J. Appl. Math. & Informatics Vol. **35**(2017), No. 3 - 4, pp. 277 - 302
https://doi.org/10.14317/jami.2017.277

NUMERICAL METHOD FOR SINGULARLY PERTURBED THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS OF REACTION-DIFFUSION TYPE

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Abstract. In this paper, we have proposed a numerical method for Singularly Perturbed Boundary Value Problems (SPBVPs) of reaction-diffusion type of third order Ordinary Differential Equations (ODEs). The SPBVP is reduced into a weakly coupled system of one first order and one second order ODEs, one without the parameter and the other with the parameter ε multiplying the highest derivative subject to suitable initial and boundary conditions, respectively. The numerical method combines boundary value technique, asymptotic expansion approximation, shooting method and finite difference scheme. The weakly coupled system is decoupled by replacing one of the unknowns by its zero-order asymptotic expansion. Finally the present numerical method is applied to the decoupled system. In order to get a numerical solution for the derivative of the solution, the domain is divided into three regions namely two inner regions and one outer region. The Shooting method is applied to two inner regions whereas for the outer region, standard finite difference (FD) scheme is applied. Necessary error estimates are derived for the method. Computational efficiency and accuracy are verified through numerical examples. The method is easy to implement and suitable for parallel computing. The main advantage of this method is that due to decoupling the system, the computation time is very much reduced.

AMS Mathematics Subject Classification : AMS 65l10 CR G1.7.
Key words and phrases : Third order singularly perturbed problems; Boundary value technique; Asymptotic expansion approximation; Shooting method; Parallel computation.

1. Introduction

The numerical treatment of Singularly Perturbed Problems (SPPs) has received significant attention in recent years. These problems arise frequently

Received March 5, 2016. Revised October 12, 2016. Accepted February 24, 2017.
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in fluid dynamics, elasticity, chemical reactor theory and many other applied areas. For long decades, a good number of research papers have been appearing in the field :'Numerical methods for singularly perturbed second order ordinary differential equations', but only few authors have developed numerical methods for singularly perturbed higher order differential equations.

Analytical treatment of SPBVPs for the higher order non-linear ODEs which have important applications in fluid dynamics is available in ([3],[7],[17],[24],[30]). O'Malley [18] discussed the existence, uniqueness and asymptotic estimates of the solution of higher order SPBVPs of the form

$$\varepsilon^{(m-n)}\{y^{(m)}+\alpha_1(x)y^{(m-1)}+...+\alpha_m(x)y\}+\beta(x)\{y^{(n)}+\beta_1(x)y^{(n-1)} \\ +....+\beta_n(x)y\}=0,$$

on the interval $\bar{\Omega}=[0,1]$ and the boundary conditions

$$y^{(\lambda_i)}(0)=l_i,\quad i=1,2,...,r,\quad y^{(\lambda_i)}(1)=l_i,\quad i=r+1,...,m,$$

with the assumption that $\beta(x)\neq 0,\ m>n$ and the coefficients are real and infinitely differentiable throughout $\bar{\Omega}$. In [19], the author discussed the asymptotic solutions of linear scalar equations of higher order.

Niederdrenk and Yserentant [17] have considered a convection-diffusion type equation and constructed a difference scheme on a variable mesh and derived conditions equivalent to stability of the discrete problem under certain assumptions.

Howes [4] established the existence and comparison results on certain boundary value problems for n^{th} order scalar nonlinear differential equations and their system analogues. He also applied this theory to several classes of singularly perturbed boundary value problems of higher order.

Michal Feckan [7] discussed the existence and asymptotic estimates solutions of SPBVPs of the type

$$\varepsilon^2 y^{(n)}=f(x,y,...,y^{(n-3)},y^{(n-2)}),\quad n\geq 3, \\ \boldsymbol{B}y=0,\quad \boldsymbol{L}y=0,\quad x\in\Omega=(0,1),$$

where $\boldsymbol{L}$ is a linear two-point boundary value condition for derivatives upto order $(n-3)$ and $\boldsymbol{B}$ has one of the following forms: $i)\,y^{(n-2)}(0)=y^{(n-2)}(1)=0$, $ii)\,y^{(n-2)}(0)=y^{(n-1)}(1)=0$, $iii)\,y^{(n-1)}(0)=y^{(n-2)}(1)=0$. Furthermore, he used an approach based on fixed point theory, Leray-Schauder degree theory and the implicit function theorem to show the existence of the solution and to investigate the asymptotic behaviour of the solution of the above BVP. In [8], Feckan considered singularly perturbed higher order ODEs and he has also established lower bounds of the number of parameters for which these equations possess a solution.

Gartland [3] considered the numerical approximation of differential operators of the form $L_\varepsilon u=\varepsilon u^{(m)}+\sum_{\nu=0}^{m-1}a_\nu u^{(\nu)}$ with out turning points. He showed that the uniform stability of the discrete boundary value problem follows from uniform stability of an associated discrete initial value problem and uniform consistency of the scheme. He has also proved that the uniform consistency requires exponential fitting or a special grid or both. Further, he has shown that a family of finite difference schemes based on an exponentially graded mesh and local polynomial basis functions are of arbitrarily high uniform order of convergence.

In [24], an iterative method is described. Further, if the order of the equation is even, then a Finite Element Method (FEM) based on standard C^{m-1} splines on a Shishkin mesh is reported in [31]. Also Semper [25], Roos [23] and O'Malley [19] have considered fourth-order equation and applied a standard FEM. In [24, 30], a FEM for convection and reaction type problems is described. In [31, 30], Sun and Stynes presented FEMs on Shishkin meshes on higher-order elliptic two point BVPs.

Motivated by the works of O'Malley, Zhao Weili, Howes and Feckan [4, 5, 7, 8, 18, 19, 38, 39], Shanthi and Ramanujam [26, 27, 28, 29] developed various computational methods for solving SPPs for fourth order ODEs subject to different types of boundary conditions.

Only very few authors have developed numerical methods for singularly perturbed third order ordinary differential equations, that too on the analytical behavior of the solution.

Zhao Weili [38] has considered a more general class of third order non-linear SPBVPs and discussed the existence, uniqueness of the solution and obtained asymptotic estimates using the theory of the differential inequalities.

In [5], Howes presented a study on the boundary and interior layer phenomena exhibited by solutions of singularly perturbed third order boundary value problems which govern the motion of thin liquid films subject to viscous, capillary and gravitational forces and are of the form $\varepsilon y''' = f(y)y' + g(x,y), \quad a < x < b$, $y(a,\varepsilon) = A, \quad y'(a,\varepsilon) = C, \quad y(b,\varepsilon) = B$. The precise conditions specifying where and when the third order derivative terms in the differential equations that can be neglected were derived and improved estimates for the actual solutions in terms of solutions of the lower order models were constructed. He also presented a technique for replacing a third order problem with an asymptotically equivalent second order one that may have wider applicability.

Nayfeh [15] presented perturbation techniques to find the asymptotic expansion solution for the third order problem considered in Howes [5]. Infact Zhao Weili [38] has derived results on third order non-linear SPPs using differential inequality theorems.

Based on the work of O'Malley, Zhao Weili, Howes and Feckan [4, 5, 7, 8, 18, 19, 38, 39], Valarmathi and Ramanujam [32, 33, 34, 35] developed various numerical methods for solving SPPs for third order ODEs subject to different types of boundary conditions. Roberts [22] has suggested a method for finding solution for third order singularly perturbed ODEs.

The fundamental idea used in this method is the Boundary Value Technique (BVT) discussed by many authors for second order, third order and fourth order ODEs [21, 32, 27] in which the authors divided the interval $[0,1]$ into two subintervals namely $[0, k\varepsilon]$ and $[k\varepsilon, 1]$ where $k\varepsilon$ is taken as the approximate width of the boundary layer. In the inner region $[0, k\varepsilon]$ they applied an EFFD scheme of [1] and a classical finite difference scheme for the outer region $[k\varepsilon, 1]$. They also presented error estimates for the numerical solution. This BVT gives an excellant portrait of the solution, especially within the boundary layers which can be seen in [27, 32].

Following the Boundary Value Technique (BVT) of Roberts [22], Vigo-Aguiar [37], Valarmathi [32] and using the basic idea underlying the method suggested in Khuri [40, 41], Jayakumar [6] and Natesan [10, 16] we in the present paper, suggest a new computational method which makes use of the zero order asymptotic expansion approximation, BVT and Shooting method to obtain a numerical solution for the derivative of SPBVPs for third order ODEs of reaction-diffusion type of the form:

$$-\varepsilon y^{'''}(x) + b(x)y'(x) + c(x)y(x) = f(x), \quad x \in \Omega, \tag{1}$$

$$y(0) = p, \quad y''(0) = q, \quad y''(1) = r. \tag{2}$$

where $0 < \varepsilon \ll 1$, $b(x)$, $c(x)$ are sufficiently smooth functions satisfying the following conditions:

$$b(x) \geq \beta, \quad \beta > 0, \tag{3}$$

$$0 \geq c(x) \geq -\gamma, \quad \gamma > 0, \tag{4}$$

$$\beta - 2\gamma \geq \gamma', \quad \text{for some } \gamma' > 0. \tag{5}$$

with $\Omega = (0, 1)$, $\Omega^0 = (0, 1]$, $\bar{\Omega} = [0, 1]$ and $y \in C^{(3)}(\Omega) \cap C^{(2)}(\bar{\Omega})$. Since the problem (1)-(2) is of singularly perturbed in nature, classical numerical methods, in general, fail to provide good approximate solution. In order to get a numerical solution for the derivative of the solution of SPBVPs (1)-(2) numerically, we divide the interval $[0, 1]$ into three subintervals $[0, \tau]$, $[\tau, 1 - \tau]$ and $[1 - \tau, 1]$.

Two inner region problems respectively defined in the intervals $[0, \tau]$, $[1 - \tau, 1]$ are solved by shooting method and the boundary value problem (BVP) corresponding to the outer region is solved based on the standard finite difference scheme. It is quite natural to take τ and $1 - \tau$ as the width of the boundary layers which can be obtained or estimated [9]. The problems defined in the intervals $[0, \tau]$, $[\tau, 1-\tau]$ and $[1-\tau, 1]$ are independent of each other. Therefore, these problems can be solved simultaneously, that is more suitable for parallel computing.

