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NON-CONVEX HYBRID ALGORITHMS FOR A FAMILY OF COUNTABLE QUASI-LIPSCHITZ MAPPINGS CORRESPONDING TO KHAN ITERATIVE PROCESS AND APPLICATIONS†

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Abstract. In this note we establish a new non-convex hybrid iteration algorithm corresponding to Khan iterative process [4] and prove strong convergence theorems of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in Hilbert spaces. Moreover, the main results are applied to get the common fixed points of finite family of quasi-asymptotically nonexpansive mappings. The results presented in this article are interesting extensions of some current results.

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1. Introduction

Fixed points of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings has become a field of interest on its on and has a various applications in related fields like image recovery, signal processing and geometry of objects [13]. Almost in all branches of mathematics we see some versions of theorems relating to fixed points of functions of special nature. As a result we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. A fixed-point iteration scheme has been applied in intensity modulated radiation therapy optimization to pre-compute

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dose-deposition coefficient matrix, see [12]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present now. Constructive fixed point theorems (for example, Banach fixed point theorem) which not only claims the existence of a fixed point but yields an algorithm, too (in the Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some mapping f that is contracting with respect to some (complete) metric on X will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and is invariably used in most of the occasions, see [5]. But it only ensures weak convergence, see [2] but more often then not, we require strong convergence in many real world problems relating to Hilbert spaces, see [1]. So mathematician are in search for the modifications of the Mann's process to control and guarantee the strong convergence, see [2, 6, 7, 8, 9, 10, 11] and references therein.

Most probably the first noticeable modification of Mann's Iteration process was proposed by Nakajo and Takahashi in [10] in 2003. They introduced this modification for only one nonexpansive mapping in a Hilbert space where Kim and Xu [6] introduced a modification for asymptotically nonexpansive mappings in the Hilbert space in 2006. In the same year Martinez-Yanes and Xu [8] introduced a variation of the Ishikawa iteration process for a nonexpansive mappings for a Hilbert space. They also gave modification of the Halpern iteration method in Hilbert space. Su and Qin [11] gave a monotone hybrid iteration process for nonexpansive mappings in a Hilbert space. Liu et al. [7] gave a novel iteration method for finite family of quasi-asymptotically pseudo-contractive mappings in a Hilbert space.

Let H be a Hilbert space and C be its nonempty, closed and convex subset of H. Let $P_C(\cdot)$ be the metric projection from H onto C.

A mapping $T : C \to C$ is said to be *nonexpensive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. Denote by $F(T)$ the set of fixed points of T. It is well known that $F(T)$ is closed and convex.

A mapping $T : C \to C$ is said to be *quasi-Lipschitz* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq L\|x - p\|$$

for all $x \in C$ and $p \in F(T)$, where $1 \leq L < \infty$ is a constant.

If $L = 1$, then T is said to be *quasi-nonexpansive*.

It is well-known that T is said to be *closed* if $x_n \to x$ and $\|Tx_n - x_n\| \to 0$ as $n \to \infty$ implies $Tx = x$. T is said to be *weak closed* if $x_n \rightharpoonup x$ and $\|Tx_n - x_n\| \to 0$ as $n \to \infty$ implies $Tx = x$. It is trivial fact that a weak closed mapping must be a closed one but converse is no longer true.

Let $\{T_n\}$ be a sequence of mappings from C into itself with a nonempty common fixed points set F. Then $\{T_n\}$ is called *uniformly closed* if for all convergent sequences $\{z_n\} \subset C$ with conditions $\|T_n z_n - z_n\| \to 0$ as $n \to \infty$, the limit of $\{z_n\}$ belongs to F.

