{m} c_{i_j}(e_{t-i_1-\cdots-i_j})\sigma^\delta_{t-i_1\cdots-i_m}. \tag{2.3}$$

From the above equation (2.3), we define

$$\hat{\sigma_t}^\delta = \sum_{k=1}^{\infty} \sum_{1\le i_1,i_2,\cdots,i_{k-1}\le p} \omega_{t-1-i_1-\cdots-i_{k-1}} \Pi_{j=1}^{k-1} c_{i_j}(e_{t-i_1-\cdots-i_j}).$$

Then $\hat{\sigma_t}^\delta$ is a nonanticipative strictly stationary solution to (1.1) and (1.2). Since $\hat{\sigma_t}^\delta$ is a solution to the equations (1.1) and (1.2), it satisfies the equation (2.2) and (2.3) . Hence we have that

$$\begin{aligned} E|\sigma_t^\delta - \hat{\sigma_t}^\delta| &= E(\sum_{1\le i_1,i_2,\cdots,i_m\le p} \Pi_{j=1}^{m} c_{i_j}(e_{t-i_1-\cdots-i_j}))E|\sigma^\delta_{t-i_1-\cdots-i_m} - \hat{\sigma}^\delta_{t-i_1-\cdots-i_m}| \\ &= \rho_0^m E|\sigma^\delta_{t-i_1-\cdots-i_m} - \hat{\sigma}^\delta_{t-i_1-\cdots-i_m}| \to 0, \end{aligned}$$

as $m\to\infty$, which implies the uniqueness of the solution. Now by independence of $\omega_{t-1-i_1\cdots-i_k}$ and $\Pi_{j=1}^{k} c_{i_j}(e_{t-i_1\cdots-i_j})$ and the assumption (A), $E(\hat{\sigma_t}^\delta) = \sum_{k=1}^{\infty} \rho_0^{k-1} E(\omega_t) = (1/(1-\rho_0))(G + E(u(x_t))) < \infty$. Take $\sigma_t^\delta = \hat{\sigma_t}^\delta$. □

In this paper, we consider the following various cases of generating function for exogenous variables:

(C1) $x_t = \rho x_{t-1} + v_t,\ \ |\rho| \le 1,\ \ (v_t) \sim iid(0, \sigma_v^2)$.

(C2) $x_t = (1-L)^{-d} v_t,\ \ |d| < 1/2,\ \ (v_t) \sim iid(0, \sigma_v^2)$.

(C3) $x_t = x_{t-1} + v_t$ and $v_t = \psi(L)\eta_t = \sum_{k=0}^{\infty} \psi_k \eta_{t-k}$ with $\psi_0 = 1, \psi(1) \ne 0$ and $\sum_{k=0}^{\infty} k^{1/2}|\psi_k| < \infty,\ \ (\eta_t) \sim iid(0, \sigma_\eta^2)$.

(C4) $x_t = x_{t-1} + v_t$ and $v_t = (1-L)^{-d}\xi_t,\ \ |d| < 1/2$, where $\xi_t$ satisfies the Assumption 1 in Davidson and De Jong (2000).

For (C1)-(C4), we assume that $(e_t)$ and $(v_t)$ are independent.

Modified GARCH($p,q$)-X model with suitable conditions for the additional covariate can explain many characteristic phenomena of financial data. Covariate given by (C1)-(C4) is allowed to be stationary short memory, stationary long memory or nonstationary long memory. For example, consider $(x_t)$ in (C2) with $v_t \sim iid\ N(0, \sigma_v^2)$. Then if $-1/2 < d \le 1/4$, $u(x_t) = x_t^2$ is a short memory process. If $1/4 < d < 1/2$, then $x_t^2$ has long memory with parameter $2d - 1/2$. On the other hand, for $(x_t)$ given in (C4), $u(x_t)$ is allowed to be a nonstationary long memory process including an integrated process.