This method is easy to implement, and further, we could give a full-fledged theory (consistency, stability, convergence and error estimates) for the same. In Section 2 some analytical results for the SPBVPs (1)-(2) are presented. Section 3 deals with derivative estimates of derivative of the solution. In Section 4 some analytical and numerical results are derived for auxiliary second order SPBVPs of reaction-diffusion type and description of the numerical method is also given. The error estimates for the method are discussed in detail in Section 5. Section

6 deals with non-linear problems. Numerical examples are presented in Section 7. Conclusions are drawn in the last section.

Through out this paper, we use C, with or without subscript to denote a generic positive constant, which is independent of N and ε. We use h_1 and h_3 for mesh sizes for the innner region problems and h_2 for mesh size for the outer region problem. We define $||.||$ of $\bar{w} = (w_1, w_2)^T \in \mathbb{R}^2$ as $||\bar{w}|| = max\{|w_1|, |w_2|\}$.

2. Preliminaries

The SPBVPs (1)-(2) can be transformed into an equivalent weakly coupled system of the form:

$$\begin{cases} P_1\bar{y}(x) \equiv y_1'(x) - y_2(x) = 0, \quad x \in \Omega^0, \\ P_2\bar{y}(x) \equiv -\varepsilon y_2''(x) + b(x)y_2(x) + c(x)y_1(x) = f(x), \quad x \in \Omega, \end{cases} \tag{6}$$

$$y_1(0) = p, \quad y_2'(0) = q, \quad y_2'(1) = r, \tag{7}$$

where $\bar{y} = (y_1, y_2)^T$, $b(x)$, $c(x)$, $f(x)$ are sufficiently smooth functions satisfying the above conditions (3)-(5). This transformation makes it possible to establish the maximum principle theorems and stability results for the continuous problem. In this section, we present a maximum principle for the above problem. Using this, a stability result is derived. Further, an asymptotic expansion approximation is constructed for the solution and a theorem is presented to establish its accuracy.

Remark 2.1. The solution of the problem (6)-(7) exhibits twin boundary layers of width $O(\sqrt{\varepsilon})$ occur at $x = 0$ and at $x = 1$ which are less severe because the boundary conditions are prescribed for the derivative of the solution [24]. The condition (3) says that the problem (6)-(7) is a non-turning point problem. The condition (4) is known as the quasi-monotonicity condition [24]. The maximum principle for the above problem (6)-(7) can be established using the conditions (3)-(5).

2.1. Maximum Principle and Stability Result.

Theorem 2.1. (Maximum Principle). *Consider the SPBVPs (6)-(7). Let* $y_1(0) \geq 0$, $y_2'(0) \geq 0$ *and* $y_2'(1) \geq 0$. *Then* $P_1\bar{y}(x) \geq 0$ *for* $x \in \Omega^0$ *and* $P_2\bar{y}(x) \geq 0$ *for* $x \in \Omega$ *implies that* $\bar{y}(x) \geq 0$ *for all* $x \in \bar{\Omega}$.

Proof. Define the test functions $\bar{s}(x) = (s_1(x), s_2(x))^T$ by

$$s_1(x) = \frac{1}{2} + \eta x^2 + x, \quad s_2(x) = 1 + \eta x, \quad x \in \bar{\Omega} \quad \text{and} \quad 0 < \eta \ll 1/2.$$

Clearly, $s_1(0) > 0$, $s_2'(0) > 0$, $s_2'(1) > 0$.
We can easily prove that $P_1\bar{s} > 0$ for $x \in \Omega^0$ and $P_2\bar{s} > 0$ for $x \in \Omega$.

Assume that the theorem is not true. We define

$$\xi = \max\left\{\max_{x\in\bar{\Omega}}\left(-\frac{y_1}{s_1}\right)(x), \quad \max_{x\in\bar{\Omega}}\left(-\frac{y_2}{s_2}\right)(x)\right\}.$$

Then, $\xi > 0$. Also $(y_1+\xi s_1)(x) \geq 0$ and $(y_2+\xi s_2)(x) \geq 0$ for $x \in \bar{\Omega}$. Furthermore, there exists a point, $x_0 \in \bar{\Omega}$ such that

$$(y_1+\xi s_1)(x_0) = 0 \quad \text{for} \quad x_0 \in \Omega^0 \quad \text{or} \quad (y_2+\xi s_2)(x_0) = 0 \quad \text{for} \quad x_0 \in \Omega.$$

Case 1: $(y_1+\xi s_1)(x_0) = 0$ for $x_0 \in \Omega^0$.
This implies that $y_1+\xi s_1$ attains its minimum at $x = x_0$.
Then,

$$0 < P_1(\bar{y}+\xi\bar{s})(x_0) = (y_1+\xi s_1)^{'}(x_0) - (y_2+\xi s_2)(x_0) \leq 0,$$

which is a contradiction.
Case 2: $(y_2+\xi s_2)(x_0) = 0$ for $x_0 \in \Omega$.
This implies that $y_2+\xi s_2$ attains its minimum at $x = x_0$.
Then,

$$0 < P_2(\bar{y}+\xi\bar{s})(x_0) = -\varepsilon(y_2+\xi s_2)^{''}(x_0)+b(x)(y_2+\xi s_2)(x_0)+c(x)(y_1+\xi s_1)(x_0) \leq 0,$$

which is a contradiction.
Hence it can be concluded that $\bar{y}(x) \geq 0, \quad \forall x \in \bar{\Omega}$. □

Lemma 2.2. (Stability Result).*If $\bar{y}(x)$ is the solution of the SPBVPs (6)-(7) then*

$$||\bar{y}(x)|| \leq C\max\{|y_1(0)|, \ |y_2'(0)|, \ |y_2'(1)|, \ \max_{x\in\Omega^0}|P_1\bar{y}(x)|, \ \max_{x\in\Omega}|P_2\bar{y}(x)|\},$$

$\forall x \in \bar{\Omega}$.

Proof.
Set $M = C\max\{|y_1(0)|, \ |y_2'(0)|, \ |y_2'(1)|, \ \max_{x\in\Omega^0}|P_1\bar{y}(x)|, \ \max_{x\in\Omega}|P_2\bar{y}(x)|\}$.

Defining two barrier functions $\bar{w}^{\pm}(x) = (w_1^{\pm}(x), w_2^{\pm}(x))^T$ by

$$w_1^{\pm}(x) = [\frac{1}{2}+\eta x^2+x]M \pm y_1(x) \quad \text{and} \quad w_2^{\pm}(x) = (1+\eta x)M \pm y_2(x).$$

We have

$$\begin{aligned} P_1\bar{w}^{\pm}(x) &= w_1^{\pm\prime}(x) - w_2^{\pm}(x) = M\eta x \pm P_1\bar{y}(x) \geq 0 \quad \text{and} \\ P_2\bar{w}^{\pm}(x) &= -\varepsilon w_2^{\pm\prime\prime}(x) + b(x)w_2^{\pm}(x) + c(x)w_1^{\pm}(x), \\ &\geq M(\beta-2\gamma) \pm P_2\bar{y}(x) \geq M\gamma' \pm P_2\bar{y}(x) \geq 0, \end{aligned}$$

by a proper choice of the constant C. Furthermore, we have

$$w_1^{\pm}(0) = M/2 \pm y_1(0) \geq 0, \quad w_2^{\pm^{'}}(0) = M\eta \pm y_2'(0) \geq 0,$$
$$w_2^{\pm\prime}(1) = M\eta \pm y_2'(1) \geq 0,$$

by a proper choice of constant C. Applying Theorem 2.1 to the barrier functions $\bar{w}^{\pm}(x)$, we get the desired result. □

2.2. Asymptotic Expansion Approximation. We use an asymptotic expansion solution of the SPBVPs (6)-(7) in the form

$$\bar{y}(x,\varepsilon) = \bar{u}_0(x) + \bar{v}_0(x) + \bar{w}_0(x) + \sqrt{\varepsilon}(\bar{u}_1(x) + \bar{v}_1(x) + \bar{w}_1(x)) + O(\varepsilon).$$

By using the method of stretching variable [14] we can get a zero order asymptotic expansion approximation of (6)-(7) in the form $\bar{y}_{as} = \bar{u}_0(x) + \bar{v}_0(x) + \bar{w}_0(x)$ where $\bar{u}_0(x) = (u_{0_1}(x), u_{0_2}(x))^T$ is the solution of the reduced problem of the BVP (6)-(7) given by

$$\begin{cases} u'_{0_1}(x) - u_{0_2}(x) = 0, \\ b(x)u_{0_2}(x) + c(x)u_{0_1}(x) = f(x), \\ u_{0_1}(0) = p. \end{cases} \tag{8}$$

$\bar{v}_0(x) = (v_{0_1}(x), v_{0_2}(x))^T$ is the left layer correction term that satisfies

$$\begin{cases} v'_{0_1}(x) - v_{0_2}(x) = 0, \\ -\varepsilon v''_{0_2}(x) + b(0)v_{0_2}(x) = 0 \end{cases} \tag{9}$$

and $\bar{v}_0(x)$ is given by

$$\begin{cases} v_{0_1}(x) = (-C_1\sqrt{\varepsilon}\exp(-x\sqrt{b(0)/\varepsilon}))/\sqrt{b(0)}, \\ v_{0_2}(x) = C_1\exp(-x\sqrt{b(0)/\varepsilon}). \end{cases} \tag{10}$$

$\bar{w}_0(x) = (w_{0_1}(x), w_{0_2}(x))^T$ is the right layer correction term that satisfies

$$\begin{cases} w'_{0_1}(x) - w_{0_2}(x) = 0, \\ -\varepsilon w''_{0_2}(x) + b(1)w_{0_2}(x) = 0 \end{cases} \tag{11}$$

and $\bar{w}_0(x)$ is given by

$$\begin{cases} w_{0_1}(x) = (C_2\sqrt{\varepsilon}\exp(-(1-x)\sqrt{b(1)/\varepsilon}))/\sqrt{b(1)}, \\ w_{0_2}(x) = C_2\exp(-(1-x)\sqrt{b(1)/\varepsilon}). \end{cases} \tag{12}$$

Note that

$$C_1 = [(q - u'_{0_2}(0)) - (r - u'_{0_2}(1))\exp(-\sqrt{b(1)/\varepsilon})]/D,$$

$$C_2 = [-(q - u'_{0_2}(0))\exp(-\sqrt{b(0)/\varepsilon}) + (r - u'_{0_2}(1))]/D,$$

where, $D = [1 - \exp(-(\sqrt{b(0)} + \sqrt{b(1)})/\sqrt{\varepsilon})]$. The following theorem gives the error bound for the difference between the solution of the SPBVPs (6)-(7) and its zero order asymptotic expansion approximation.

