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Estimation of entropy of the inverse weibull distribution under generalized progressive hybrid censored data

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Abstract

The inverse Weibull distribution (IWD) can be readily applied to a wide range of situations including applications in medicines, reliability and ecology. It is generally known that the lifetimes of test items may not be recorded exactly. In this paper, therefore, we consider the maximum likelihood estimation (MLE) and Bayes estimation of the entropy of a IWD under generalized progressive hybrid censoring (GPHC) scheme. It is observed that the MLE of the entropy cannot be obtained in closed form, so we have to solve two non-linear equations simultaneously. Further, the Bayes estimators for the entropy of IWD based on squared error loss function (SELF), precautionary loss function (PLF), and linex loss function (LLF) are derived. Since the Bayes estimators cannot be obtained in closed form, we derive the Bayes estimates by revoking the Tierney and Kadane approximate method. We carried out Monte Carlo simulations to compare the classical and Bayes estimators. In addition, two real data sets based on GPHC scheme have been also analysed for illustrative purposes.

Keywords: Generalized progressive hybrid censoring, inverse weibull distribution, maximum likelihood estimation, Tierney and Kadane approximation.

1. Introduction

Entropy, which is one of the important terms in statistical mechanics, was originally defined in physics especially in the second law of thermodynamics. The differential entropy $H(f)$ (Cover and Thomas, 2005) of the random variable X is given by

$$H(f) = -\int_{-\infty}^{\infty} f(x)\log f(x)dx,$$

where $f(x)$ denote a probability density function (pdf) of random variable X. Kang *et al.* (2012) considered the approximate maximum likelihood estimators of entropy of a double exponential distribution based on multiply Type II censoring. Cho *et al.* (2014) provided the Bayes estimators of entropy of a Rayleigh distriution based on doubly-generalized Type II hybrid censoring scheme. Cho *et al.* (2015a) provided the Bayes estimators of entropy of

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a Weibull distribution based on generalized progressive hybrid censoring scheme. Lee and Cho (2015) provided the Bayes estimators of entropy of a exponential distribution based on multiply Type II censored competing risks data.

The IWD can be readily applied to a wide range of situations including applications in reliability, medicines and ecology. Keller *et al*. (1985) derived the IWD by investigating failures of mechanical components subject to degradation. Khan *et al*. (2008) presented some important theoretical properties of the IWD. The cumulative distribution function (cdf) is given by

$$F(x;\alpha,\beta) = exp\left(-\beta x^{-\alpha}\right), \quad x>0, \quad \alpha>0, \quad \beta>0, \tag{1.1}$$

where α and β are the scale and shape parameters, respectively. Note that when $\alpha = 1$, we have the Frechet distribution function. Also, when $\beta = 1$ and $\beta = 2$, the IWDs are referred to as the inverse exponential and inverse Raleigh distribution, respectively.

Let us consider a life-testing experiment where n items is kept under observation until failure. These items could be some components, system or computer chips in reliability study experiments, or they could be patients put under certain clinical or drug conditions. However, it is generally known that the lifetimes of test items may not be recorded exactly. Also, there are situations wherein the withdrawal of items prior to failure is prearranged in order to decrease the cost or time associated with experience.

Therefore, the aim of this paper is to propose the classical and Bayes estimation of the entropy of a IWD under GPHC scheme. However, we observed that the MLE of the entropy cannot be derived in closed form. So we have to solve two non-linear equations simultaneously. Also, we derive the Bayes estimation of the entropy based on flexible priors. The Bayes estimators for the entropy of IWD based on SELF, PLF and LLF are derived. Since the Bayes estimators cannot be obtained in closed form, we derive the Bayes estimates by revoking the Tierney and Kadane approximate method.

The rest of this paper is organized as follows. In section 2, we derives a classical and Bayes estimators of the entropy of IWD based on GPHC scheme. In section 3, Monte Carlo simulations are conducted to compare the results among classical and Bayes estimators, and real data set based on GPHC scheme are analysed for illustrative purposes.

2. Entropy estimation

Let X be a random variable with the cdf, $F(x)$, and the pdf, $f(x)$. Then, the differential entropy of X is given by

$$H(f) = E[-logf(x)] = -\int_0^\infty f(x)logf(x)dx = -\alpha\beta[A+B+C],$$

where A, B and C are obtained below:

$$\begin{aligned}
A &= \int_0^\infty log(\alpha\beta)x^{-(\alpha+1)}e^{-\beta x^{-\alpha}}dx = \frac{log(\alpha\beta)}{\alpha\beta}, \\
B &= -(\alpha+1)\int_0^\infty log(x)x^{-(\alpha+1)}e^{-\beta x^{-\alpha}}dx \\
&= \frac{\alpha+1}{\alpha^2\beta}\left[\int_0^\infty log(u_2)e^{-u_2}du_2 - \int_0^\infty log(\beta)e^{-u_2}du_2\right] = -\frac{\alpha+1}{\alpha^2\beta}\left(\gamma+log\beta\right),
\end{aligned}$$

and

$$C = -\beta \int_0^\infty x^{-2\alpha-1} e^{-\beta x^{-\alpha}} dx = -\frac{1}{\alpha\beta} \int_0^\infty u_2 e^{-\beta u_2} du_2 = -\frac{1}{\alpha\beta},$$

where γ is the Euler-Mascheroni constant. Finally, Shannon entropy reduces to

$$H(f) = 1 + \left(1 + \frac{1}{\alpha}\right)(\gamma + \log\beta) - \log(\alpha\beta).$$