In 1953, we have Mann iterative scheme [5]:

$$x_{n+1} = (1 - a_n)x_n n + a_n T(x_n), \quad n = 0, 1, 2, \ldots.$$

In [3], Guan et al. established the following non-convex hybrid iteration algorithm corresponding to Mann iterative scheme:

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1 - a_n)x_n + a_n T_n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq (1 + (L_n - 1)a_n)\|x_n - z\| \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

and proved some strong convergence results relating to common fixed points for a uniformly closed asymptotic family of countable quasi-Lipschitz mappings in H. They applied their results for the finite case to obtain fixed points.

In 2013, Khan iterative scheme [4] was defined as the following process:

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = (1 - \alpha_n)x_n + \alpha_n T(x_n), \quad n = 0, 1, 2, \ldots. \end{cases}$$

In this article, we established a kind of non-convex hybrid iteration algorithms corresponding to Khan iterative scheme and prove relevant strong convergence theorems of common fixed points for uniformly closed asymptotically family of countable quasi-Lipschitz mappings in Hilbert spaces. An application of presented algorithm is also given.

2. Main results

In this section we give our main results.

Definition 2.1. Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\}$ be a family of countable quasi-L_n-Lipschitz mappings from C into itself. $\{T_n\}$ is said to be *asymptotically* if $\lim_{n\to\infty} L_n = 1$.

The following lemmas are well known and useful for our conclusions

Proposition 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$, $z = P_C x$ if and only if we have $\langle x - z, z - y\rangle \geq 0$ for all $y \in C$.*

Proposition 2.3. *Let C be a closed convex subset of a Hilbert space H and let $\{T_n\}$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Then the common fixed point set F is closed and convex.*

Proposition 2.4. *Let C be a closed convex subset of a Hilbert space H. For any given $x_0 \in H$, we have $p = P_C x_0$ if and only if $\langle p - z, x_0 - p\rangle \geq 0$ for all $z \in C$.*

Theorem 2.5. *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\} : C \to C$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Assume that $\alpha_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = T_n[(1-\alpha_n)x_n + \alpha_n T_n x_n], \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [(1 + (L_n - 1)\alpha_n)]L_n\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, where $\overline{co}C_n$ denotes the closed convex closure of C_n for all $n \geq 1$ and $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. we split the proof into seven steps.

Step 1. It is obvious that $\overline{co}C_n$ and Q_n are closed and convex for all $n \geq 0$. Next, we show that $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$. Indeed, for each $p \in F \cap A$, we have

$$\begin{aligned} \|y_n - p\| &= \|T_n[(1-\alpha_n)x_n + \alpha_n T_n x_n] - p\| \\ &= \|(1-\alpha_n)(T_n x_n - p) + \alpha_n(T_n^2 x_n - p)\| \\ &\leq (1-\alpha_n)L_n\|x_n - p\| + \alpha_n L_n^2\|x_n - p\| \\ &= [(1 + (L_n - 1)\alpha_n)]L_n\|x_n - p\| \end{aligned}$$

and $p \in A$, so $p \in C_n$ which implies that $F \cap A \subset C_n$ for all $n \geq 0$. therefore, $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$.

Step 2. We show that $F \cap A \subset \overline{co}C_n \cap Q_n$ for all $n \geq 0$. it suffices to show that $F \cap A \subset Q_n$ for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$ we have $F \cap A \subset C = Q_0$. Assume that $F \cap A \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $\overline{co}C_n \cap Q_n$, from Proposition 2.2, we have

$$\langle x_{n+1} - z, x_{n+1} - x_0\rangle \leq 0, \quad \forall z \in \overline{co}C_n \cap Q_n$$

as $F \cap A \subset \overline{co}C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in F \cap A$. This together with the definition of Q_{n+1} implies that $F \cap A \subset Q_{n+1}$. Hence the $F \cap A \subset \overline{co}C_n \cap Q_n$ holds for all $n \geq 0$.

Step 3. We prove $\{x_n\}$ is bounded. Since F is a nonepmty closed convex subset of C, there exists a unique element $z_0 \in F$ such that $z_0 = P_F x_0$. From $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in \overline{co}C_n \cap Q_n$. As $z_0 \in F \cap A \subset \overline{co}C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$$

for each $n \geq 0$. This implies that $\{x_n\}$ is bounded.