Theorem 2.3. *The zero order asymptotic expansion approximation* $\bar{y}_{as} = \bar{u}_0(x) + \bar{v}_0(x) + \bar{w}_0(x)$ *of the solution* $\bar{y}(x)$ *of the SPBVPs (6)-(7) defined by (8)-(12) satisfies the inequality*

$$||\bar{y}(x) - \bar{y}_{as}(x)|| \le C\sqrt{\varepsilon}, \quad \forall x \in \bar{\Omega}.$$

Proof. It is easy to prove that

$$|(y_1 - y_{1as})(0)| \le C\sqrt{\varepsilon}, \quad |(y_2 - y_{2as})'(0)| = 0 \quad \text{and} \quad |(y_2 - y_{2as})'(1)| = 0.$$

Further applying the differential operators it is easy to check with the following expressions:

$$\text{we have} \quad |P_1(\bar{y} - \bar{y}_{as})(x)| = 0 \quad \text{and}$$

$$\begin{aligned}
|P_2(\bar{y} - \bar{y}_{as})(x)| &= |f(x) - P_2\bar{y}_{as}(x)|, \\
&= |f(x) - \{-\varepsilon(u_{0_2} + v_{0_2} + w_{0_2})''(x) \\
&+ b(x)(u_{0_2} + v_{0_2} + w_{0_2})(x) + c(x)(u_{0_1} + v_{0_1} + w_{0_1})(x)\}|, \\
&\le \varepsilon|u''_{0_2}(x)| + |\frac{x\sqrt{b(0)}}{\sqrt{\varepsilon}}|[\frac{\sqrt{\varepsilon}}{\sqrt{b(0)}}]|b'(\theta_1)||v_{0_2}(x)| \\
&+ |\frac{(1-x)\sqrt{b(1)}}{\sqrt{\varepsilon}}|[\frac{\sqrt{\varepsilon}}{\sqrt{b(1)}}]|b'(\theta_2)||w_{0_2}(x)| \\
&\quad + |c(x)|(|v_{0_1}(x)| + |w_{0_1}(x)|),
\end{aligned}$$

where $0 < \theta_1 < x$ and $1 - x < \theta_2 < 1$. Using the fact that $t\exp(-t) \le \exp(-t/2), \quad \forall t \ge 0$, the above expression reduces to

$$\begin{aligned}
|P_2(\bar{y} - \bar{y}_{as})(x)| &\le C\varepsilon + C\sqrt{\varepsilon}[\exp(-(x/2)\sqrt{b(0)/\varepsilon}) \\
&\quad + \exp(-((1-x)/2)\sqrt{b(1)/\varepsilon})], \\
&\le C\sqrt{\varepsilon}.
\end{aligned}$$

From the stability result given by Lemma 2.2 it follows that

$$||\bar{y}(x) - \bar{y}_{as}(x)|| \le C\sqrt{\varepsilon}, \quad \forall x \in \bar{\Omega}.$$

□

Corollary 2.4. *If $y_1(x)$ is the solution of the SPBVPs (6)-(7) and $u_{0_1}(x)$ is solution of the problem (8) then $|y_1(x) - u_{0_1}(x)| \le C\sqrt{\varepsilon}, \quad \forall x \in \bar{\Omega}$.*

Proof. From the above theorem, $|y_1(x) - (u_{0_1}(x) + v_{0_1}(x))| \le C\sqrt{\varepsilon}.$

$$\begin{aligned}
\text{Consider, } |y_1(x) - u_{0_1}(x)| &= |y_1(x) - u_{0_1}(x) + v_{0_1}(x) - v_{0_1}(x)|, \\
&\le |y_1(x) - (u_{0_1}(x) + v_{0_1}(x))| + |v_{0_1}(x)|, \\
&\le C_1\sqrt{\varepsilon} + C_2\sqrt{\varepsilon}, \\
&\le C\sqrt{\varepsilon}.
\end{aligned}$$

□

3. Estimates of derivatives

Theorem 3.1. *Let $\bar{y}(x)$ be the solution of the SPBVPs (6)-(7). Then $y_2(x)$ satisfy*

$$|y_2^{(k)}(x)| \quad \leq \quad C(1+\varepsilon^{-(k/2)}e(x,\beta)) \tag{13}$$

for $0 \leq k \leq 3$, *where,* $e(x,\beta) = e^{-x\sqrt{\beta/\varepsilon}} + e^{-(1-x)\sqrt{\beta/\varepsilon}}, \quad x \in \bar{\Omega}$.

Proof. Consider the BVP

$$\varepsilon y_2''(x) + b(x)y_2(x) + c(x)y_1(x) = f(x), \quad y_2'(0) = q, \quad y_2'(1) = r.$$

Rewrite this BVP as

$$\varepsilon y_2''(x) + b(x)y_2(x) = f(x) - c(x)y_1(x), \quad y_2'(0) = q, \quad y_2'(1) = r.$$

Then, $y_1 \in C^{(2)}(\bar{\Omega})$ and using the procedure adopted in [12] we have $|y_2^{(k)}(x)| \leq C(1+\varepsilon^{-(k/2)}e(x,\beta))$, as required. □

4. Some analytical and numerical results for second order SPBVPs

We present some results for the following SPBVPs which are needed for the rest of the paper. Consider the auxiliary second order SPBVPs

$$Ly_2^\star(x) \equiv -\varepsilon {y_2^\star}''(x) + b(x)y_2^\star(x) = f(x) - c(x)u_{0_1}(x), \quad x \in \Omega, \tag{14}$$

$$B_0y_2^\star(0) \equiv {y_2^\star}'(0) = q, \quad B_1y_2^\star(1) \equiv {y_2^\star}'(1) = r, \tag{15}$$

where $u_{0_1}(x)$ is defined as in (8), $b(x)$ and $f(x)$ are sufficiently smooth and $b(x) \geq \beta, \quad \beta > 0, \quad 0 \geq c(x) \geq -\gamma, \quad \gamma > 0$.

4.1. Analytical Results.

Theorem 4.1. (Maximum Principle).*Consider the SPBVPs (14)-(15). Let $y_2^\star(x)$ be a smooth function satisfying $B_0y_2^\star(0) \geq 0$, $B_1y_2^\star(1) \geq 0$ and $Ly_2^\star(x) \geq 0$ for $x \in \Omega$. Then, $y_2^\star(x) \geq 0$, $\forall x \in \bar{\Omega}$.*

Proof. Please refer [1]. □

Lemma 4.2. *If $y_2^\star(x)$ is the solution of the SPBVPs (14)-(15) then*

$$|y_2^\star(x)| \leq C\max\{|B_0y_2^\star(0)|, \ |B_1y_2^\star(1)|, \ \max_{x\in\Omega}|Ly_2^\star(x)|\}, \ \forall x \in \bar{\Omega}.$$

Proof. Define the barrier functions $\psi^\pm(x)$ as

$$\psi^\pm(x) = A'(1+\eta' x) \pm y_2^\star(x), \quad x \in \bar{\Omega},$$

where $A' = C\max\{|B_0y_2^\star(0)|, \ |B_1y_2^\star(1)|, \ \max_{x\in\Omega}|Ly_2^\star(x)|\}$ and $0 < \eta' \ll 1/2$.

It is easy to check that $B_0\psi^\pm(0) \geq 0$, $B_1\psi^\pm(1) \geq 0$ and $L\psi^\pm(x) \geq 0$ for a proper choice of the constant C. Applying Theorem 4.1 to $\psi^\pm(x)$, the required stability bound is obtained. □

Theorem 4.3. *If $\bar{y}(x)$ and $y_2^{\star}(x)$ are solutions of the SPBVPs (6)-(7) and (14)-(15) respectively, then*

$$|y_2(x) - y_2^{\star}(x)| \le C\sqrt{\varepsilon}, \quad \forall x \in \bar{\Omega}.$$

Proof. The second component $y_2(x)$ of the solution $\bar{y}(x)$ of the BVP (6)-(7), satisfies the BVP

$$-\varepsilon {y_2}''(x) + b(x)y_2(x) = f(x) - c(x)y_1(x), \quad x \in \Omega, \quad y_2^{'}(0) = q, \quad y_2^{'}(1) = r.$$

Further, the function $w(x) = y_2(x) - y_2^{\star}(x)$ satisfies the BVP

$$-\varepsilon w''(x) + b(x)w(x) = -c(x)[y_1(x) - u_{0_1}(x)], \quad x \in \Omega, \quad w'(0) = 0, \quad w'(1) = 0.$$

From the stability result as given in Doolen [1] we have,

$$|w(x)| \le C|y_1(x) - u_{0_1}(x)|.$$

$$\text{From Theorem 2.3,} \quad |y_1(x) - y_{1as}(x)| \le C\sqrt{\varepsilon}.$$

$$\text{That is,} \quad |y_1(x) - u_{0_1}(x) - v_{0_1}(x) - w_{0_1}(x)| \le C\sqrt{\varepsilon}.$$

$$\text{Then, } |y_1(x) - u_{0_1}(x)| - |v_{0_1}(x) + w_{0_1}(x)| \le |y_1(x) - u_{0_1}(x) - v_{0_1}(x) - w_{0_1}(x)|$$

$$\text{implies that,} \quad |y_1(x) - u_{0_1}(x)| \le |v_{0_1}(x) + w_{0_1}(x)| + C\sqrt{\varepsilon} \le C\sqrt{\varepsilon}.$$

$$\text{That is} \quad |y_1(x) - u_{0_1}(x)| \le C\sqrt{\varepsilon}.$$

Therefore

$$|w(x)| \le C\sqrt{\varepsilon}.$$

Hence,

$$|y_2(x) - y_2^{\star}(x)| \le C\sqrt{\varepsilon}, \quad \forall x \in \bar{\Omega}.$$