2.1. Maximum likelihood estimation

This section deals with deriving MLEs of the unknown parameters of a IWD. As a consequence, MLE of entropy will also be obtained. Suppose that $X_{1:m:n}$, $X_{2:m:n}$, $\cdots$, $X_{m:m:n}$ denote the observed values of such a progressively Type II censored sample. The integer $m, k \in \{1, 2, \cdots, n\}$ is pre-fixed such that $k < m$. Also, $(R_1, R_2, \cdots, R_m)$ are pre-fixed integers satisfying $\sum_{i=1}^m R_i + m = n$. And $T \in (0, \infty)$ is a pre-fixed time point. Using Cho *et al*. (2015b) and Eq (1.1), the likelihood functions of α and β are given by

Case I

$$L_1(\alpha,\beta) = C_1(\alpha\beta)^k x_{k:m:n}^{-(\alpha+1)} e^{-\beta x_{k:m:n}^{-\alpha}} \left[1 - e^{-\beta x_{k:m:n}^{-\alpha}}\right]^{R_k^*} \prod_{i=1}^{k-1} x_{i:m:n}^{-(\alpha+1)} e^{-\beta x_{i:m:n}^{-\alpha}} \times \left[1 - e^{-\beta x_{i:m:n}^{-\alpha}}\right]^{R_i},$$

Case II

$$L_2(\alpha,\beta) = C_2(\alpha\beta)^D \prod_{i=1}^{D} x_{i:m:n}^{-(\alpha+1)} e^{-\beta x_{i:m:n}^{-\alpha}} \left[1 - e^{-\beta x_{i:m:n}^{-\alpha}}\right]^{R_i} \left[1 - e^{-\beta T^{-\alpha}}\right]^{R_{D+1}^*},$$

Case III

$$L_3(\alpha,\beta) = C_3(\alpha\beta)^m \prod_{i=1}^{m} x_{i:m:n}^{-(\alpha+1)} e^{-\beta x_{i:m:n}^{-\alpha}} \left[1 - e^{-\beta x_{i:m:n}^{-\alpha}}\right]^{R_i},$$

where $C_1 = \left[\prod_{i=1}^k \sum_{k=i}^m (R_k+1)\right]$, $C_2 = \left[\prod_{i=1}^D \sum_{k=i}^m (R_k+1)\right]$, $C_3 = [\prod_{i=1}^m \sum_{k=i}^m (R_k+1)]$, $R_k^* = n - k - \sum_{i=1}^{k-1} R_i$, and $R_{D+1}^* = n - D - \sum_{i=1}^{D} R_i$.

Therefore, likelihood functions can be combined as

$$L(\alpha,\beta) \propto (\alpha\beta)^s \prod_{i=1}^{s} x_{i:m:n}^{-(\alpha+1)} e^{-\beta x_{i:m:n}^{-\alpha}} \left[1 - e^{-\beta x_{i:m:n}^{-\alpha}}\right]^{R_i} \eta(\alpha,\beta),$$

where $s = k$, $\eta(\alpha,\beta) = 1$ and $R_k = n - k - \sum_{i=1}^{k-1} R_i$ for Case I, $s = D$ and $\eta(\alpha,\beta) = [1 - e^{-\beta T^{-\alpha}}]^{R_{D+1}^*}$ for Case II, and $s = m$ and $\eta(\alpha,\beta) = 1$ for Case III.

Hence, the log-likelihood function becomes

$$l(\alpha,\beta) \propto s log\alpha\beta - (\alpha+1)\sum_{i=1}^{s} log x_{i:m:n} - \beta\sum_{i=1}^{s} x_{i:m:n}^{-\alpha} + \sum_{i=1}^{s} log\left[1-e^{-\beta x_{i:m:n}^{-\alpha}}\right]^{R_i} + log\eta(\alpha,\beta).$$

Differentiating the log-likelihood function partially with respect to α and β and then equating to zero, we have

$$\frac{\partial l(\alpha,\beta)}{\partial\alpha} = \frac{s}{\alpha} + \sum_{i=1}^{s}\left[\beta x_{i:m:n}^{-\alpha} log x_{i:m:n} - log x_{i:m:n} - R_i\frac{x_{i:m:n}^{-\alpha} log x_{i:m:n} e^{-\beta x_{i:m:n}^{-\alpha}}}{1-e^{-\beta x_{i:m:n}^{-\alpha}}}\right] + \eta_\alpha^1(\alpha,\beta) = 0, \tag{2.1}$$

and

$$\frac{\partial l(\alpha,\beta)}{\partial\beta} = \frac{s}{\beta} - \sum_{i=1}^{s} x_{i:m:n}^{-\alpha} + \sum_{i=1}^{s} R_i\frac{x_{i:m:n} e^{-\beta x_{i:m:n}^{-\alpha}}}{1-e^{-\beta x_{i:m:n}^{-\alpha}}} + \eta_\beta^1(\alpha,\beta) = 0. \tag{2.2}$$