Let $\xrightarrow{d}$ denote the convergence in distribution. Likewise, we use $\xrightarrow{p}$ to signify the convergence in probability. For notational simplicity, we define

$$\omega_t := \sum_{j=1}^{q} g_j(e_{t+1-j}) + u(x_t),$$

and

$$z_{t,k} := \Pi_{j=1}^{k} c_{i_j}(e_{t-i_1\cdots-i_j}) - \Pi_{j=1}^{k} E(c_{i_j}(e_{t-i_1\cdots-i_j})).$$

**Lemma 2.1** (stationary case) Suppose that the assumption (A) and either (C1) with $|\rho| < 1$ or (C2) hold. Then as $n \to \infty$,

$$\frac{1}{n}\sum_{t=1}^{n} \omega_t \xrightarrow{p} G + E(u(x_t)), \tag{2.4}$$

$$\frac{1}{n}\sum_{t=1}^{n} \omega_{t-1-i_1-\cdots-i_k} z_{t,k} \xrightarrow{p} 0. \tag{2.5}$$

**Proof**: Under the assumptions, it is known that $(x_t)$ is strictly stationary and ergodic. The equation (2.4) is obtained by applying the ergodic theorem. Note that for fixed $k \geq 1$, $(z_{t,k})$ is a mean zero process which is strictly stationary ergodic and $z_{t,k}$ is independent of $\omega_{t-1-i_1-\cdots-i_k}$. Apply the ergodic theorem again to have the equation (2.5). □

**Theorem 2.2** Consider the process $y_t$ defined by (1.1), (1.2), and the exogenous process $(x_t)$ generated by the assumption either (C1) with $|\rho| < 1$ or (C2). If the assumption (A) and $E(u(x)) < \infty$ hold, then as $n \to \infty$,

$$\frac{1}{n}\sum_{t=1}^{n} |y_t|^{\delta} \xrightarrow{p} \frac{G + E(u(x_t))}{1-\rho_0} E|e_t|^{\delta}.$$

**Proof**: By the assumption (A),

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{1\leq i_1,i_2,\cdots,i_{k-1}\leq p} \Pi_{j=1}^{k-1} E(c_{i_j}(e_{t-i_1\cdots-i_j})) &= \sum_{k=1}^{\infty} \rho_0^{k-1} \\ &= \frac{1}{1-\rho_0}. \end{aligned} \tag{2.6}$$

From (2.1), (2.4), (2.5), and (2.6), we have that as $n \to \infty$,

$$\frac{1}{n}\sum_{t=1}^{n} \sigma_t^{\delta} = \frac{1}{n}\sum_{t=1}^{n}(\sum_{k=1}^{\infty} \sum_{1\leq i_1,i_2,\cdots,i_{k-1}\leq p} \omega_{t-1-i_1\cdots-i_{k-1}} \Pi_{j=1}^{k-1} c_{i_j}(e_{t-i_1\cdots-i_j}))$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \sum_{1\le i_1,i_2,\cdots,i_{k-1}\le p} \frac{1}{n}\sum_{t=1}^{n} \omega_{t-1-i_1\cdots-i_{k-1}} z_{t,k-1} \\
&\quad + \sum_{k=1}^{\infty} \sum_{1\le i_1,i_2,\cdots,i_{k-1}\le p} \frac{1}{n}\sum_{t=1}^{n} \omega_{t-1-i_1\cdots-i_{k-1}} \Pi_{j=1}^{k-1} E(c_{i_j}(e_{t-i_1\cdots-i_j})) \\
&\xrightarrow{p} \frac{1}{1-\rho_0}(G + E(u(x_t))). \qquad (2.7)
\end{aligned}
$$

Also, by applying the ergodic theorem, the equation (2.7), and the independence of $\sigma_t^\delta$ and $e_t^\delta$, we obtain that

$$
\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n}|y_t|^\delta &= \frac{1}{n}\sum_{t=1}^{n}\sigma_t^\delta(|e_t|^\delta - E(|e_t|^\delta)) + E|e_t|^\delta \frac{1}{n}\sum_{t=1}^{n}\sigma_t^\delta \\
&\xrightarrow{p} \frac{G + E(u(x_t))}{1-\rho_0} E|e_t|^\delta.
\end{aligned}
$$

□

Note that $(x_t)$ given by (C3) or (C4) is a nonstationary integrated process. The behavior of the model $y_t$ with a nonstationary $(x_t)$ given by (C3) or (C4) depends on the function $u(\cdot)$ in (1.2). We assume that the function $u(\cdot)$ is H-regular. For the definitions of regular and H-regular function, see Park and Phillips (1999,2001). A H-regular function $u(\cdot)$ with the asymptotic order $k$ and the limit homogeneous function $h$ can be written as $u(\lambda x) = k(\lambda)h(x) + r(x,\lambda)$, $h$ is locally integrable and $r$ is such that (a) $|r(x,\lambda)| \le a(\lambda)p(x)$, where $\limsup_{\lambda\to\infty} a(\lambda)/k(\lambda) = 0$ and $p$ is locally integrable, or (b) $|r(x,\lambda)| \le b(\lambda)p(x)q(\lambda x)$, where $\limsup_{\lambda\to\infty} b(\lambda)/k(\lambda) < \infty$ and $q$ is locally integrable and vanishes at infinity, i.e., $q(x) \to 0$ as $|x| \to \infty$. We make the assumption on $u(x)$ in (1.2).