□

4.2. Description of the method. Step 1: An asymptotic approximation is derived for the solution of (6)-(7) which is given by (8)-(12).
Step 2: The first component of the solution $\bar{y}$ of the SPBVPs (6)-(7), namely y_1 is approximated by the first component of the solution of the reduced problem namely u_{0_1} given by (8). Then replacing y_1 appearing in the second equation of (6) by u_{0_1} and taking the same boundary values, one gets the auxiliary SPBVPs (14)-(15). The solution of this problem is taken as an approximation to y_2 which is the second equation of (6) which has to be solved.
Step 3: In order to solve the auxiliary second order problem (14)-(15) numerically, we divide the interval $[0, 1]$ into three subintervals $[0, \tau]$, $[\tau, 1 - \tau]$ and $[1 - \tau, 1]$. The subintervals $[0, \tau]$, $[1 - \tau, 1]$ are respectively called left and right inner regions, whereas the subinterval $[\tau, 1 - \tau]$ is called outer region, where, $\tau = \min\{\frac{1}{4}, \sqrt{\frac{\varepsilon}{\beta}} \ln N\}$. Then, from the SPBVPs (14)-(15) three problems namely left inner region problem, right inner region problem and outer region problem are derived. To find the boundary condition at $x = \tau$, a zero

order asymptotic expansion is used.
The left inner region problem for (14)-(15) is given by

$$\begin{cases} \varepsilon y_2''(x) - b(x)y_2(x) = f(x) + c(x)u_{0_1}(x), \quad x \in (0,\tau), \\ -y_2'(0) = q, \quad y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau) + w_{0_2}(\tau). \end{cases} \tag{16}$$

The outer region problem for (14)-(15) is given by

$$\begin{cases} \varepsilon y_2''(x) - b(x)y_2(x) = f(x) + c(x)u_{0_1}(x), \quad x \in (\tau, 1-\tau), \\ y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau) + w_{0_2}(\tau), \\ y_2(1-\tau) = u_{0_2}(1-\tau) + v_{0_2}(1-\tau) + w_{0_2}(1-\tau). \end{cases} \tag{17}$$

The right inner region problem for (14)-(15) is given by

$$\begin{cases} \varepsilon y_2''(x) - b(x)y_2(x) = f(x) + c(x)u_{0_1}(x), \quad x \in (1-\tau, 1), \\ y_2(1-\tau) = u_{0_2}(1-\tau) + v_{0_2}(1-\tau) + w_{0_2}(1-\tau), \quad y_2'(1) = r. \end{cases} \tag{18}$$

Step 4: The left inner region problem (16) is solved by the Shooting method using the initial conditions $\breve{y}_2(0) = u_{0_2}(0) + v_{0_2}(0) + w_{0_2}(0), \quad \breve{y}_2'(0) = q$. Here, Shooting method in the sense that BVP (16) is replaced by the IVP (19) on the interval $[0,\tau]$.
Step 5: The right inner region problem (18) is solved by the Shooting method using the initial conditions $\tilde{y}_2(1) = u_{0_2}(1) + v_{0_2}(1) + w_{0_2}(1), \quad \tilde{y}_2'(1) = r$. Here, Shooting method in the sense that BVP (18) is replaced by the IVP (22) on the interval $[\tau, 1]$.
Step 6: The outer region problem (17) subject to boundary conditions $y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau) + w_{0_2}(\tau), \quad y_2(1-\tau) = u_{0_2}(1-\tau) + v_{0_2}(1-\tau) + w_{0_2}(1-\tau)$ is solved by standard FD scheme.
Step 7: After solving both the inner region problems and the outer region problem, we combine their solutions to obtain an approximate solution y_2 for the derivative of the original problem (1)-(2) over the interval $\bar{\Omega}$.

4.3. Numerical Schemes.

4.3.1. Left Inner Region Problem. Using **Step 4** for the BVP (16), we get the following IVP

$$\begin{cases} -\varepsilon \breve{y}_2''(x) + b(x)\breve{y}_2(x) = f(x) - c(x)u_{0_1}(x), \quad x \in (0,\tau], \\ \breve{y}_2(0) = \bar{q} = u_{0_2}(0) + v_{0_2}(0) + w_{0_2}(0), \quad \breve{y}_2'(0) = q. \end{cases} \tag{19}$$

This IVP is equivalent to the following system:

$$\begin{cases} P_1^*\bar{y}^* \equiv y_1^{*'}(x) - y_2^*(x) = 0, \\ P_2^*\bar{y}^* \equiv -\varepsilon y_2^{*\prime}(x) + b(x)y_1^*(x) = f^*(x), \quad x \in (0,\tau], \\ y_1^*(0) = \bar{q}, \quad y_2^*(0) = q. \end{cases} \tag{20}$$

where $f^*(x) = f(x) - c(x)u_{0_1}(x), \quad u_{0_1}(x)$ is defined as in (8), $\quad \bar{y}^* = (y_1^*, y_2^*)^T$, $b(x) \geq \beta, \quad \beta > 0, \quad 0 \geq c(x) \geq -\gamma, \quad \gamma > 0$.

Theorem 4.4. (Maximum Principle). *Consider the IVP (20). Let $y_1^*(0) \geq 0$, $y_2^*(0) \geq 0$ and $P_1^*\bar{y}^*(x) \geq 0$, $P_2^*\bar{y}^*(x) \geq 0$ for $x \in (0,\tau]$. Then, $\bar{y}^*(x) \geq 0$, $\forall x \in [0,\tau]$.*

Proof. Please refer [20]. □

Lemma 4.5. (Stability Result). *If $\bar{y}^*(x)$ is the solution of the IVP (20). Then*

$$||\bar{y}^*(x)|| \leq C \max\{|y_1^*(0)|, \ |y_2^*(0)|, \ \max_{x\in(0,\tau]} |P_1^*\bar{y}^*(x)|, \ \max_{x\in(0,\tau]} |P_2^*\bar{y}^*(x)|\},$$

for all $x \in [0,\tau]$.

Proof.

$$\text{Set} \quad A' = C \max\{|y_1^*(0)|, \ |y_2^*(0)|, \ \max_{x\in(0,\tau]} |P_1^*\bar{y}^*(x)|, \ \max_{x\in(0,\tau]} |P_2^*\bar{y}^*(x)|\}.$$

Defining two barrier functions $\bar{\chi}^{*\pm}(x) = (\chi_1^{*\pm}(x), \chi_2^{*\pm}(x))^T$ by

$$\chi_1^{*\pm}(x) = A'(1+x+x^2) \pm y_1^*(x) \quad \text{and} \quad \chi_2^{*\pm}(x) = A' \pm y_2^*(x).$$

We have

$$P_1^*\bar{\chi}^{*\pm}(x) = \chi_1^{*\pm\prime}(x) - \chi_2^{*\pm}(x) = A'(2x) \pm P_1^*\bar{y}^*(x) \geq 0 \quad \text{and}$$
$$P_2^*\bar{\chi}^{*\pm}(x) = -\varepsilon\chi_2^{*\pm\prime}(x) + b(x)\chi_1^{*\pm}(x) \geq \beta A' \pm P_2^*\bar{y}^*(x) \geq 0,$$

by a proper choice of C. Furthermore, we have

$\chi_1^{*\pm}(0) = A' \pm y_1^*(0) \geq 0$, $\quad \chi_2^{*\pm}(0) = A' \pm y_2^*(0) \geq 0$, by a proper choice of C.

Applying Theorem 4.4 to the barrier functions $\bar{\chi}^{*\pm}(x)$, we get the desired result. □

Theorem 4.6. *Consider the solution $\bar{y}^*(x)$ of the IVP (20). Then $y_1^*(x)$ and $y_2^*(x)$ satisfy*

$$|y_1^{*(k)}(x)| \leq C\varepsilon^{-(k-1)/2}e(x,\beta), \quad |y_2^{*(k)}(x)| \leq C\varepsilon^{-(k)/2}e(x,\beta) \ \textit{for} \ 0 \leq k \leq 2,$$

$x \in (0,\tau]$, where $e(x,\beta) = e^{-x\sqrt{\beta/\varepsilon}} + e^{-(1-x)\sqrt{\beta/\varepsilon}}$.

Proof. For $k=0$, the result follows from Lemma 4.5. From (20), it is evident that $|y_1^{*'}(x)| \leq Ce(x,\beta)$ and $|y_2^{*'}(x)| \leq C\varepsilon^{-1/2}e(x,\beta)$. Differentiating the equations (20) once and using the above estimates of $|y_1^{*'}(x)|$ and $|y_2^{*'}(x)|$, it is found that $|y_1^{*''}(x)| \leq C\varepsilon^{-1/2}e(x,\beta)$ and $|y_2^{*''}(x)| \leq C\varepsilon^{-1}e(x,\beta)$. □

Applying Euler's finite difference scheme for (20), we get

$$\begin{cases} P_1^{*N/4}\bar{y}_i^* = D^-y_{1,i}^* - y_{2,i}^* = 0, \\ P_2^{*N/4}\bar{y}_i^* = -\varepsilon D^-y_{2,i}^* + b(x_i)y_{1,i}^* = f^*(x_i) \quad \text{for } 1 \leq i \leq N/4, \\ y_{1,0}^* = \bar{q}, \ y_{2,0}^* = q, \end{cases} \tag{21}$$

where, $D^-y_{j,i}^* = (y_{j,i}^* - y_{j,i-1}^*)/h_1$, $j=1,2$, $\quad h_1 = \dfrac{4\tau}{N}$, $x_i = ih_1$, $\quad 1 \leq i \leq N/4$.

Here, τ is the transition parameter $\tau = \min\{\frac{1}{4}, \sqrt{\frac{\varepsilon}{\beta}} \ln N\}$. This fitted mesh is denoted by $\bar{\Omega}_\tau^{N/4}$.