Here, for Case II,

$$\eta_\alpha^1(\alpha,\beta) = -R_{D+1}^*\frac{\beta T^{-\alpha} log T e^{-\beta T^{-\alpha}}}{1-e^{-\beta T^{-\alpha}}} \text{ and } \eta_\beta^1(\alpha,\beta) = R_{D+1}^*\frac{T^{-\alpha}e^{-\beta T^{-\alpha}}}{1-e^{-\beta T^{-\alpha}}},$$

for Case I and Case III,

$$\eta_\alpha^1(\alpha,\beta) = \eta_\beta^1(\alpha,\beta) = 0.$$

The MLEs of α and β are the solution of Eqs (2.1) and (2.2). However, solutions for α and β are not available. Therefore, we propose to use the Newton-Raphson algorithm to solve it. See for example the work of Kwon *et al*. (2014), Lee *et al*. (2014) and Shin *et al*. (2014). Using the MLEs of α and β, say $\hat{\alpha}$ and $\hat{\beta}$, the MLE of entropy function is obtained as

$$\hat{H} = 1 + \left(1+\frac{1}{\hat{\alpha}}\right)\left(\gamma + log\hat{\beta}\right) - log\left(\hat{\alpha}\hat{\beta}\right).$$

2.2. Bayes estimation using Tierney and Kadane approximation

In this section, we obtain the Bayes estimators for the entropy function of the IWD under GPHC scheme using Tierney and Kadane approximation. We obtain estimator under three different loss functions defined as

$$\text{SELF}: L_s(\hat{\theta},\ \theta) = (\hat{\theta}-\theta)^2,$$
$$\text{PLF}: L_p(\hat{\theta},\ \theta) = (\theta-\hat{\theta})^2/\hat{\theta},$$
$$\text{LLF}: L_l(\hat{\theta},\ \theta) = exp[c(\hat{\theta}-\theta)] - c(\hat{\theta}-\theta) - 1.$$

Let $X_{1:m:n}, X_{2:m:n}, \cdots, X_{s:m:n}$ denote a GPHC sample of IWD(α, β). There does not exist any conjugate prior distribution for α and β. Therefore, we assumed that priors of α and β are independent, and the parameters α and β follow the gamma(a_1, b_1) and gamma(a_2, b_2) prior distributions with $a_1 > 0$, $a_2 > 0$, $b_1 > 0$ and $b_2 > 0$. Therefore, the joint prior distribution of α and β is obtained as

$$\pi(\alpha,\beta) \propto \alpha^{a_1-1}\beta^{a_2-1}e^{-b_1\alpha-b_2\beta}, \alpha > 0, \beta > 0.$$

Note that, when $a_1 = a_2 = b_1 = b_2 = 0$, the prior distribution is the diffuse prior of α and β. Then, the joint density of the parameters and data is obtained as follows.

$$L(\alpha,\beta|\mathbf{X})\,\pi(\alpha,\beta) \propto \alpha^{s+a_1-1}\beta^{s+a_2-1}\prod_{i=1}^{s} x_{i:m:n}^{-(\alpha+1)} e^{-\beta x_{i:m:n}^{-\alpha}-b_1\alpha-b_2\beta}\left[1-e^{-\beta x_{i:m:n}^{-\alpha}}\right]^{R_i}\eta(\alpha,\beta),$$

where $\mathbf{X} = (X_{1:m:n}, X_{2:m:n}, \cdots, X_{s:m:n})$.

Then, we can derive the Bayes estimator of entropy under LLF. It is derived as

$$\tilde{H}_L = -\frac{1}{c}log\left[\frac{\int_0^\infty exp\{\ -c[1+(1+1/\alpha)(\gamma+log\beta)-log(\alpha\beta)]\}L(\alpha,\beta|\mathbf{X})\pi(\alpha,\beta)\,d\alpha d\beta}{\int_0^\infty\int_0^\infty L(\alpha,\beta|\mathbf{X})\pi(\alpha,\beta)\,d\alpha d\beta}\right].$$