STEP 4. We show that $\{x_n\}$ converges strongly to a point of C (we show that $\{x_n\}$ is a Cauchy sequence). As $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \subset Q_n$ and $x_n = P_{Q_n} x_0$ (Proposition 2.4), we have

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$$

for every $n \geq 0$, which together with the boundedness of $\|x_n - x_0\|$ implies that there exsists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_n - x_0 \rangle \leq 0$ and hence

$$\begin{aligned} \|x_{n+m} - x_n\|^2 &= \|(x_{n+m} - x_0) - (x_n - x_0)\|^2 \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \to 0, \ n \to \infty \end{aligned}$$

for any $m \geq 1$. Therefore $\{x_n\}$ is a Cauchy sequence in C, then there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

STEP 5. We show that $y_n \to q$, as $n \to \infty$. Let

$$D_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + L_n^2(L_n - 1)(L_n + 1)\}.$$

From the definition of D_n, we have

$$\begin{aligned} D_n &= \{z \in C : \langle y_n - z, y_n - z \rangle \leq \langle x_n - z, x_n - z \rangle + L_n^2(L_n - 1)(L_n + 1)\} \\ &= \{z \in C : \|y_n\|^2 - 2\langle y_n, z \rangle + \|z\|^2 \leq \|x_n\|^2 - 2\langle x_n, z \rangle + \|z\|^2 \\ &\qquad + L_n^2(L_n - 1)(L_n + 1)\} \\ &= \{z \in C : 2\langle x_n - y_n, z \rangle \leq \|x_n\|^2 - \|y_n\|^2 + L_n^2(L_n - 1)(L_n + 1)\}. \end{aligned}$$

This implies that D_n is closed and convex, for all $n \geq 0$. Next, we show that $C_n \subset D_n$, $n \geq 0$.

In fact, for any $z \in C_n$, we have

$$\begin{aligned} \|y_n - z\|^2 &\leq [1 + (L_n - 1)\alpha_n]^2 L_n^2 \|x_n - z\|^2 \\ &= \|x_n - z\|^2 L_n^2 + [2(L_n - 1)\alpha_n + (L_n - 1)^2 \alpha_n^2] L_n^2 \|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 L_n^2 + [2(L_n - 1) + (L_n - 1)^2] L_n^2 \|x_n - z\|^2 \\ &= \|x_n - z\|^2 L_n^2 + (L_n - 1) + (L_n + 1) L_n^2 \|x_n - z\|^2. \end{aligned}$$

From

$$C_n = \{z \in C : \|y_n - z\| \leq [1 + (L_n - 1)\alpha_n] L_n \|x_n - z\|\} \cap A, \quad n \geq 0,$$

we have $C_n \subset A$, $n \geq 0$. Since A is convex, we also have $\overline{co}C_n \subset A$, $n \geq 0$. Consider $x_n \in \overline{co}C_{n-1}$, we know that

$$\begin{aligned} \|y_n - z\| &\leq \|x_n - z\|^2 + L_n^2(L_n - 1)(L_n + 1)\|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 + L_n^2(l_n - 1)(L_n + 1). \end{aligned}$$

This implies that $z \in D_n$ and hence $C_n \subset D_n$, $n \geq 0$. Sinnce D_n is convex, we have $\overline{co}(C_n) \subset D_n$, $n \geq 0$. Therefore

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + L_n^2(L_n - 1)(L_n - 1) \to 0$$

as $n \to \infty$. That is, $y_n \to q$ as $n \to \infty$.

Step 6. We show that $q \in F$. From the definition of y_n, we have

$$(1 + \alpha_n T_n)\|T_n x_n - x_n\| = \|y_n - x_n\| \to 0$$

as $n \to \infty$. Since $\alpha_n \in (a, 1] \subset [0, 1]$, from the above limit we have

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0.$$

Since $\{T_n\}$ is uniformly closed and $x_n \to q$, we have $q \in F$.