Theorem 4.7. (Discrete Maximum Principle). *Consider the discrete IVP (21). Let* $y^*_{1,0} \geq 0, \quad y^*_{2,0} \geq 0$. *Then* $P_1^{*N/4} \bar{y}^*_i \geq 0$ *and* $P_2^{*N/4} \bar{y}^*_i \geq 0$ *for* $1 \leq i \leq N/4$, *implies that* $\bar{y}^*_i \geq 0$ *for* $0 \leq i \leq N/4$.

Proof. Please refer [20]. □

Lemma 4.8. (Stability Result).*Consider the discrete IVP (21). If* $\bar{y}^*_i$ *is any mesh function, then*

$$||\bar{y}^*_i|| \leq C \max\{|y^*_{1,0}|, \ |y^*_{2,0}|, \ \max_{1 \leq i \leq N/4} |P_1^{*N/4} \bar{y}^*_i|, \ \max_{1 \leq i \leq N/4} |P_2^{*N/4} \bar{y}^*_i|\},$$

for $0 \leq i \leq N/4$.

Proof.

$$\text{Set } M' = C \max\{|y^*_{1,0}|, \ |y^*_{2,0}|, \ \max_{1 \leq i \leq N/4} |P_1^{*N/4} \bar{y}^*_i|, \ \max_{1 \leq i \leq N/4} |P_2^{*N/4} \bar{y}^*_i|\},$$

Defining two barrier functions $\bar{\chi}^{*\pm}_i = (\chi^{*\pm}_{1,i}, \chi^{*\pm}_{2,i})^T$ by

$$\chi^{*\pm}_{1,i} = M'\{1 + x_i + x_i^2\} \pm y^*_{1,i} \quad \text{and} \quad \chi^{*\pm}_{2,i}(x) = M' \pm y^*_{2,i}, \quad 0 \leq i \leq N/4.$$

Then, applying Theorem 4.7 to $\bar{\chi}^{*\pm}_i$ for a proper selection of the constant C, we can obtain the desired bound for $\bar{y}^*_i$. □

4.3.2. Right Inner Region Problem. Using **Step 5** for the BVP (18), we get the following IVP

$$\begin{cases} -\varepsilon \tilde{y}_2^{''}(x) + b(x)\tilde{y}_2(x) = f(x) - c(x)u_{0_1}(x), \quad x \in [1-\tau, 1), \\ \tilde{y}_2(1) = \bar{r} = u_{0_2}(1) + v_{0_2}(1) + w_{0_2}(1), \quad \tilde{y}_2'(1) = r. \end{cases} \tag{22}$$

This IVP is equivalent to the following system:

$$\begin{cases} P_1^{**}\bar{y}^{**} \equiv y_1^{**'}(x) - y_2^{**}(x) = 0, \\ P_2^{**}\bar{y}^{**} \equiv -\varepsilon y_2^{**\prime}(x) + b(x)y_1^{**}(x) = f^*(x), \quad x \in [1-\tau, 1), \\ y_1^{**}(1) = \bar{r}, \quad y_2^{**}(1) = r. \end{cases} \tag{23}$$

where $f^*(x) = f(x) - c(x)u_{0_1}(x)$, $\quad u_{0_1}(x)$ is defined as in (8), $\quad \bar{y}^{**} = (y_1^{**}, y_2^{**})^T$, $b(x) \geq \beta, \quad \beta > 0, \quad 0 \geq c(x) \geq -\gamma, \quad \gamma > 0$.

Theorem 4.9. (Maximum Principle).*Consider the IVP (23). Let* $y_1^{**}(1) \geq 0$, $y_2^{**}(1) \geq 0$ *and* $P_1^{**}\bar{y}^{**}(x) \geq 0$ *and* $P_2^{**}\bar{y}^{**}(x) \geq 0$ *for* $x \in [1-\tau, 1)$. *Then,* $\bar{y}^{**}(x) \geq 0$ *for* $x \in [1-\tau, 1]$.

Proof. Please refer [20]. □

Lemma 4.10. (Stability Result).*If $\bar{y}^{**}(x)$ is the solution of the IVP (23). Then,*

$$||\bar{y}^{**}(x)|| \leq C\max\{|y_1^{**}(1)|,\, |y_2^{**}(1)|,\, \max_{x\in[1-\tau,1)}|P_1^{**}\bar{y}^{**}(x)|,\, \max_{x\in[1-\tau,1)}|P_2^{**}\bar{y}^{**}(x)|\},$$

$\forall x \in [1-\tau, 1]$.

Proof.

Set $A'' = C\max\{|y_1^{**}(1)|,\, |y_2^{**}(1)|,\, \max_{x\in[1-\tau,1)}|P_1^{**}\bar{y}^{**}(x)|,\, \max_{x\in[1-\tau,1)}|P_2^{**}\bar{y}^{**}(x)|\}$.

Defining two barrier functions $\bar{\chi}^{**\pm}(x) = (\chi_1^{**\pm}(x), \chi_2^{**\pm}(x))^T$ by

$$\chi_1^{**\pm}(x) = A''(1+2x) \pm y_1^{**}(x) \quad \text{and} \quad \chi_2^{**\pm}(x) = A'' \pm y_2^{**}(x).$$

We have

$$P_1^{**}\bar{\chi}^{**\pm}(x) = \chi_1^{**\pm\prime}(x) - \chi_2^{**\pm}(x) = A'' \pm P_1^{**}\bar{y}^{**}(x) \geq 0 \quad \text{and}$$
$$P_2^{**}\bar{\chi}^{**\pm}(x) = -\varepsilon\chi_2^{**\pm\prime}(x) + b(x)\chi_1^{**\pm}(x) \geq \beta A'' \pm P_2^{**}\bar{y}^{**}(x) \geq 0,$$

by a proper choice of C. Furthermore, we have

$$\chi_1^{**\pm}(1) = 3A'' \pm y_1^{**}(1) \geq 0, \quad \chi_2^{**\pm}(1) = A'' \pm y_2^{**}(1) \geq 0,$$

by a proper choice of C. Applying Theorem 4.9 to the barrier functions $\bar{\chi}^{**\pm}(x)$, we get the desired result. □

Theorem 4.11. *Consider the solution $\bar{y}^{**}(x)$ of the IVP (23). Then $y_1^{**}(x)$ and $y_2^{**}(x)$ satisfy*

$$|y_1^{**(k)}(x)| \leq C\varepsilon^{-(k-1)/2}e(x,\beta), \quad |y_2^{**(k)}(x)| \leq C\varepsilon^{-(k/2)}e(x,\beta)$$

for $0 \leq k \leq 2$, $x \in [1-\tau, 1)$, *where* $e(x,\beta) = e^{-x\sqrt{\beta/\varepsilon}} + e^{-(1-x)\sqrt{\beta/\varepsilon}}$.

Proof. Proof is similar as Theorem 4.6. □

Applying Euler's finite difference scheme for (23), we get

$$\begin{cases} P_1^{**N/4}\bar{y}^{**} \equiv D^+y_{1,i}^{**} - y_{2,i}^{**} = 0, \\ P_2^{**N/4}\bar{y}^{**} \equiv -\varepsilon D^+y_{2,i}^{**} + b(x_i)y_{1,i}^{**} = f^*(x_i) \quad \text{for} \quad 0 \leq i \leq N/4-1, \\ y_{1,N/4}^{**} = \bar{r}, \quad y_{2,N/4}^{**} = r. \end{cases} \tag{24}$$

where, $D^+y_{j,i} = (y_{j,i+1} - y_{j,i})/h_3,\ j = 1,2,\quad h_3 = \dfrac{4\tau}{N},\quad x_i = (1-\tau) + ih_3$, $0 \leq i \leq N/4 - 1$. Here, τ is the transition parameter defined as before. This fitted mesh is denoted by $\bar{\Omega}_\tau^{N/4}$.

Theorem 4.12. (Discrete Maximum Principle). *Consider the discrete IVP (24). Let $y_{1,N/4}^{**} \geq 0$, $y_{2,N/4}^{**} \geq 0$. Then $P_1^{**N/4}\bar{y}_i^{**} \geq 0$ and $P_2^{**N/4}\bar{y}_i^{**} \geq 0$ for $0 \leq i \leq N/4-1$ implies that $\bar{y}_i^{**} \geq 0$ for $0 \leq i \leq N/4$.*

Proof. Please refer [20]. □

Lemma 4.13. (Stability Result).*Consider the discrete IVP (24). If* $\bar{y}_i^{**}$ *is any mesh function, then*

$$||\bar{y}_i^{**}|| \leq C \max\{|y_{1,N/4}^{**}|,\, |y_{2,N/4}^{**}|, \max_{0\leq i\leq N/4-1} |P_1^{**N/4}\bar{y}_i^{**}| \max_{0\leq i\leq N/4-1} |P_2^{**N/4}\bar{y}_i^{**}|\},$$

for $0 \leq i \leq N/4$.

Proof.

Set $A'' = C\max\{|y_{1,N/4}^{**}|, |y_{2,N/4}^{**}|, \max_{0\leq i\leq N/4-1} |P_1^{**N/4}\bar{y}_i^{**}|, \max_{0\leq i\leq N/4-1} |P_2^{**N/4}\bar{y}_i^{**}|\}$,

Defining the barrier functions $\bar{\chi}_i^{**\pm} = (\chi_{1,i}^{**\pm}, \chi_{2,i}^{**\pm})^T$ by

$$\chi_{1,i}^{**\pm} = A''\{1+2x_i\} \pm y_{1,i}^{**} \quad \text{and} \quad \chi_{2,i}^{**\pm}(x) = A'' \pm y_{2,i}^{**} \quad \text{for} \quad 0 \leq i \leq N/4.$$

Applying Theorem 4.12 to $\bar{\chi}_i^{**\pm}$ for a proper selection of the constant C, we can obtain the desired bounds for $\bar{y}_i^{**}$. □

4.3.3. Outer Region Problem. The outer region problem for (14)-(15) is given by

$$Ly_2(x) = \begin{cases} -\varepsilon y_2''(x) + b(x)y_2(x) = f(x) - c(x)u_{0_1}(x), \quad x \in (\tau, 1-\tau), \\ B_0y_2(0) = y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau) + w_{0_2}(\tau) = q^*, \\ B_0y_2(1) = y_2(1-\tau) = u_{0_2}(1-\tau) + v_{0_2}(1-\tau) + w_{0_2}(1-\tau) = r^*, \end{cases} \tag{25}$$

where $b(x)$ and $f(x)$ are sufficiently smooth and $b(x) \geq \beta$, $\beta > 0$, $0 \geq c(x) \geq -\gamma$, $\gamma > 0$.