It is easily observed that above equation is in the form of ratio of two integrals for which simplified closed forms are not available. Thus we use Tierney and Kadane approximation method to approximate all the Bayes estimators. Tierney and Kadane (1986) introduced an easily computable approximation for the posterior mean and variance of a non-negative parameter or more generally, of a smooth function of the parameter that is non-zero on the interior of the parameter space. For detail, let g be a smooth, positive function on the parameter space. The posterior expectation of $g(\alpha,\beta)$ is obtained as

$$\hat{g} = E(g(\alpha,\beta)|\mathbf{X}) = \int\int g(\alpha,\beta)\pi(\alpha,\beta|\mathbf{X})d\alpha d\beta = \frac{\int\int e^{nv^*(\alpha,\beta)}d\alpha d\beta}{\int\int e^{nv(\alpha,\beta)}d\alpha d\beta},$$

where

$$v(\alpha,\beta) = \frac{logL(\alpha,\beta)+log\pi(\alpha,\beta)}{n} \quad \text{and} \quad v^*(\alpha,\beta) = v(\alpha,\beta) + \frac{logg(\alpha,\beta)}{n}.$$

For the (α,β), the Bayes estimator using Tierney and Kadane approximation of $g(\alpha,\beta)$ can be obtained as

$$\hat{g} = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}}e^{nv^*(\hat{\alpha}_{v^*},\hat{\beta}_{v^*})-nv(\hat{\alpha}_v,\hat{\beta}_v)},$$

where $(\hat{\alpha}_v, \hat{\beta}_v)$ and $(\hat{\alpha}_{v^*}, \hat{\beta}_{v^*})$ maximize the $v(\alpha,\beta)$ and $v^*(\alpha,\beta)$, respectively. $|\Sigma^*|$ and $|\Sigma|$ denote the minus of inverse of Hessians of $v(\alpha,\beta)$ and $v^*(\alpha,\beta)$ at $(\hat{\alpha}_v, \hat{\beta}_v)$ and $(\hat{\alpha}_{v^*}, \hat{\beta}_{v^*})$, respectively. In our problem, we observe that

$$\begin{aligned} v(\alpha,\beta) =& \frac{1}{n}\Bigg[(s+a_1-1)log\alpha + (s+a_2-1)log\beta - (\alpha+1)\sum_{i=1}^{s} logx_{i:m:n} \\ & - \beta\sum_{i=1}^{s} x_{i:m:n}^{-\alpha} + \sum_{i=1}^{s} R_i log\left[1-e^{-\beta x_{i:m:n}^{-\alpha}}\right] + log\eta(\alpha,\beta) - (b_1\alpha+b_2\beta)\Bigg]. \end{aligned}$$

Then, $(\hat{\alpha}_v, \hat{\beta}_v)$ is computed by solving the following equations

$$\frac{\partial v(\alpha,\beta)}{\partial \alpha} = - b_1 + \frac{s+a_1-1}{\alpha} - \sum_{i=1}^{s} logx_{i:m:n} + \beta \sum_{i=1}^{s} \left[x_{i:m:n}^{-\alpha} logx_{i:m:n} - R_i \frac{x_{i:m:n}^{-\alpha} logx_{i:m:n} e^{-\beta x_{i:m:n}^{-\alpha}}}{1-e^{-\beta x_{i:m:n}^{-\alpha}}} \right] + \eta_{\alpha}^{1}(\alpha,\beta) = 0,$$

and

$$\frac{\partial v(\alpha,\beta)}{\partial \beta} = - b_2 + \frac{s+a_2-1}{\beta} - \sum_{i=1}^{s} x_{i:m:n}^{-\alpha} + \sum_{i=1}^{s} R_i \frac{x_{i:m:n}^{-\alpha} e^{-\beta x_{i:m:n}^{-\alpha}}}{1-e^{-\beta x_{i:m:n}^{-\alpha}}} + \eta_{\beta}^{1}(\alpha,\beta) = 0.$$

Also, we compute $|\Sigma|$ and it is given by

$$|\Sigma| = \left[\frac{\partial^2 v(\alpha,\beta)}{\partial \alpha^2} \frac{\partial^2 v(\alpha,\beta)}{\partial \beta^2} - \frac{\partial^2 v(\alpha,\beta)}{\partial \alpha \partial \beta} \frac{\partial^2 v(\alpha,\beta)}{\partial \beta \partial \alpha} \right]^{-1},$$

where

$$\frac{\partial^2 v(\alpha,\beta)}{\partial \alpha^2} = \frac{1}{n} \left[-\frac{s+a_1-1}{\alpha^2} - \beta \sum_{i=1}^{s} \left\{ R_i x_{i:m:n}^{-\alpha} \left(logx_{i:m:n}\right)^2 e^{-\beta x_{i:m:n}^{-\alpha}} \times \frac{\beta x_{i:m:n}^{-\alpha} + e^{-\beta x_{i:m:n}^{-\alpha}} - 1}{\left(1-e^{-\beta x_{i:m:n}^{-\alpha}}\right)^2} + x_{i:m:n}^{-\alpha} \left(logx_{i:m:n}\right)^2 \right\} + \eta_{\alpha^2}^{2}(\alpha,\beta) \right],$$

