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# THE DYNAMICS OF POSITIVE SOLUTIONS OF A HIGHER ORDER FRACTIONAL DIFFERENCE EQUATION WITH ARBITRARY POWERS

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Abstract. The purpose of this paper is to investigate the local asymptotic stability of equilibria, the periodic nature of solutions, the existence of unbounded solutions and the global behavior of solutions of the fractional difference equation

$$x_{n+1} = \frac{\alpha x_{n-(k+1)}}{\beta + \gamma x^p_{n-k} x^q_{n-(k+2)}},\ n = 0, 1, ...$$

where the parameters $\alpha, \beta, \gamma, p, q$ are non-negative numbers and the initial values $x_{-(k+2)}, x_{-(k+1)}, ..., x_{-1}, x_0 \in \mathbb{R}^+$.

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## 1. Introduction

In the past twenty years, many papers appeared focusing on the investigation of the qualitative analysis of solutions of difference equations and their systems (see [1, 5, 6, 10, 11, 15] and the references cited therein). One of the reasons for this is a requirement for some techniques which can be used in studying equations arising in mathematical models describing real life cases in population dynamics, economics, probability theory, genetics, psychology and so on. That is, the theory of difference equations gets a central position in applicable analysis. Hence, it is very valuable to get the behavior of solutions of fractional difference equations and to discuss the local aysmptotic stability of their equilibrium points and global behavior of solutions.

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According to us, it is of a great importance to investigate not only non-linear difference equations, but also those equations which contain powers of arbitrary positive degrees (see [8, 9, 16, 17] and the references cited therein).

The purpose of this paper is to study the local asymptotic stability of equilibrium points, the periodic nature of solutions, the existence of unbounded solutions and the global behavior of solutions of the following fractional difference equation

$$x_{n+1} = \frac{\alpha x_{n-(k+1)}}{\beta + \gamma x_{n-k}^{p} x_{n-(k+2)}^{q}}, \ n \in \mathbb{N} \tag{1.1}$$

where the parameters $\alpha, \beta, \gamma, p, q$ are non-negative numbers and the initial values $x_{-(k+2)}, x_{-(k+1)}, \ldots, x_{-1}, x_0$ are arbirary positive numbers such that the denominator is always positive.

In [4], El-Owaidy *et al.* investigated the global behavior of the following rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{p}}, \ n \in \mathbb{N},$$

with non-negative parameters and non-negative initial values.

By generalizing the results due to El-Owaidy *et al.* [4], in [3], Chen *et al.* studied the dynamical behavior of the following rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma x_{n-l}^{p}}, \ n \in \mathbb{N}$$

where $k, l \in \mathbb{N}$, the parameters are positive real numbers and the initial values $x_{-\max\{k,l\}}, \ldots, x_{-1}, x_0 \in (0, \infty)$.

In [2], Ahmed studied the dynamical behavior of the following rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma \prod_{i=l}^{k} x_{n-2i}^{p_i}}, n \in \mathbb{N}$$

where the parameters are non-negative real numbers and the initial values are non-negative real numbers.

In [7], Erdogan *et. al.* investigated the dynamical behavior of positive solutions of the following higher-order difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma \sum_{k=1}^{t} x_{n-2k} \prod_{k=1}^{t} x_{n-2k}}, n \in \mathbb{N}$$

where the parameters are non-negative real numbers and the initial values are non-negative real numbers.

In [14], Karatas investigated the global behavior of the equilibria of the following difference equation

$$x_{n+1} = \frac{A x_{n-m}}{B + C \prod_{i=0}^{2k+1} x_{n-i}}, n \in \mathbb{N}$$

where the parameters are non-negative real numbers and the initial values are non-negative real numbers.

If some parameters of Eq.(1.1) are zero, then five equations emerge, that is, if $\alpha = 0$ in Eq.(1.1), then it is trivial, if $\beta = 0$ in Eq.(1.1), then it can be reduced to a linear difference equation by the change of variables $x_n = e^{y_n}$. If $\gamma = 0$ in Eq.(1.1), then it is linear and finally, the case $p = 0$ or $q = 0$ was investigated in [3].