(B) $u(x)$ in (1.2) is a H-regular function with the asymptotic order $k$ and the limit homogeneous function $h$ and $h$ is regular.

Now, consider the case where $(x_t)$ is given by the assumption (C3). Recall that if $\eta_t$ is iid $(0, \sigma_\eta^2)$, then

$$
W_n(r) := \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}\frac{\eta_t}{\sigma_\eta} \xrightarrow{d} W(r), \quad 0 \le r \le 1,
$$

where $W(r)$ is the standard Brownian motion on the unit interval $[0,1]$.

Under the assumption (C3), Phillips and Solo (1992) show that

$$
\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]} v_t \xrightarrow{d} \psi(1)\sigma_\eta W(r), \quad 0 \le r \le 1.
$$

If in addition, $h$ is regular, then

$$
\frac{1}{n}\sum_{t=1}^{n} h\left(\frac{x_t}{\sqrt{n}}\right) \xrightarrow{d} \int_0^1 h(\sigma_v W(r))dr, \qquad (2.8)
$$

where $\sigma_v = \psi(1)\sigma_\eta$ (Theorem 3.2 in Park and Phillips, 1999).

**Lemma 2.2** (nonstationary case) Suppose the assumption (A), (B), and (C3) hold. If $k(\sqrt{n}) \to \infty$ as $n \to \infty$, then as $n \to \infty$,

$$\frac{1}{nk(\sqrt{n})}\sum_{t=1}^{n}\omega_t \xrightarrow{d} \int_0^1 h(\sigma_v W(r))dr. \tag{2.9}$$

**Proof**: Assumption (B) implies that $u(\lambda x) = k(\lambda)h(x) + o(k(\lambda))$ for all large $\lambda$. The second part of the assumption (A) implies that

$$\frac{1}{nk(\sqrt{n})}\sum_{t=1}^{n}\sum_{j=1}^{q} g_j(e_{t+1-j}) = o_p(1).$$

Hence from (2.8) we obtain that

$$\begin{aligned}
\frac{1}{nk(\sqrt{n})}\sum_{t=1}^{n}\omega_t &= \frac{1}{nk(\sqrt{n})}\sum_{t=1}^{n}u(x_t) + o_p(1)\\
&= \frac{1}{nk(\sqrt{n})}\sum_{t=1}^{n}\Big(k(\sqrt{n})h(\frac{x_t}{\sqrt{n}}) + o_p(k(\sqrt{n}))\Big)\\
&= \frac{1}{n}\sum_{t=1}^{n}h(\frac{x_t}{\sqrt{n}}) + o_p(1)\\
&\xrightarrow{d} \int_0^1 h(\sigma_v W(r))dr.
\end{aligned}$$

□

**Theorem 2.3** Suppose that the assumption (A), (B), and (C3) hold. If $k(\sqrt{n}) \to \infty$ as $n \to \infty$ and $E(c_i(e_0))^l < \infty$ for some $l \geq 2$, then as $n \to \infty$,

$$(1) \quad \sum_{t=1}^{n}\omega_{t-1-i_1-\cdots-i_k}z_{t,k} = o_p(nk(\sqrt{n})), \tag{2.10}$$

$$(2) \quad \frac{1}{nk(\sqrt{n})}\sum_{t=1}^{n}|y_t|^\delta \xrightarrow{d} \frac{E|e_t|^\delta}{1-\rho_0}\int_0^1 h(\sigma_v W(r))dr.$$

**Proof**: (1) Since $z_{t,k-1}$ is a mean zero $p(k-1)$-dependent process and $x_t$ and $z_{t,k}$ are independent, using Lemma A.2 in Park and Phillips (2001) to get $\sum_{t=1}^{n}u(x_t)z_{t,k-1} = o_p(nk(\sqrt{n}))$ and the result follows. (2) Use the equation (2.9) and (2.10) to obtain that