$$\frac{\partial^2 v(\alpha,\beta)}{\partial \beta^2} = \frac{1}{n} \left[-\frac{s+a_2-1}{\beta^2} - \sum_{i=1}^{s} R_i \frac{x_{i:m:n}^{-2\alpha} e^{-\beta x_{i:m:n}^{-\alpha}}}{\left(1-e^{-\beta x_{i:m:n}^{-\alpha}}\right)^2} + \eta_{\beta^2}^{2}(\alpha,\beta) \right]$$

and

$$\frac{\partial^2 v(\alpha,\beta)}{\partial \alpha \partial \beta} = \frac{1}{n} \left[\sum_{i=1}^{s} R_i x_{i:m:n}^{-\alpha} logx_{i:m:n} e^{-\beta x_{i:m:n}^{-\alpha}} \frac{\beta x_{i:m:n}^{-\alpha} + e^{-\beta x_{i:m:n}^{-\alpha}} - 1}{\left(1-e^{-\beta x_{i:m:n}^{-\alpha}}\right)^2} + \sum_{i=1}^{s} x_{i:m:n}^{-\alpha} logx_{i:m:n} + \eta_{\alpha\beta}^{2}(\alpha,\beta) \right].$$

Here, for Case II,

$$\eta_{\alpha^2}^{2}(\alpha,\beta) = -\beta R_{D+1}^{*} T^{-\alpha} \left(logT\right)^2 e^{-\beta T^{-\alpha}} \frac{\beta T^{-\alpha} + e^{-\beta T^{-\alpha}} - 1}{\left[1-e^{-\beta T^{-\alpha}}\right]^2},$$

$$\eta_{\alpha\beta}^{2}(\alpha,\beta) = R_{D+1}^{*} T^{-\alpha} logT e^{-\beta T^{-\alpha}} \frac{\beta T^{-\alpha} + e^{-\beta T^{-\alpha}} - 1}{\left[1-e^{-\beta T^{-\alpha}}\right]^2}, \eta_{\beta^2}^{2}(\alpha,\beta) = -R_{D+1}^{*} \frac{T^{-2\alpha} e^{-\beta T^{-\alpha}}}{\left(1-e^{-\beta T^{-\alpha}}\right)^2},$$

for Cases I and III,

$$\eta^2_\alpha(\alpha,\beta) = \eta^2_\beta(\alpha,\beta) = \eta^2_{\alpha\beta}(\alpha,\beta) = 0.$$

In order to compute Bayes estimator of entropy function under LLF, we take $g(\alpha,\beta) = exp\{-c[1+(1+1/\alpha)(\gamma+log\beta) - log(\alpha\beta)]\}$. Then, $v^*_L(\alpha,\beta)$ is obtained as

$$v^*_L(\alpha,\beta) = v(\alpha,\beta) - \frac{c}{n}\left[1+\left(1+\frac{1}{\alpha}\right)(\gamma+log\beta) - log(\alpha\beta)\right].$$

Now solving the following equation

$$\frac{\partial v^*_L(\alpha,\beta)}{\partial\alpha} = \frac{\partial v(\alpha,\beta)}{\partial\alpha} + \frac{c}{n\alpha^2}(\alpha+\gamma+log\beta) = 0$$

and

$$\frac{\partial v^*_L(\alpha,\beta)}{\partial\beta} = \frac{\partial v(\alpha,\beta)}{\partial\beta} - \frac{c}{n\alpha\beta} = 0,$$

we obtain $(\hat{\alpha}_{v^*_L}, \hat{\beta}_{v^*_L})$. Then, we compute $|\Sigma^*_L|$ and it is given by

$$|\Sigma^*_L| = \left[\frac{\partial^2 v^*_L(\alpha,\beta)}{\partial\alpha^2}\frac{\partial^2 v^*_L(\alpha,\beta)}{\partial\beta^2} - \frac{\partial^2 v^*_L(\alpha,\beta)}{\partial\alpha\partial\beta}\frac{\partial^2 v^*_L(\alpha,\beta)}{\partial\beta\partial\alpha}\right]^{-1},$$

where

$$\frac{\partial^2 v^*_L(\alpha,\beta)}{\partial\alpha^2} = \frac{\partial^2 v(\alpha,\beta)}{\partial\alpha^2} - \frac{c}{n\alpha^3}(\alpha+2\gamma+2log\beta),\quad \frac{\partial^2 v^*_L(\alpha,\beta)}{\partial\beta^2} = \frac{\partial^2 v(\alpha,\beta)}{\partial\beta^2} + \frac{c}{n\alpha\beta^2}$$

and

$$\frac{\partial^2 v^*_L(\alpha,\beta)}{\partial\alpha\partial\beta} = \frac{\partial^2 v(\alpha,\beta)}{\partial\alpha\partial\beta} + \frac{c}{n\alpha^2\beta}.$$

Then, the Bayes estimator of entropy function under LLF is obtained by

$$\hat{H}_L = -\frac{1}{c}log\left[\sqrt{\frac{|\Sigma^*_L|}{|\Sigma|}}e^{nv^*_L(\hat{\alpha}_{v^*_L},\hat{\beta}_{v^*_L}) - nv(\hat{\alpha}_v,\hat{\beta}_v)}\right].$$

Similarly, we can obtain the Bayes estimators $\hat{H}_S$ and $\hat{H}_P$ using Tierney and Kadane approximation of entropy under SELF and PLF, respectively.