Theorem 4.14. (Maximum Principle).*Consider the BVP (25). Let* $y_2(x)$ *be a smooth function satisfying* $B_0y_2(0) \geq 0$, $B_1y_2(1) \geq 0$ *and* $Ly_2(x) \geq 0$ *for* $x \in (\tau, 1-\tau)$. *Then,* $y_2(x) \geq 0$ *for* $x \in [\tau, 1-\tau]$.

Proof. Please refer [1]. □

Lemma 4.15. (Stability result).*If* $y_2(x)$ *is the solution of the BVP (25) then*

$$|y_2(x)| \leq C\max\{|B_0y_2(0)| + |B_1y_2(1)| + \max_{x\in(\tau,1-\tau)} |Ly_2(x)|\}, \quad \forall x \in [\tau, 1-\tau].$$

Proof. [1]. □

To solve this BVP, we apply standard FD scheme defined by

$$\begin{cases} L^{N/2}y_{2,i} := -\varepsilon\delta^2 y_{2,i} + b(x_i)y_{2,i} = f(x_i) - c(x_i)u_{0_1}(x_i), \quad 1 \leq i \leq N/2-1, \\ B_0^{N/2}y_{2,0} = y_{2,0} = q^*, \quad B_1^{N/2}y_{2,N} = y_{2,N/2} = r^*, \end{cases} \tag{26}$$

where $\delta^2 y_{2,i} = (y_{2,i+1} - 2y_{2,i} + y_{2,i-1})/h_2^2$, $x_i = \tau + ih_2$ and $h_2 = 2(1-2\tau)/N$, $1 \leq i \leq N/2-1$.

Theorem 4.16. (Discrete Maximum Principle). *Consider the discrete BVP (26). If* $B_0^{N/2}y_{2,0} \geq 0$, $B_1^{N/2}y_{2,N/2} \geq 0$ *and* $L^{N/2}y_{2,i} \geq 0$ *for* $1 \leq i \leq N/2-1$. *Then* $y_{2,i} \geq 0$ *for* $1 \leq i \leq N/2$.

Proof. Please refer [1]. □

Lemma 4.17. (Discrete Stability Result). *If* $y_{2,i}$ *is the solution of the BVP (26) then*

$$|y_{2,i}| \leq C\max\{|B_0^{N/2}y_{2,0}|+|B_1^{N/2}y_{2,N/2}|+\max_{1\leq i\leq N/2-1}|L^{N/2}y_{2,i}|\}, for\, 0 \leq i \leq N/2.$$

Proof. Please refer [1]. □

5. Error Estimates

In this section, we derive error estimates for the solution of (14)-(15).

5.1. Inner region problems. In order to derive error estimate for the solution of the inner region problems we prove the following theorems.

Error estimates for Left Inner Region Problem.

Theorem 5.1. *Let* $\bar{y}^* = (y_1^*, {y_2}^*)^T$ *and* $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ *be, respectively, the solutions of (20) and (21). Then,*

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq CN^{-1}\ln N \quad for \quad 0 \leq i \leq N/4, \quad x_i \in \bar{\Omega}_\tau^{N/4}.$$

Proof. From Lemma 4.1 in [9] and Theorem 4.6 it is clear that for each i, the consistency errors due to $\bar{y}^*$ with $P_1^{*N/4}$ and $P_2^{*N/4}$ are bounded as given below.

$$\begin{aligned} |P_1^{*N/4}(\bar{y}^*(x_i) - \bar{y}_i^*)| &= |(D^- - D)y_1^*(x_i)|, \\ &= \frac{h_1}{2}|y_1^{*''}(t)|, \\ &= \frac{h_1}{2\sqrt{\varepsilon}}e(x,\beta), \end{aligned} \tag{27}$$

$$\begin{aligned} \text{and } |P_2^{*N/4}(\bar{y}^*(x_i) - \bar{y}_i^*)| &= \varepsilon|(D^- - D)y_2^*(x_i)|, \\ &= \frac{\varepsilon h_1}{2}|y_2^{*''}(t)|, \\ &= \frac{h_1}{2}e(x,\beta), \end{aligned} \tag{28}$$

for some point t satisfying, $x_{i-1} \leq t \leq x_i$, where $e(x,\beta) = e^{-x\sqrt{\beta/\varepsilon}}+e^{-(1-x)\sqrt{\beta/\varepsilon}}$. Since $\tau = \min\{\frac{1}{4}, \sqrt{\frac{\varepsilon}{\beta}}\ln N\}$, the argument is considered for two cases $\tau = \frac{1}{4}$ and $\tau = \sqrt{\frac{\varepsilon}{\beta}}\ln N$ separately.

Case 1: $\tau = \dfrac{1}{4}$. Note that $\dfrac{1}{4} \leq \sqrt{\dfrac{\varepsilon}{\beta}} \ln N$ implies $\varepsilon^{-1/2} \leq C \ln N$.

From (27) and (28) and using $h_1 \leq CN^{-1}$. we have

$$\begin{cases} |P_1^{*N/4}(\bar{y}^*(x_i) - \bar{y}_i^*)| \leq CN^{-1} \ln N. \\ |P_2^{*N/4}(\bar{y}^*(x_i) - \bar{y}_i^*)| \leq CN^{-1} \leq CN^{-1} \ln N. \end{cases} \tag{29}$$

Case 2: $\tau = \sqrt{\dfrac{\varepsilon}{\beta}} \ln N$.

From (27) and (28), we have

$$\begin{cases} |P_1^{*N/4}(\bar{y}^*(x_i) - \bar{y}_i^*)| \leq CN^{-1} \ln N. \\ |P_2^{*N/4}(\bar{y}^*(x_i) - \bar{y}_i^*)| \leq CN^{-1} \leq CN^{-1} \ln N. \end{cases} \tag{30}$$

Since $y_1^*(0) = y_{0,1}^*, \quad y_2^*(0) = y_{0,2}^*$ by the discrete stability result given by Lemma 4.8 it follows that

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq CN^{-1} \ln N.$$

□

Theorem 5.2. *Let $\bar{y}^* = (y_1^*, y_2^*)^T$ and $\bar{y}^{*1} = (y_1^{*1}, y_2^{*1})^T$ be, respectively, the solutions of the IVPs*

$$\begin{cases} y_1^{*'} - y_2^* = 0, \\ -\varepsilon y_2^{*'} + b(x) y_1^* = f(x) - c(x) u_{0_1}(x), \quad x \in \Omega, \\ y_1^*(0) = \alpha', \quad y_2^*(0) = \beta'. \end{cases} \tag{31}$$

and

$$\begin{cases} y_1^{*1'} - y_2^{*1} = 0, \\ -\varepsilon y_2^{*1'} + b(x) y_1^{*1} = f(x) - c(x) u_{0_1}(x), \quad x \in \Omega, \\ y_1^{*1}(0) = \alpha' + O(\varepsilon), \quad y_2^{*1}(0) = \beta', \end{cases} \tag{32}$$

then $\quad ||\bar{y}^*(x) - \bar{y}^{*1}(x)|| \leq C\sqrt{\varepsilon}$.

Proof. Let $\bar{w} = \bar{y}^* - \bar{y}^{*1}$. Then $\bar{w}$ satisfies

$$\begin{cases} w_1' - w_2 = 0, \\ -\varepsilon w_2' + b(x) w_1 = 0, \quad x \in \Omega, \\ w_1(0) = O(\varepsilon), \quad w_2(0) = 0. \end{cases} \tag{33}$$

Using the maximum principle for the system (33) as in Doolan [1], we have

$$||\bar{y}^*(x) - \bar{y}^{*1}(x)|| \leq C\sqrt{\varepsilon}, \quad \forall x \in \Omega.$$

□

Theorem 5.3. *Let $\bar{y}^* = (y_1^*, y_2^*)^T$ be the solution of the IVP (20). Further, let $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ be the numerical solution of the IVP (32) after applying the Euler's finite difference scheme as given in (21). Then,*

$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq C\sqrt{\varepsilon} + CN^{-1} \ln N \quad$ *for* $\quad 0 \leq i \leq N/4 \quad$ *and* $\quad x_i \in \bar{\Omega}_\tau^{N/4}$.

Proof. From Theorem 5.1, $||\bar{y}^{*1}(x_i) - \bar{y}_i^*|| \leq CN^{-1}\ln N$.
From Theorem 5.2, $||\bar{y}^*(x_i) - \bar{y}^{*1}(x_i)|| \leq C\sqrt{\varepsilon}$.
Using these estimates in the inequality,

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq ||\bar{y}^*(x_i) - \bar{y}^{*1}(x_i)|| + ||\bar{y}^{*1}(x_i) - \bar{y}_i^*||,$$

where $\bar{y}^{*1}(x)$ is the solution of the system (32), this theorem gets proved. □

The SPBVPs (14)-(15) is equivalent to the following IVP

$$\begin{cases} -\varepsilon y_2''(x) + b(x)y_2(x) = f(x) - c(x)u_{0_1}(x), & x \in \Omega, \\ y_2(0) = q^*, \quad y_2'(0) = q, \end{cases} \tag{34}$$

where q^* is the asymptotic value of the solution of the BVP (14)-(15) at $x = 0$. Because of uniqueness of the solutions of the IVP (34) and the BVP (14)-(15), we have the following result on the error estimate for the left inner region problem.