Note that Eq.(1.1) can be reduced to the following fractional difference equation

$$y_{n+1} = \frac{r y_{n-(k+1)}}{1 + y_{n-k}^{p} y_{n-(k+2)}^{q}},\ n \in \mathbb{N} \tag{1.2}$$

by the change of variables $x_n = (\frac{\beta}{\gamma})^{\frac{1}{p+q}} y_n$ with $r = \frac{\alpha}{\beta}$. So, in order to study Eq.(1.1), we will investigate Eq.(1.2).

As far as we examine, there is exactly no paper dealing with Eq.(1.1). Therefore, in this paper, we focus on Eq.(1.1) in order to fill in the gap.

## 2. Preliminaries

For the completeness in the paper, we find useful to remind some basic concepts of the difference equations theory as follows:

Let $I$ be an interval of real numbers and let $f : I^{k+3} \to I$ be a continuously differentiable function. Then for any condition $x_{-(k+2)}, x_{-(k+1)}, ..., x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-(k+1)}, x_{n-(k+2)}),\ n \in \mathbb{N} \tag{2.1}$$

has a unique positive solution $\{x_n\}_{n=-(k+2)}^{\infty}$.

**Definition 2.1.** *An equilibrium point of Eq.(2.1) is a point $\overline{x}$ that satisfies*

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

*The point $\overline{x}$ is also said to a fixed point of the function $f$.*

**Definition 2.2.** *Let $\overline{x}$ be a positive equilibrium of (2.1).*
*(a) $\overline{x}$ is stable if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every positive solution $\{x_n\}_{n=-(k+2)}^{\infty}$ of (2.1) with $\sum_{i=-(k+2)}^{0} |x_i - \overline{x}| < \delta$, $|x_n - \overline{x}| < \varepsilon$ holds for $n \in \mathbb{N}$.*
*(b) $\overline{x}$ is locally asymptotically stable if $\overline{x}$ is stable and there is $\gamma > 0$ such that $\lim x_n = \overline{x}$ holds for every positive solution $\{x_n\}_{n=-(k+2)}^{\infty}$ of (2.1) with $\sum_{i=-(k+2)}^{0} |x_i - \overline{x}| < \gamma$.*
*(c) $\overline{x}$ is a global attractor if $\lim x_n = \overline{x}$ holds for every positive solution $\{x_n\}_{n=-(k+2)}^{\infty}$ of (2.1).*
*(d) $\overline{x}$ is globally asymptotically stable if $\overline{x}$ is both stable and global attractor.*

**Definition 2.3.** *The linearized equation of (2.1) about the equilibrium point $\overline{x}$ is*

$$y_{n+1} = \zeta_0 y_n + \zeta_1 y_{n-1} + \cdots + \zeta_{k+2} y_{n-(k+2)},\ n \in \mathbb{N} \tag{2.2}$$

*where*

$$\zeta_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, ..., \overline{x}),\ \zeta_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, ..., \overline{x}), \ldots,\ \zeta_{k+2} = \frac{\partial f}{\partial x_{n-(k+2)}}(\overline{x}, \overline{x}, ..., \overline{x}).$$

*The characteristic equation of* (2.2) *is*

$$\lambda^{k+3} - \zeta_0\lambda^{k+2} - \zeta_1\lambda^{k+1} - ... - \zeta_{k+1}\lambda - \zeta_{k+2} = 0. \tag{2.3}$$

**Definition 2.4.** *Any solution* $\{x_n\}_{n=-(k+2)}^{\infty}$ *of Eq.*(2.1) *is said to non-oscillatory solution if there exists* $n_0 \geq -(k+2)$ *such that either*

$$x_n > \overline{x} \text{ for all } n \geq n_0$$

*or*

$$x_n < \overline{x} \text{ for all } n \geq n_0.$$

*Also,* $\{x_n\}_{n=-(k+2)}^{\infty}$ *is said to an oscillatory solution if it is not a non-oscillatory solution.*

The following result, known as the *Linearized Stability Theorem*, is very useful in determining the local stability character of the equilibrium point $\overline{x}$ of equation (2.1).