3. Illustrative examples and simulation results

3.1. Illustrative example

In order to analyze the real life data set, we use the proposed estimators in the above section. The real life data set were from the data on failure times of aircraft windshields

(Blischke and Murthy (2000)). Blischke and Murthy (2000) have examined the goodness-of-fit of the data to IWD and they found that the IWD fits the data. Here, we consider the case when the data are progressively Type II censored with the following schemes: $m = 77$ and $R_{77} = 10$, $R_1 = \cdots = R_{76} = 0$. And, we take Case I: $k = 60$ and $T = 3.0$, Case II: $k = 60$ and $T = 3.5$ and Case III: $k = 60$ and $T = 4.5$. The Bayes estimators based on the diffuse prior ($a_1 = a_2 = b_1 = b_2 = 0.00001$) are obtained. And, the Bayes estimator based on the LLF with $c = 1.5$, 2.0 and 2.5 is obtained. Table 3.1 shows the estimates of the entropy under GPHC scheme. In Table 3.1, we have tabulated entropy of the respective MLE in the third column of the table. In the other columns, the Bayes estimates of entropy using Tierney and Kadane approximation method are tabulated. We observed that Bayes estimates of entropy under SELF are marginally smaller than the corresponding Bayes estimates of entropy under PLF and LLF.

Table 3.1 Estimates of entroy for example.

Case	T	$\hat{H}$	$\hat{H}_S$	$\hat{H}_P$	$\hat{H}_L$		
					$c = 1.5$	$c = 2.0$	$c = 2.5$
Case I	3.0	2.5292	2.5261	2.5262	2.5286	2.5283	2.5280
Case II	3.5	2.4264	2.4228	2.4229	2.4264	2.4246	2.4244
Case III	4.5	2.3333	2.3292	2.3293	2.3309	2.3307	2.3305

3.2. Simulation results

Since the performance of the different methods cannot be compared theoretically, we carried out Monte Carlo simulations to compare the different methods. We consider various n, m, k and T, and three different progressively Type II censoring schemes (PCS), namely; Scheme I : $R_m = n - m$ and $R_i = 0$ for $i \neq m$. Scheme II : $R_1 = n - m$ and $R_i = 0$ for $i \neq 1$. Scheme III : $R_1 = R_m = (n - m)/2$ and $R_i = 0$ for $i \neq 1$ and m.

In each cases, we take $\alpha = 2$ and $\beta = 2$, and we replicate the process 10,000 times. The associated MLEs are computed using a Newton-Raphson method. All Bayes estimates are calculated with respect to the diffuse prior ($a_1 = a_2 = b_1 = b_2 = 0.00001$) distribution. Bayes estimates of entropy are obtained with respect to SELF, PLF and LLF. And, the Tierney and Kadane approximation method have been used to derive approximate closed forms for Bayes estimates. Under LLF, Bayes estimates are obtained for $c = 1.5, 2.0$ and 2.5. And, various schemes have been taken into consideration to calculate bias and mean squared error (MSE) of all estimates. And these results are tabulated in Table 3.2. We present discussions based on the MSEs and biases.

In Table 3.2, MSEs and biases of all estimates of entropy are presented for various n, m, k, T and PCS. In general, the MSE and bias decrease as sample size increases. For fixed sample size, the MSE and bias decrease generally as the number of progressive censored data decreases. For fixed sample size and progressive censoring data size, the MSE and bias decrease generally as the time T increases. For fixed time T, sample and progressive censoring data size, the MSE and bias decrease generally as the number of guarantee sample size k increases. It is also observed that the MLE for schemes I and III behaves quite similarly in terms of bias and MSE. The MLE for scheme II has larger MSE and bias than the corresponding MLE for the other two schemes. Also, we can observe that the Bayes estimates of entropy under SELF and PLF behave almost similar in terms of MSEs and biases. This holds true for all presented schemes. However, we can observe that Bayes estimates of entropy

Table 3.2 The relative MSEs and biases of entropy estimators with MLE and Bayes estimator.