$$\begin{aligned}
\frac{1}{nk(\sqrt{n})}\sum_{t=1}^{n}\sigma_t^\delta &= \frac{1}{nk(\sqrt{n})}\sum_{t=1}^{n}[\sum_{k=1}^{\infty}\sum_{1\leq i_1,\cdots,i_{k-1}\leq p}\omega_{t-1-i_1-\cdots-i_{k-1}}z_{t,k-1}\\
&\quad+\sum_{k=1}^{\infty}\sum_{1\leq i_1,\cdots,i_{k-1}\leq p}\omega_{t-1-i_1-\cdots-i_{k-1}}\Pi_{j=1}^{k-1}E(c_{i_j}(e_{t-i_1-\cdots-i_j}))]\\
&\xrightarrow{d} \frac{1}{1-\rho_0}\int_0^1 h(\sigma_v W(r))dr.
\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{nk(\sqrt{n})}\sum_{t=1}^{n}|y_t|^\delta &= \frac{1}{nk(\sqrt{n})}(\sum_{t=1}^{n}\sigma_t^\delta(|e_t|^\delta - E|e_t|^\delta) + \sum_{t=1}^{n}\sigma_t^\delta E|e_t|^\delta) \\ &= o_p(1) + E|e_t|^\delta \frac{1}{nk(\sqrt{n})}\sum_{t=1}^{n}\sigma_t^\delta \\ &\xrightarrow{d} \frac{E|e_t|^\delta}{1-\rho_0}\int_0^1 h(\sigma_v W(r))dr.\end{aligned}$$

□

Now consider the process $(x_t)$ obtained under the assumption (C4). Recall that from Theorem 3.1 and 4.2 in Davidson and De Jong (2000), we have that

$$\frac{1}{\sigma_{nv}}\sum_{t=1}^{[nr]} v_t \xrightarrow{d} W_d(r), \quad 0 \le r \le 1$$

and

$$\frac{1}{n}\sum_{t=1}^{n}\frac{x_t}{\sigma_{nv}} \xrightarrow{d} \int_0^1 W_d(r)dr, \tag{2.11}$$

where $\sigma_{nv}^2 = E(\sum_{t=1}^n v_t)^2 = O_p(n^{1+2d})$. Apply the continuous mapping theorem to (2.11) to obtain that for a continuous function $h$,

$$\frac{1}{n}\sum_{t=1}^{n}h(\frac{x_t}{\sigma_{nv}}) \xrightarrow{d} \int_0^1 h(W_d(r))dr. \tag{2.12}$$

Here $W_d(r)$ denotes a fractional Brownian motion.

Moreover,

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}h(\frac{x_t}{\sigma_{nv}})u_t \xrightarrow{d} \int_0^1 h(W_d(r))dU,$$

if zero mean process $u_t$ satisfies the condition in Remark A1 in Han (2015).

**Lemma 2.3** Assume (A), (B) and (C4) and $k(\sigma_{nv}) \to \infty$ as $n \to \infty$. Then

$$\frac{1}{nk(\sigma_{nv})}\sum_{t=1}^{n} w_t \xrightarrow{d} \int_0^1 h(W_d(r))dr. \tag{2.13}$$

**Proof**:

$$\begin{aligned}\frac{1}{nk(\sigma_{nv})}\sum_{t=1}^{n} w_t &= \frac{1}{nk(\sigma_{nv})}\sum_{t=1}^{n}k(\sigma_{nv})h(\frac{x_t}{\sigma_{nv}}) + o_p(1) \\ &\xrightarrow{d} \int_0^1 h(W_d(r)dr.\end{aligned}$$

□

**Theorem 2.4** Assume (A), (B), and (C4). If, in addition, $E(c_i(e_0))^l < \infty$ for some $l \geq 2$ and $k(\sigma_{nv}) \to \infty$ as $n \to \infty$, then

$$(1) \quad \sum_{t=1}^{n} \omega_{t-1-i_1-\cdots-i_k} z_{t,k} = o_p(nk(\sigma_{nv})).$$

$$(2) \quad \frac{1}{nk(\sigma_{nv})} \sum_{t=1}^{n} |y_t|^\delta \xrightarrow{d} \frac{E|e_t|^\delta}{1-\rho_0} \int_0^1 h(W_d(r))dr.$$

**Proof**: (1) Note that under the assumptions, for fixed $k$, $z_{t,k}$ is a mean zero, covariance stationary and $L_2$-NED of size $-1/2$, that is, $z_{t,k}$ satisfies all conditions for Theorem 4.1, 4.2 in Davidson and De Jong (2000). Therefore we have that $n^{-1/2}\sum_{t=1}^n z_{t,k} \xrightarrow{d} W$ and

$$n^{-1/2} \sum_{t=1}^{n} h(\frac{x_t}{\sigma_{nv}}) z_{t,k-1} \xrightarrow{d} \int_0^1 h(W_d(r))dW. \tag{2.14}$$