**Theorem 2.1.** *(Linearized Stability Theorem) Consider Eq.*(2.1) *such that* $\overline{x}$ *is a fixed point of* $f$*. If all roots of the function* $f$ *about* $\overline{x}$ *lie inside the open unit disk* $|\lambda| < 1$*, then* $\overline{x}$ *is locally asymtotically stable. If one of them has a modulus greater than one, then* $\overline{x}$ *is unstable. The fixed point* $\overline{x}$ *of* $f$ *is called a saddle point if* $f$ *has roots both inside and outside the unit disk. If any root of* $f$ *has absolute value equal to one, then the fixed point* $\overline{x}$ *of* $f$ *is called a non-hyperbolic point.*

For other basic knowledge about difference equations, the reader is referred to [12, 13].

## 3. Main Results

In this section we prove our main results.

**Theorem 3.1.** *We have the following cases for the equilibrium points of Eq.*(1.2)*;*

- i $\overline{y}_0 = 0$ is always the equilibrium point of Eq.(1.2).
- ii If $r > 1$, then Eq.(1.2) has the positive equilibrium $\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$.
- iii If $r < 1$ and $\frac{1}{p+q}$ is an even positive integer, then Eq.(1.2) has the positive equilibrium $\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$ which is always in the interval $(0, 1)$.

*Proof.* The proof is easily obtained from the definition of equilibrium point. □

In the following theorems, we investigate the local asymptotic behavior of the equilibria and the global behavior of solutions of Eq.(1.2) with $r, p, q > 0$ and positive initial conditions.

**Theorem 3.2.** *For the local asymptotic stability of equilibria of Eq.*(1.2)*, we obtain the following results;*
*(i) If* $r < 1$*, then the zero equilibrium point* $\overline{y}_0$ *is locally asymptotically stable.*
*(ii) If* $r > 1$*, then the zero equilibrium point is locally unstable.*
*(iii) If* $r = 1$*, then the zero equilibrium point is non-hyperbolic point.*
*(iv) If* $r > 1$ *and* $k \in 2\mathbb{Z}^+$*, then the positive equilibrium point* $\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$ *is locally unstable.*
*(v) If* $r \in (0,1)$ *and* $\frac{1}{p+q}$ *is an even positive integer, then the positive equilibrium point* $\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$ *is locally unstable.*

*Proof.* The linearized equation associated with Eq.(1.2) about zero equilibrium is

$$z_{n+1} - rz_{n-(k+1)} = 0,\ n \in \mathbb{N}.$$

The characteristic polynomial of Eq.(1.2) about zero equilibrium is

$$\lambda^{k+3} - r\lambda = 0.$$

So, the proof of $(i)$, $(ii)$ and $(iii)$ follows immediately from Linearized Stability Theorem.
For the proof $(iv)$ suppose that $r > 1$, then the linearized equation associated with Eq.(1.2) about $\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$ is

$$z_{n+1} + p(1 - \frac{1}{r})z_{n-k} - z_{n-(k+1)} + q(1 - \frac{1}{r})z_{n-(k+2)} = 0,\ n \in \mathbb{N}.$$

Therefore, the characteristic polynomial of Eq.(1.2) about the equilibrium $\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$ is

$$\lambda^{k+3} + p(1 - \frac{1}{r})\lambda^2 - \lambda + q(1 - \frac{1}{r}) = 0.$$

If we set the function as follows;

$$h(\lambda) = \lambda^{k+3} + p(1 - \frac{1}{r})\lambda^2 - \lambda + q(1 - \frac{1}{r}) = 0,$$

then, it is clear that

$$h(-1) = \frac{(p+q)(r-1)}{r} > 0$$

and

$$\lim_{\lambda \to -\infty} h(\lambda) = -\infty,$$

so, $h(\lambda)$ has at least a root in the interval $(-\infty, -1)$. This completes the proof.
For the proof $(v)$ we assume that $r < 1$, then the linearized equation associated with Eq.(1.2) about $\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$ is

$$t_{n+1} + p(1 - \frac{1}{r})t_{n-k} - t_{n-(k+1)} + q(1 - \frac{1}{r})t_{n-(k+2)} = 0,\ n \in \mathbb{N}.$$