n	m	k	Sch.	$\hat{H}$	$\hat{H}_S$	$\hat{H}_P$	$\hat{H}_L$		
							$c = 1.5$	$c = 2.0$	$c = 2.5$
						$T = 1.5$			
20	10	4	I	.2667(-.0932)	.2409(.0750)	.2444(.0833)	.2328(.0975)	.2262(.0885)	.2204(.0798)
			II	.3044(-.1766)	.2157(.0316)	.2175(.0438)	.2040(.0662)	.1985(.0531)	.1937(.0408)
			III	.2757(-.0972)	.2425(.0886)	.2470(.0987)	.2351(.1142)	.2272(.1028)	.2203(.0921)
	14	5	I	.2238(-.1056)	.1944(.0621)	.1972(.0697)	.1905(.0843)	.1855(.0758)	.1810(.0677)
			II	.2508(-.1380)	.1949(.0466)	.1969(.0561)	.1884(.0748)	.1838(.0645)	.1797(.0547)
			III	.2377(-.1118)	.2015(.0657)	.2044(.0743)	.1970(.0904)	.1917(.0808)	.1869(.0717)
		7	I	.1927(-.1276)	.1524(.0258)	.1534(.0326)	.1486(.0485)	.1461(.0413)	.1438(.0343)
			II	.2274(-.1642)	.1625(-.0156)	.1621(-.0082)	.1532(.0105)	.1518(.0030)	.1506(-.0042)
			III	.2066(-.1455)	.1541(.0082)	.1546(.0154)	.1482(.0328)	.1462(.0253)	.1444(.0181)
	18	9	I	.1750(-.1356)	.1324(-.0092)	.1324(-.0034)	.1271(.0118)	.1260(.0060)	.1250(.0003)
			II	.1799(-.1401)	.1335(-.0174)	.1332(-.0115)	.1270(.0039)	.1262(-.0019)	.1255(-.0075)
			III	.1774(-.1385)	.1328(-.0137)	.1326(-.0079)	.1268(.0075)	.1259(.0016)	.1250(-.0040)
		11	I	.1646(-.1256)	.1265(-.0225)	.1261(-.0177)	.1206(-.0049)	.1201(-.0096)	.1197(-.0141)
			II	.1676(-.1232)	.1301(-.0261)	.1296(-.0213)	.1237(-.0088)	.1233(-.0134)	.1229(-.0179)
			III	.1665(-.1249)	.1284(-.0246)	.1279(-.0198)	.1223(-.0072)	.1218(-.0118)	.1214(-.0163)
40	10	4	I	.2381(-.1677)	.1816(-.0500)	.1807(-.0466)	.1697(-.0113)	.1692(-.0153)	.1687(-.0191)
			II	.2665(-.1269)	.2104(-.0047)	.2107(.0010)	.2045(.0474)	.2012(.0397)	.1984(.0325)
			III	.2222(-.0945)	.1954(.0009)	.1957(.0042)	.1911(.0339)	.1892(.0299)	.1874(.0261)
	20	8	I	.1191(-.0266)	.1177(.0389)	.1181(.0407)	.1195(.0590)	.1185(.0568)	.1176(.0546)
			II	.1556(-.0770)	.1373(.0046)	.1375(.0075)	.1360(.0354)	.1349(.0317)	.1339(.0282)
			III	.1357(-.0313)	.1324(.0425)	.1329(.0449)	.1348(.0670)	.1335(.0641)	.1322(.0612)
		10	I	.1168(-.0285)	.1146(.0368)	.1150(.0387)	.1162(.0569)	.1154(.0547)	.1145(.0526)
			II	.1506(-.0921)	.1283(.0210)	.1282(.0185)	.1243(.0063)	.1238(.0034)	.1234(.0005)
			III	.1257(-.0420)	.1189(.0300)	.1192(.0322)	.1201(.0540)	.1192(.0513)	.1183(.0486)
	34	12	I	.1068(-.0587)	.0998(.0069)	.1000(.0086)	.1004(.0265)	.0998(.0244)	.0992(.0224)
			II	.1107(-.0671)	.1006(.0018)	.1008(.0037)	.1006(.0234)	.1000(.0211)	.0995(.0189)
			III	.1086(-.0640)	.1002(.0033)	.1004(.0051)	.1005(.0239)	.0999(.0217)	.0994(.0196)
		14	I	.1001(-.0670)	.0908(-.0033)	.0909(-.0016)	.0907(.0158)	.0903(.0139)	.0899(.0120)
			II	.1044(-.0802)	.0913(-.0155)	.0913(-.0137)	.0899(.0049)	.0896(.0028)	.0899(.0049)
			III	.1019(-.0748)	.0907(-.0103)	.0908(-.0086)	.0900(.0094)	.0896(.0075)	.0893(.0055)
						$T = 2.0$			
20	10	4	I	.2140(-.1252)	.1713(-.0027)	.1711(.0030)	.1632(.0176)	.1617(.0119)	.1603(.0064)
			II	.2634(-.0862)	.2355(.0589)	.2385(.0682)	.2266(.0837)	.2203(.0735)	.2147(.0637)
			III	.2166(-.0810)	.1950(.0363)	.1963(.0427)	.1885(.0556)	.1850(.0488)	.1819(.0423)
	14	5	I	.1514(-.0748)	.1402(.0238)	.1410(.0284)	.1373(.0383)	.1355(.0336)	.1338(.0289)
			II	.2060(-.0825)	.1905(.0381)	.1922(.0446)	.1854(.0573)	.1817(.0503)	.1783(.0435)
			III	.1713(-.0714)	.1604(.0358)	.1617(.0412)	.1569(.0521)	.1542(.0464)	.1516(.0408)
		7	I	.1506(-.0749)	.1390(.0234)	.1398(.0280)	.1363(.0380)	.1345(.0332)	.1328(.0286)
			II	.1833(-.1003)	.1580(.0162)	.1587(.0223)	.1536(.0358)	.1514(.0294)	.1493(.0232)
			III	.1593(-.0779)	.1454(.0285)	.1463(.0338)	.1425(.0450)	.1404(.0395)	.1384(.0341)
	18	9	I	.1340(-.0603)	.1246(.0312)	.1253(.0357)	.1229(.0454)	.1213(.0408)	.1199(.0362)
			II	.1432(-.0752)	.1281(.0207)	.1287(.0255)	.1257(.0363)	.1242(.0313)	.1228(.0265)
			III	.1386(-.0690)	.1263(.0248)	.1269(.0294)	.1242(.0396)	.1227(.0348)	.1212(.0302)
		11	I	.1238(-.0726)	.1092(.0144)	.1095(.0186)	.1070(.0283)	.1060(.0240)	.1051(.0199)
			II	.1352(-.0876)	.1143(.0001)	.1143(.0045)	.1109(.0150)	.1101(.0106)	.1094(.0063)
			III	.1284(-.0815)	.1106(.0061)	.1107(.0104)	.1078(.0205)	.1069(.0162)	.1061(.0119)
40	10	4	I	.2364(-.1694)	.1791(-.0519)	.1781(-.0485)	.1670(-.0132)	.1666(-.0171)	.1661(-.0209)
			II	.2498(-.0587)	.2223(.0301)	.2234(.0346)	.2213(.0705)	.2176(.0645)	.2142(.0587)
			III	.1910(-.1225)	.1553(-.0349)	.1548(.0321)	.1476(-.0039)	.1471(-.0071)	.1467(-.0102)
	20	8	I	.0950(-.0623)	.0849(-.0126)	.0848(-.0113)	.0830(.0028)	.0829(.0014)	.0827(.0000)
			II	.1329(-.0313)	.1275(.0260)	.1279(.0281)	.1283(.0482)	.1273(.0456)	.1263(.0430)
			III	.1008(-.0308)	.0968(.0161)	.0969(.0175)	.0968(.0318)	.0963(.0302)	.0959(.0285)
		10	I	.0950(-.0623)	.0849(-.0126)	.0848(-.0113)	.0830(.0028)	.0829(.0014)	.0827(.0000)
			II	.1245(-.0388)	.1171(.0176)	.1173(.0197)	.1171(.0395)	.1164(.0370)	.1157(.0346)
			III	.1006(-.0308)	.0965(.0160)	.0966(.0174)	.0965(.0317)	.0961(.0301)	.0956(.0285)
	34	12	I	.0640(-.0426)	.0608(-.0028)	.0609(-.0017)	.0608(.0089)	.0606(.0078)	.0604(.0066)
			II	.0821(-.0450)	.0785(-.0010)	.0786(.0003)	.0785(.0127)	.0782(.0113)	.0779(.0099)
			III	.0736(-.0437)	.0704(-.0019)	.0704(-.0007)	.0704(.0108)	.0701(.0095)	.0699(.0083)
		14	I	.0640(-.0426)	.0608(-.0028)	.0609(-.0017)	.0608(.0089)	.0606(.0078)	.0604(.0066)
			II	.0811(-.0456)	.0773(-.0017)	.0774(-.0004)	.0773(.0120)	.0770(.0106)	.0768(.0092)
			III	.0732(-.0441)	.0698(-.0023)	.0698(-.0011)	.0698(.0103)	.0695(.0091)	.0693(.0078)