Theorem 5.4. *Let $y_2^\star(x_i)$ be the solution of the BVP (14)-(15). Further, let $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ be the numerical solution of the IVP (21). Then,*

$$|y_2^\star(x_i) - y_{1,i}^*| \leq C\sqrt{\varepsilon} + CN^{-1}\ln N \quad \textit{for} \quad 0 \leq i \leq N/4, \quad x_i \in \bar{\Omega}_\tau^{N/4}.$$

Proof. Consider the inequality

$$|y_2^\star(x_i) - y_{1,i}^*| \leq |y_2^\star(x_i) - y_1^{*1}(x_i)| + |y_1^{*1}(x_i) - y_{1,i}^*|,$$

where $y_1^{*1}(x)$ is the solution of the system (32). The proof follows from Theorem 5.2 and Theorem 5.3. □

Theorem 5.5. *Let $\bar{y}$ be the solution of the BVP (6)-(7) and let $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ be the numerical solution of the IVP (21). Then,*

$$|y_2(x_i) - y_{1,i}^*| \leq C\sqrt{\varepsilon} + CN^{-1}\ln N \quad \textit{for} \quad 0 \leq i \leq N/4, \quad x_i \in \bar{\Omega}_\tau^{N/4}.$$

Proof. Consider the inequality,

$$|y_2(x_i) - y_{1,i}^*| \leq |y_2(x_i) - y_2^\star(x_i)| + |y_2^\star(x_i) - y_{1,i}^*|,$$

where $y_2^\star(x)$ is the solution of the BVP(14)-(15). The proof follows from Theorem 4.3 and Theorem 5.4. □

Error estimates for Right Inner Region Problem.

Theorem 5.6. *Let $\bar{y}^{**} = (y_1^{**}, y_2^{**})^T$ and $\bar{y}_i^{**} = (y_{1,i}^{**}, y_{2,i}^{**})^T$ be, respectively, the solutions of (23) and (24). Then,*

$$||\bar{y}^{**}(x_i) - \bar{y}_i^{**}|| \leq CN^{-1}\ln N \quad \textit{for} \quad 0 \leq i \leq N/4, \quad x_i \in \bar{\Omega}_\tau^{N/4}.$$

Proof. Proof is similar as Theorem 5.1. □

Theorem 5.7. *Let $\bar{y}^{***} = (y_1^{**}, y_2^{**})^T$ and $\bar{y}^{**1} = (y_1^{**1}, y_2^{**1})^T$ be, respectively, the solutions of the IVPs*

$$\begin{cases} {y_1^{**}}' - y_2^{**} = 0, \\ -\varepsilon {y_2^{**}}' + b(x)y_1^{**} = f(x) - c(x)u_{0_1}(x), \quad x \in \Omega, \\ y_1^{**}(1) = \alpha'', \quad y_2^{**}(1) = \beta'' \end{cases} \tag{35}$$

and

$$\begin{cases} {y_1^{**1}}' - y_2^{**1} = 0, \\ -\varepsilon {y_2^{**1}}' + b(x)y_1^{**1} = f(x) - c(x)u_{0_1}(x), \quad x \in \Omega, \\ y_1^{**1}(1) = \alpha'' + O(\varepsilon), \quad y_2^{**1}(1) = \beta'' \end{cases} \tag{36}$$

then, $||\bar{y}^{**}(x) - \bar{y}^{**1}(x)|| \leq C\sqrt{\varepsilon}$.

Proof. Proof is similar as Theorem 5.2. □

Theorem 5.8. *Let $\bar{y}^{**} = (y_1^{**}, y_2^{**})^T$ be the solution of the IVP (35). Further, let $\bar{y}_i^{**} = (y_{1,i}^{**}, y_{2,i}^{**})^T$ be the numerical solution of the IVP (36) after applying the Euler's finite difference scheme as given in (24). Then,*

$$||\bar{y}^{**}(x_i) - \bar{y}_i^{**}|| \leq C\sqrt{\varepsilon} + CN^{-1}\ln N \quad \textit{for} \quad 0 \leq i \leq N/4 \quad \textit{and} \quad x_i \in \bar{\Omega}_\tau^{N/4}.$$

Proof. From Theorem 5.7, $||\bar{y}^{**}(x_i) - \bar{y}^{**1}(x_i)|| \leq C\sqrt{\varepsilon}$.
From Theorem 5.6, $||\bar{y}^{**1}(x_i) - \bar{y}_i^{**}|| \leq CN^{-1}\ln N$.
Using these estimates in the inequality,

$$||\bar{y}^{**}(x_i) - \bar{y}_i^{**}|| \leq ||\bar{y}^{**}(x_i) - \bar{y}^{**1}(x_i)|| + ||\bar{y}^{**1}(x_i) - \bar{y}_i^{**}||,$$

where $\bar{y}^{**1}(x)$ is the solution of the system (36), this theorem gets proved. □

The BVP (14)-(15) is equivalent to the following IVP

$$\begin{cases} -\varepsilon y_2''(x) + b(x)y_2(x) = f(x) - c(x)u_{0_1}(x), \quad x \in \Omega, \\ y_2(1) = r^*, \quad y_2'(1) = r, \end{cases} \tag{37}$$

where r^* is the asymptotic value of the solution of the BVP (14)-(15) at $x = 1$. Because of uniqueness of the solution of the IVP (37) and the BVP (14)-(15), we have the following result on the error estimate for the right inner region problem.

Theorem 5.9. *Let $y_2^\star(x_i)$ be the solution of the BVP (14)-(15). Further, let $\bar{y}_i^{**} = (y_{1,i}^{**}, y_{2,i}^{**})^T$ be the numerical solution of the IVP (24). Then,*

$$|y_2^\star(x_i) - y_{1,i}^{**}| \leq C\sqrt{\varepsilon} + CN^{-1}\ln N \quad \textit{for} \quad 0 \leq i \leq N/4, \quad x_i \in \bar{\Omega}_\tau^{N/4}.$$

Proof. Consider the inequality

$$|y_2^\star(x_i) - y_{1,i}^{**}| \leq |y_2^\star(x_i) - y_1^{**1}(x_i)| + |y_1^{**1}(x_i) - y_{1,i}^{**}|,$$

where $y_1^{**1}(x)$ is the solution of the BVP (36). The proof follows from Theorem 5.7 and Theorem 5.8. □

Theorem 5.10. *Let $\bar{y}$ be the solution of the BVP (6)-(7) and let $\bar{y}_i^{**} = (y_{1,i}^{**}, y_{2,i}^{**})^T$, be the numerical solution of the IVP (24). Then,*

$$|y_2(x_i) - y_{1,i}^{**}| \leq C\sqrt{\varepsilon} + CN^{-1}\ln N \quad \textit{for} \quad 0 \leq i \leq N/4, \quad x_i \in \bar{\Omega}_\tau^{N/4}.$$

Proof. Consider the inequality,

$$|y_2(x_i) - y_{1,i}^{**}| \leq |y_2(x_i) - y_2^\star(x_i)| + |y_2^\star(x_i) - y_{1,i}^{**}|,$$

where $y_2^\star(x)$ is the solution of the system (14)-(15). The proof follows from Theorem 4.3 and Theorem 5.9. □

5.2. Outer Region Problem. Adopting the method of analysis provided in [2] the following theorems can be proved.

Theorem 5.11. *Let $y_2(x_i)$ be the solution of the BVP (25) and the solution $y_{2,i}$ of the BVP (26) satisfy*

$$|y_2(x_i) - y_{2,i}| \leq CN^{-1}\ln N \quad \textit{for} \quad 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}$$

Proof. Please refer [2]. □

Theorem 5.12. *Let $y_2^\star(x_i)$ be the solution of the BVP (14)-(15) and $y_{2,i}$ be the numerical solution of the BVP (25) after applying the standard FD scheme as given in (26). Then,*

$$|y_2^\star(x_i) - y_{2,i}| \leq C\sqrt{\varepsilon} + CN^{-1}\ln N \quad \textit{for} \quad 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. From Theorem 4.3, $|y_2^\star(x_i) - y_2(x_i)| \leq C\sqrt{\varepsilon}$.
From Theorem 5.11, $|y_2(x_i) - y_{2,i}| \leq CN^{-1}\ln N$.
Using these estimates in the inequality,

$$|y_2^\star(x_i) - y_{2,i}| \leq |y_2^\star(x_i) - y_2(x_i)| + |y_2(x_i) - y_{2,i}|,$$

where $y_2(x_i)$ is the solution of the BVP (25), this theorem gets proved. □

Theorem 5.13. *Let $\bar{y}$ be the solution of the BVP (6)-(7) and $y_{2,i}$ be the numerical approximation obtained for $y_2(x_i)$ from the BVP (25) after applying the standard FD scheme as given in (26). Then,*

$$|y_2(x_i) - y_{2,i}| \leq C\sqrt{\varepsilon} + CN^{-1}\ln N \quad \textit{for} \quad 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. From Theorem 4.3, $|y_2(x_i) - y_2^\star(x_i)| \leq C\sqrt{\varepsilon}$,
From Theorem 5.12, $|y_2^\star(x_i) - y_{2,i}| \leq CN^{-1}\ln N$.
Using these estimates in the inequality,

$$|y_2(x_i) - y_{2,i}| \leq |y_2(x_i) - y_2^\star(x_i)| + |y_2^\star(x_i) - y_{2,i}|,$$

where $y_2^\star(x_i)$ is the solution of the BVP (14)-(15), this theorem gets proved. □

6. Non-linear problem

Consider the quasi-linear BVP

$$-\varepsilon y^{'''}(x) = F(x, y, y'), \quad x \in \Omega, \tag{38}$$

$$y(0) = p, \quad y''(0) = q, \quad y''(1) = r. \tag{39}$$

where $F(x, y, y')$ is a smooth function such that

$$\begin{cases} F_{y'}(x, y, y') \geq \beta, \quad \beta > 0, \\ 0 \geq F_y(x, y, y') \geq -\gamma, \quad \gamma > 0, \quad \beta - 2\gamma \geq \eta', \\ \text{for some} \quad \eta' > 0. \end{cases} \tag{40}$$

Assume that the reduced problem $F(x, y, y') = 0,\ y(0) = p$ has a solution $y_0 \in C^{(3)}(\bar{\Omega})$. Then (38)-(39) has a unique solution and has less severe twin boundary layers of width $O(\sqrt{\varepsilon})$ near $x = 0$ and $x = 1$ ([18, 38]). Analytical results such as existence, uniqueness and asymptotic behavior of the solution of (38)-(39) can be found in [7, 8, 18, 32, 38].