From (2.14), we have that

$$\begin{aligned} \frac{1}{nk(\sigma_{nv})} \sum_{t=1}^{n} \omega_{t-1-i_1-\cdots-i_{k-1}} z_{t,k-1} &= o_p(1) + \frac{1}{nk(\sigma_{nv})} \sum_{t=1}^{n} u(x_t) z_{t,k-1} \\ &= o_p(1) + \frac{1}{\sqrt{n}} (\frac{1}{\sqrt{n}} \sum_{t=1}^{n} h(\frac{x_t}{\sigma_{nv}})) \\ &\xrightarrow{p} 0. \end{aligned}$$

(2) By the same process used in the proof of Theorem 3 and (2.13), we have that

$$\frac{1}{nk(\sigma_{nv})} \sum_{t=1}^{n} |y_t|^\delta \xrightarrow{d} \frac{E|e_t|^\delta}{1-\rho_0} \int_0^1 h(W_d(r))dr.$$

□

***Remark 2.1*** Recall that if the assumption (C2) holds, then $x_t = \sum_{j=0}^\infty \theta_j v_{t-j}$, where $\theta_j = \frac{\Gamma(d+j)}{\Gamma(d)\Gamma(j+1)} \sim \frac{1}{\Gamma(d)} j^{d-1}$ and $x_t$ is strictly stationary with $\sum_{j=0}^\infty \theta_j^2 < \infty$ and $\sum_{j=0}^\infty \theta_j^4 < \infty$, hence $E(x_t^2) < \infty$ and $E(x_t^4) < \infty$. (C1) with $\rho = 1$ is a special case of (C3) with $\psi_0 = 1, \psi_k = 0$ if $k \neq 0$. It is known $(1-L)^{-d}\eta_t = \sum_{k=0}^\infty \psi_k \eta_{t-k}$ with $\psi_k \sim (1/\Gamma(d))k^{d-1}$. In order to $\sum_{k=0}^\infty k^{1/2} k^{d-1} < \infty$, the condition $d < -1/2$ is necessary. Thus (C4) is not the case of (C3). In fact $\xi_t$ in (C4) is not iid but $L_2$-NED.

In comparison with the previous results of Theorem 2, we consider the following limiting behavior of $(x_t)$ given by the assumption (C2) with $E(v_t^4) < \infty$.

**Theorem 2.5** If $x_t$ is defined by the assumption (C2) with $E(v_t^4) < \infty$, then the FCLT holds for $x_t^2$ with $-1/2 < d < -1/4$.

$$\frac{1}{\sigma_n} \sum_{t=1}^{[nr]} (x_t^2 - E(x_t^2)) \xrightarrow{d} W,$$

where $\sigma_n^2 = Var(\sum_{t=1}^n x_t^2)$.

**Proof**: Define $E_{t-m}^{t+m}(X) := E(X|\sigma(v_{t-m},\cdots,v_t,\cdots,,v_{t+m}))$. Under the assumption (C2), $x_t = \sum_{j=1}^{\infty}\theta_j v_{t-j},\ \ \theta_j \sim \frac{1}{\Gamma(d)} j^{d-1}$.

Note that

$$\begin{aligned} E_{t-m}^{t+m}(x_t^2) &= \sum_{j=1}^{m}\theta_j^2 v_{t-j}^2 + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\theta_i\theta_j v_{t-i}v_{t-j} \\ &\quad + E_{t-m}^{t+m}(\sum_{j=m+1}^{\infty}\theta_j^2 v_{t-j}^2 + 2\sum_{i=1}^{\infty}\sum_{i<j,j=m+1}^{\infty}\theta_i\theta_j v_{t-i}v_{t-j}). \end{aligned}$$

Using Minkowski inequality and Cauchy-Schwarz inequality yields that

$$\begin{aligned} \|x_t^2 - E_{t-m}^{t+m}(x_t^2)\|_2 &\le \sum_{j=m+1}^{\infty}\theta_j^2\|v_{t-j}^2-\sigma_v^2\|_2 + 2\sum_{i=1}^{\infty}\sum_{i<j,j=m+1}\theta_i\theta_j\|v_{t-i}v_{t-j}\|_2 \\ &\le \|v_t^2-\sigma_v^2\|_2\sum_{j=m+1}^{\infty}\theta_j^2 + 2\|v_t\|_4^2(\sum_{i=1}^{m}\sum_{j=m+1}^{\infty}\theta_i\theta_j + \sum_{i=m+1}^{\infty}\sum_{j=i+1}^{\infty}\theta_i\theta_j). \end{aligned}$$