Therefore, the characteristic polynomial of Eq.(1.2) about the equilibrium $\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$ is

$$\lambda^{k+3} + p(1-\frac{1}{r})\lambda^2 - \lambda + q(1-\frac{1}{r}) = 0.$$

If we set the function as follows;

$$g(\lambda) = \lambda^{k+3} + p(1-\frac{1}{r})\lambda^2 - \lambda + q(1-\frac{1}{r}) = 0,$$

then, it is clear that

$$g(1) = \frac{(p+q)(r-1)}{r} < 0$$

and

$$\lim_{\lambda\to\infty} g(\lambda) = \infty,$$

so, $g(\lambda)$ has at least a root in the interval $(1,\infty)$. This completes the proof. □

**Theorem 3.3.** *Assume that* $r < 1$*, then the zero equilibrium point* $\overline{y}_0$ *of Eq.*(1.2) *is globally asymptotically stable.*

*Proof.* We know by Theorem 3.2($i$) that the zero equilibrium point of Eq.(1.2) is locally asymptotically stable, hence, it suffices to show that

$$\lim_{n\to\infty} y_n = 0$$

for any positive solution $\{y_n\}_{n=-(k+2)}^{\infty}$ of Eq.(1.2).
From Eq.(1.2), we have the following inequlity for all $n \geq 0$;

$$0 \leq y_{n+1} = \frac{r y_{n-(k+1)}}{1 + y_{n-k}^{p} y_{n-(k+2)}^{q}} \leq r y_{n-(k+1)}.$$

In this sense, we have the following inequlities for $i \in \{0,1,2,...\}$;

$$\begin{aligned} y_{i(k+2)+1} &\leq r^{i+1} y_{-k-1}, \\ y_{i(k+2)+2} &\leq r^{i+1} y_{-k}, \\ &\vdots \\ y_{i(k+2)+k+2} &\leq r^{i+1} y_0. \end{aligned}$$

Since $r < 1$, we have

$$\lim_{i\to\infty} r^{i+1} = 0,$$

hence, we obtain that

$$\lim_{n\to\infty} y_n = 0.$$

Thus, the proof is complete. □

**Theorem 3.4.** *Assume* $k$ *is an even positive integer, then, Eq.*(1.2) *possesses eventual prime period two solutions if and only if* $r = 1$.

*Proof.* Let

$$..., \Phi, \Psi, \Phi, \Psi, \Phi, \Psi, ...$$

a period two solution of Eq.(1.2). Then, we eventually have

$$\Phi = \frac{r\Phi}{1 + \Psi^{p+q}} \text{ and } \Psi = \frac{r\Psi}{1 + \Phi^{p+q}}. \tag{3.1}$$

such that $\Phi \neq \Psi$. If both $\Phi$ and $\Psi$ are non-zero, then we obtain from (3.1) that $\Phi = \Psi = (r-1)^{\frac{1}{p+q}}$, which is a contradiction. Hence, either $\Phi$ or $\Psi$ must be equal to zero. Assume that $\Phi = 0$ which implies that $(r-1)\Psi = 0$, so $r = 1$.
In contrast, if $r = 1$, then choose the initial conditions such as $y_{-(k+2)} = y_{-k} = \ldots = y_0 = 0$ and $y_{-(k+1)} = y_{-(k-1)} = y_{-1} = \delta > 0$ or such as $y_{-(k+2)} = y_{-k} = \ldots = y_0 = \delta > 0$ and $y_{-(k+1)} = y_{-(k-1)} = y_{-1} = 0$. We can see by induction that

$$\ldots, 0, \delta, 0, \delta, \ldots$$

is the prime period two solution of Eq.(1.2). □

**Theorem 3.5.** *Assume that $k$ is an odd positive integer, then, Eq.(1.2) has no eventual prime period two solutions.*

*Proof.* Assume that Eq.(1.2) has the prime period two solution

$$..., x, y, x, y, ...$$

then, we eventually have

$$x = \frac{ry}{1 + x^{p+q}} \text{ and } y = \frac{rx}{1 + y^{p+q}}$$

such that $x \neq y$. Obviously, $x = 0$ implies $y = 0$ or vice versa. This case is impossible. So we obtain that both $x$ and $y$ are greater than zero. So we have

$$\begin{aligned} x^{p+q}(1 + x^{p+q})^{p+q+1} + r^{p+q}(1 - r^2 + x^{p+q}) &= 0, \\ y^{p+q}(1 + y^{p+q})^{p+q+1} + r^{p+q}(1 - r^2 + y^{p+q}) &= 0, \end{aligned}$$