In order to obtain a numerical solution of (38)-(39), first Newton's method of quasi-linearisation is applied [1] and the problem is linearized. Consequently, we get a sequence $\{y^{[m]}\}_0^\infty$ of successive approximations with a proper choice of initial guess $y^{[0]}$ (Here also $y^0(x) = p + qx$ is a good initial approximation). We define $y^{[m+1]}$ for each fixed non-negative integer m, to be the solution of the following linear problem:

$$\begin{cases} -\varepsilon(y^{'''}(x))^{[m+1]} + b^m(x)(y'(x))^{[m+1]} + c^m(x)(y(x))^{[m+1]} = F^{[m]}(x), \\ y^{[m+1]}(0) = p, \quad (y''(x))^{[m+1]}(0) = q, \quad (y''(x))^{[m+1]}(1) = r, \end{cases} \tag{41}$$

where

$$\begin{cases} b^{[m]}(x) = F_{y'}(x, y^{[m]}, (y')^{[m]}), \\ c^{[m]}(x) = F_y(x, y^{[m]}, (y')^{[m]}), \\ F^{[m]}(x) = F(x, y^{[m]}, (y')^{[m]}) - (y')^{[m]} F_{y'}(x, y^{[m]}, (y')^{[m]}) \\ \qquad\qquad -(y)^{[m]} F_y(x, y^{[m]}, (y')^{[m]}). \end{cases} \tag{42}$$

and for each m, $b^{[m]}(x)$, $c^{[m]}(x)$ satisfy (40)

Remark 6.1. If the initial guess $y^{[0]}$ is sufficiently close to the solution $y(x)$ of (38)-(39), then, following the method of proof given in [1], one can prove that the sequence $\{y^{[m]}\}_0^\infty$ converges to $y(x)$. From (40), it follows that for each fixed m:

$$b^{[m]}(x) = F_{y'}(x, y^{[m]}, (y')^{[m]}) \geq \beta, \quad \beta > 0,$$
$$0 \geq c^{[m]}(x) = F_y(x, y^{[m]}, (y')^{[m]}) \geq -\gamma, \quad \gamma > 0,$$
$$\beta - 2\gamma \geq \eta', \quad \text{for some} \quad \eta' > 0.$$

Remark 6.2. The solution of the reduced problem of (38)-(39) or a suitable approximation will be taken as the initial guess $y^{[0]}$ to generate the successive approximations $\{y^{[m]}\}_0^\infty$.

Remark 6.3. For the above Newton's quasi-linearisation process the following convergence criterion is used.

$$|y^{[m+1]}(x_j) - y^{[m]}(x_j)| \leq \delta, \quad x_j \in \bar{\Omega}, \quad m \geq 0.$$

7. Illustrations

In this section, we present two examples to illustrate the method described in this paper. Let Y^N be a numerical approximation for the exact solution y on the mesh Ω^N and N is the number of mesh points. We compute the maximum point-wise errors using

$$E_\varepsilon^N = \max_{x \in \bar{\Omega}^N} \mid Y^N(x_j) - y(x_j) \mid \quad \text{and} \quad E^N = \max_{\varepsilon} E_\varepsilon^N.$$

Then, the order of convergence is given by

$$p^* = \min_N p^N \quad \text{where,} \quad p^N = log_2 \left\{ \frac{E^N}{E^{2N}} \right\}.$$

Example 7.1. Consider the BVP

$$-\varepsilon y^{'''}(x) + (x+2)y'(x) - y(x) = \varepsilon^{3/4}(log(x+2)),$$
$$y(0) = 1, \quad y''(0) = 0, \quad y''(1) = 1.$$

The numerical result is presented in Table 1.

Example 7.2. Consider the BVP

$$-\varepsilon y^{'''}(x) + 4(y')^2(x) - 4y(x) = \varepsilon^{5/2}(x + e^{-x}),$$
$$y(0) = 0, \quad y''(0) = 1, \quad y''(1) = 0.$$

This BVP is linearised using the Newton's Method of quasi-linearisation. The numerical result is presented in Table 2. The initial approximation for y_1 is taken to be $y^0(x) = x$.

8. Conclusions

In this paper, we presented a numerical method to solve third-order SP-BVPs for ODEs subject to particular type of boundary conditions by adopting the techniques of [6, 21, 32, 37] and [10]-[13], [36] who used to solve second-order and third-order SPBVPs for ODEs. The boundary conditions help us to reduce the given third order ordinary differential equation into a weakly coupled system of one first order and one second order equation subject to initial and boundary conditions, respectively. It is quite natural that one would to expect better solution of the problem in the interval $[\tau, 1-\tau]$. But our numerical experiments show that this method gives good solution only in the neighbourhood

TABLE 1. *Maximum pointwise errors E_ε^N, E^N and $p^*$for the Example 7.1.*

ε	Number of mesh points N				
	64	128	256	512	1024
2^{-6}	7.8320e-006	4.1259e-006	4.0088e-006	3.7689e-007	2.1038e-007
2^{-7}	3.8882e-006	2.8736e-006	1.8870e-007	1.2494e-007	1.7760e-007
2^{-8}	1.9941e-006	1.5318e-006	6.5349e-007	5.8470e-007	1.4380e-007
2^{-9}	9.5705e-007	7.2589e-007	4.8175e-008	2.8835e-008	1.7690e-008
2^{-10}	4.7452e-008	3.6795e-008	2.4587e-008	1.5368e-008	8.4450e-008
2^{-6}	3.3552e-004	1.6376e-004	8.2380e-005	4.1690e-005	2.1345e-005
2^{-7}	1.6276e-004	8.1380e-005	4.0690e-005	2.0345e-005	1.0173e-005
2^{-8}	8.1380e-005	4.0690e-005	2.0345e-005	1.0173e-005	5.0863e-006
2^{-9}	4.0690e-005	2.0345e-005	1.0173e-005	5.0863e-006	2.5431e-006
2^{-10}	2.0345e-005	1.0173e-005	5.0863e-006	2.5431e-006	1.2716e-006
2^{-6}	7.7913e-006	6.5522e-006	4.6232e-006	3.7669e-006	2.6242e-006
2^{-7}	7.3962e-006	6.5065e-006	4.3068e-006	2.6232e-006	1.3118e-006
2^{-8}	6.9758e-007	5.8787e-007	2.1533e-007	1.3118e-007	6.5589e-007
2^{-9}	6.4879e-007	4.9389e-007	1.0767e-007	6.5589e-008	3.2795e-008
2^{-10}	5.7440e-008	4.6944e-008	4.3843e-008	3.2895e-008	1.3697e-008
E^N	3.3552e-004	1.6376e-004	8.2380e-005	4.1690e-005	2.1345e-005
p	1.0348e+000	9.9122e-001	9.8259e-001	p^*9.6580e-001	
The order of convergence=9.6580e-001					
CPU time(sec.)=7.4219e+000					

of $x = 0$ and $x = 1$. Of course, an approximate solution can be improved by taking better approximate initial condition as said in Section 4. This is the reason for taking the solution of the IVP only in the interval $[0, \tau]$. In [32], both inner and outer region problems are BVPs, whereas in our case the inner region problem is an IVP and the outer region problem is a BVP. Naturally IVPs can be treated more easily compared with BVPs. Though the present method yields almost the same order of convergence as given in [32], the method produces very good reduction on the maximum-pointwise error compared with [32]. The main advantage of this paper is that due to decoupling the system, the size of the matrix to be inverted is reduced from $2N - 1$ to $N - 1$. This results in a good reduction of the computation time. Error estimates derived in Section 5 show first order convergence. Our numerical experiments show that this method gives good approximate solution especially with in the boundary layer regions which can be seen from the numerical results presented in Table 1 and Table 2. In all the tables the numerical results appearing in the rows 1-5 and 11-15 correspond to the left and right boundary layers, respectively. The rest of the rows namely 6-10 correspond to the outer region.

Table 2. *Maximum pointwise errors E_ε^N, E^N and p^* for the Example 7.2.*

ε	Number of mesh points N				
	64	128	256	512	1024
2^{-6}	8.1361e-007	6.1746e-007	4.1030e-007	8.4383e-007	4.4263e-007
2^{-7}	4.0181e-007	3.1373e-007	2.1014e-007	1.2291e-007	7.1810e-008
2^{-8}	2.1090e-007	1.5186e-007	1.1007e-007	6.1956e-008	3.5415e-008
2^{-9}	1.1045e-007	7.5933e-008	5.1036e-008	3.1478e-008	1.7712e-008
2^{-10}	5.1225e-008	3.7967e-008	2.5118e-008	2.5039e-008	1.8522e-008
2^{-6}	1.2877e-004	6.6463e-005	3.3757e-005	1.7010e-005	0.9080e-005
2^{-7}	6.9374e-005	1.1627e-005	9.0215e-00	2.5499e-007	3.4951e-007
2^{-8}	4.5921e-007	1.7017e-007	2.8588e-008	2.2584e-008	1.5193e-008
2^{-9}	6.5193e-008	5.5193e-008	4.5193e-008	3.0816e-008	2.6496e-008
2^{-10}	5.6496e-008	4.0496e-008	3.6496e-008	2.6496e-008	1.7611e-008
2^{-6}	2.9596e-004	1.6452e-004	9.1354e-005	5.0554e-005	2.7827e-005
2^{-7}	6.4798e-006	5.2259e-007	4.5679e-007	2.7827e-007	1.3914e-007
2^{-8}	6.3991e-007	4.1131e-007	2.2841e-007	1.3915e-007	1.9569e-008
2^{-9}	6.2095e-007	4.0565e-007	3.1420e-007	6.9568e-008	5.4784e-008
2^{-10}	5.8497e-008	5.0282e-008	4.7099e-008	3.4784e-008	1.7392e-008
E^N	1.2877e-004	6.6463e-005	3.3757e-005	1.7010e-005	0.9080e-005
p	9.5417e-001	9.7736e-001	9.8880e-001	p^*9.0562e-001	
The order of convergence= 9.0562e-001					
CPUtime(sec.) =1.1781e+001					

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