Step 7. We claim that $q = z_0 = P_F x_0$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exists a positive integer N, if $n > N$ then $\|x_0 - x_n\| > \|x_0 - z_0\|$, which leads to

$$\begin{aligned}\|z_0 - x_n\|^2 &= \|z_0 - x_n + x_n - x_0\|^2 \\ &= \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle.\end{aligned}$$

It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$ which implies that $z_0 \overline{\in} Q_n$, so that $z_0 \overline{\in} F$, this is a contradiction. This completes the proof. □

In [3], we show an example of C_n which does not involve a convex subset.

Corollary 2.6. *Let C be a closed convex subset of a Hilbert space H, and let $T : C \to C$ be a closed quasi-nonexpansive mapping from C into itself. Assume that $\alpha_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = T[(1 - \alpha_n)x_n + \alpha_n T x_n], \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$, where $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. Take $T_n \equiv T$ and $L_n \equiv 1$ in Theorem 2.5, in this case, C_n is closed and convex, for all $n \geq 0$, by using Theorem 2.5, we obtain Corollary 2.6. □

Corollary 2.7. *Let C be a closed convex subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping from C into itself. Assume that $\alpha_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = T[(1 - \alpha_n)x_n + \alpha_n T x_n], \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$, *where* $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

3. Application to family of quasi-asymptotically nonexpansive mappings

In this section, we will apply the above result to study the following finit family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$. Let

$$\|T_i^j x - p\| \leq k_{i,j}\|x - p\|, \quad \forall x \in C,\, p \in F,$$

where F denotes the common fixed point set of $\{T_n\}_{n=0}^{N-1}$, $\lim_{j\to\infty} k_{i,j} = 1$ for all $0 \leq i \leq N-1$. The finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$ is said to be *uniformly L-Lipschitz* if

$$\|T_i^j x - T_i^j y\| \leq L_{i,j}\|x - y\|, \quad \forall x, y \in C$$

for all $i = 0, 1, 2, ..., N-1$ and $j \geq 1$, where $L \geq 1$.

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\}_{n=0}^{N-1} : C \to C$ be a uniformly L-Lipschitz finit family of asymptotically quasi-nonexpansive mappings with nonempty common fixed point set F. Assume that $\alpha_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = T_{i(n)}^{j(n)}[(1-\alpha_n)x_n + \alpha_n T_{i(n)}^{j(n)} x_n], \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (k_{i(n),j(n)} - 1)\alpha_n] \\ \qquad\qquad \times k_{i(n),j(n)}\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, where $\overline{co}C_n$ denotes the closed convex closure of C_n for all $n \geq 1$, $n = (j(n)-1)N + i(n)$ for all $n \geq 0$ and$A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. It is sufficient to prove the following two conclusions.

Conclusion 1. $\{T_{n=0}^{N-1}\}_{n=0}^{\infty}$ is a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself.

Conclusion 2. $F = \bigcap_{n=0}^{N} F(T_n) = \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, where $F(T_n)$ denotes the fixed point set of the mappings T_n. □

Corollary 3.2. *Let C be a closed convex subset of a Hilbert space H, and let $T : C \to C$ be a L-Lipschitz asymptotically quasi-nonexpansive mappings with nonempty common fixed point set F. Assume that $\alpha_n \in (a, 1]$ holds for some*

$a \in (0,1)$. Then $\{x_n\}$ generated by

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = T^n[(1-\alpha_n)x_n + \alpha_n T^n x_n], \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (k_n - 1)\alpha_n]k_n \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, where $\overline{co}C_n$ denotes the closed convex closure of C_n for all $n \geq 1$ and $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. Take $T_n \equiv T$ in Theorem 3.1, we obtain Corollary 3.2. □

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