Now,

$$\sum_{j=m+1}^{\infty}\theta_j^2 \sim \frac{1}{\Gamma(d)}\int_{m+1}^{\infty}x^{2(d-1)}dx = O(m^{2d-1}).$$

Similarly, we have that

$$\sum_{i=1}^{m}\sum_{j=m+1}^{\infty}\theta_i\theta_j = O(m^{2d}),$$

and hence

$$\|x_t^2 - E_{t-m}^{t+m}(x_t^2)\|_2 \le O(m^{2d}).$$

Therefore, $\{x_t^2\}$ is $L_2$-NED of size $-1/2$ if $-1/2<d<-1/4$.

Next, by simple calculation, $\sum_{h=1}^{\infty}Cov(x_t^2,x_{t+h}^2) = (\mu_4-\mu_2^2)(\sum_{i=0}^{\infty}\theta_i^2\theta_{h+i}^2) + 4\mu_2^2\sum_{i<j}\theta_i\,\theta_j\theta_{h+i}\theta_{h+j}$. Adopt the integral approximation to get $\sum_{h=1}^{\infty}Cov(x_t^2,x_{t+h}^2)<\infty$, which implies that $n^{-1}Var(\sum_{t=1}^{n}x_t^2)$ converges as $n\to\infty$. Thus, $\{x_t^2 - E(x_t^2)\}$ satisfies the assumption of Theorem 1.2 in Davidson (2002), and the conclusion follows. □

If $c_i = g_j = 0\ \forall i,j (i=1,\cdots,p, j=1,\cdots,q)$, $u(x)=\omega+\pi x^2$, and $\delta=2$ in (1.2) and $x_t$ satisfies the condition in Theorem 5, then we have that

$$\begin{aligned} \frac{1}{\sigma_n}\sum_{t=1}^{[nr]}(y_t^2 - E(y_t^2)) &= \frac{1}{\sigma_n}\sum_{t=1}^{[nr]}\pi(e_{t-1}^2-1)(x_{t-1}^2-E(x_{t-1}^2)) \\ &\quad + \frac{1}{\sigma_n}\sum_{t=1}^{[nr]}\pi(x_{t-1}^2 - E(x_{t-1}^2)) \\ &\xrightarrow{d} \pi W. \end{aligned}$$

□

As an example, we consider the following GARCH$(p, q)$-X model defined by

$$y_t = \sigma_t e_t, \tag{2.15}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 + \pi x_{t-1}^2, \tag{2.16}$$

where $\omega > 0, \alpha_i, \beta_j \geq 0 (i = 1, \cdots, p, \quad j = 1, \cdots, q), \pi \geq 0$ are constants and $(e_t)$ is iid with mean zero. Here $c_i(e_t) := \alpha_i e_t^2 + \beta_i,\ \omega_t := \omega + \pi x_t^2$. Assume that $\rho_0 = \sum_{j=1}^{p} (\alpha_j + \beta_j) < 1$ and $E(c_i(e_0))^l < \infty$ for some $l > 2$.

GARCH(1,1)-X model with $\alpha_i = \beta_j = 0, i \geq 2, j \geq 2$ in (2.16) is considered by Han (2015) and various asymptotics are proved. Note that $u(x) = \pi x^2$ is a H-regular function with the asymptotic order $k(\lambda) = \pi \lambda^2$ and the limit homogenous function $h(x) = x^2$. If $c_i(\cdot) = 0, g_j(\cdot) = 0\ \forall i, j, \delta = 2$ and $x_t$ is generated by the assumption (C3), then the process generated by (1.1) and (1.2) is the model studied in Park (2002). Asymptotic behavior of the sample variance, autocorrelation, kurtosis, and leptokurtosis are examined under the proper condition given to $u(\cdot)$.

Apply the previous results to GARCH$(p, q)$-X model of (2.15) and (2.16) to obtain that, for example, under the assumption (C2),

$$\frac{1}{n} \sum_{t=1}^{n} y_t^2 \ \xrightarrow{p}\ \frac{1}{1-\delta} (\omega + \pi E(x_t^2))$$

and under the assumption (C3),

$$\frac{1}{n\sigma_{n\xi}^2} \sum_{t=1}^{n} y_t^2 \ \xrightarrow{d}\ \frac{\pi}{1-\delta} \int_0^1 (W_d(r))^2 dr.$$

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