under LLF are better than the Bayes estimates of entropy under SELF and PLF in terms of MSEs. Also, the problem of selecting a suitable loss function is concerned, it can be seen that LLF emerges as the best loss function. Among Bayes estimates of entropy, we observed that Bayes estimates obtained using the LLF for the choice $c = 2.5$ have overall lower MSEs. Therefore, for Bayes estimating the entropy under LLF, the choice $c = 2.5$ seems to be a reasonable choice for Tierney and Kadane estimates.

4. Conclusions

In this paper, we consider the classical and Bayes estimation of the entropy of a IWD under GPHC scheme. We observed that the MLE of the entropy cannot be derived in closed form, so we have to solve two non-linear equations simultaneously. We further consider the Bayes estimation of the entropy based on flexible priors. The Bayes estimators for the entropy of IWD based on the symmetric and asymmetric loss functions. Since the Bayes estimators cannot be obtained in closed form, we derive the Bayes estimates by revoking the Tierney and Kadane approximate method. The Bayes estimators of entropy are superior to the MLE in terms of MSEs and biases. The choice of LLF seems to be a reasonable choice for Bayes estimation of entropy. For Bayes estimating the entropy under LLF, the choice $c = 2.5$ seems to be a reasonable for Tierney and Kadane estimates. Although we focused on the entropy estimate of the IWD based on GPHC scheme, estimation of the entropy from other distributions based on GPHC scheme is of potential interest in future research.

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