that is, $x$ and $y$ are two distinct positive roots of $f(z) = z^{p+q}(1 + z^{p+q})^{p+q+1} + r^{p+q}(1 - r^2 + z^{p+q}) = 0$. Obviously, when $r \leq 1$, the $f(z)$ has no positive roots. Now, let $r > 1$ and set $1 + z^{p+q} = w$. Then the function, $g(w) = w^{p+q+2} - w^{p+q+1} + wr^{p+q} - r^{p+q+2}$, $w > 1$, has at least two distinct positive roots. However, $g'(w) = w^{p+q}\left[(p+q+2)w - (p+q+1)\right] + r^{p+q} > 0$ for any $w \in (1, \infty)$, which indicates that $g(w)$ is strictly increasing in the interval $(1, \infty)$. This implies that the function $g(w)$ does not have two distinct positive roots at all in the interval $(1, \infty)$. Thus, Eq.(1.2) does not have the prime period two solution when $r > 1$. This completes the proof. □

For the oscillatory solution of Eq.(1.2) , we have the following results.

**Theorem 3.6.** *Suppose that $r > 1$, $k \in 2\mathbb{Z}^+$ and let $\{y_n\}_{n=-(k+2)}^{\infty}$ be any solution of Eq.(1.2) such that*

$$y_{-(k+2)}, y_{-k}, \ldots, y_{-2}, y_0 \geq \overline{y}_1 \text{ and } y_{-(k+1)}, y_{-(k-1)}, \ldots, y_{-3}, y_{-1} < \overline{y}_1 \quad (3.2)$$

*or*

$$y_{-(k+2)}, y_{-k}, \ldots, y_{-2}, y_0 < \overline{y}_1 \text{ and } y_{-(k+1)}, y_{-(k-1)}, \ldots, y_{-3}, y_{-1} \geq \overline{y}_1 \quad (3.3)$$

*holds. Then, $\{y_n\}_{n=-(k+2)}^{\infty}$ oscillates about the positive equilibrium point $\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$ with semicycles of length one.*

*Proof.* Suppose that $r > 1$ and the case (3.2) holds for the solution $\{y_n\}_{n=-(k+2)}^{\infty}$. From Eq.(1.2), it is clear that

$$y_1 = \frac{ry_{-(k+1)}}{1 + y_{-k}^p y_{-(k+2)}^q} < \overline{y}_1 = (r-1)^{\frac{1}{p+q}},$$

$$y_2 = \frac{ry_{-k}}{1 + y_{-(k-1)}^p y_{-(k+1)}^q} > \overline{y}_1 = (r-1)^{\frac{1}{p+q}},$$

$$\ldots$$

$$y_{k+1} = \frac{ry_{-1}}{1 + y_0^p y_{-2}^q} < \overline{y}_1 = (r-1)^{\frac{1}{p+q}}, \; y_{k+2} = \frac{ry_0}{1 + y_1^p y_{-1}^q} > \overline{y}_1 = (r-1)^{\frac{1}{p+q}}.$$

So, the proof follows by induction. For the case (3.3) the proof is similar and will be omitted. □

**Corollary 3.7.** *Suppose that $r < 1, k \in 2\mathbb{Z}^+, \frac{1}{p+q} \in 2\mathbb{Z}^+$ and let $\{y_n\}_{n=-(k+2)}^{\infty}$ be any solution of Eq.(1.2) such that*

$$y_{-(k+2)}, y_{-k}, \ldots, y_{-2}, y_0 \geq \overline{y}_2 \text{ and } y_{-(k+1)}, y_{-(k-1)}, \ldots, y_{-3}, y_{-1} < \overline{y}_2$$

*or*

$$y_{-(k+2)}, y_{-k}, \ldots, y_{-2}, y_0 < \overline{y}_2 \text{ and } y_{-(k+1)}, y_{-(k-1)}, \ldots, y_{-3}, y_{-1} \geq \overline{y}_2$$

*holds. Then, $\{y_n\}_{n=-(k+2)}^{\infty}$ oscillates about the positive equilibrium point $\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$ with semicycles of length one.*

In respect of the unbounded solutions of Eq.(1.2), the following result is reproduced.

**Theorem 3.8.** *Assume $r > 1$, $k \in 2\mathbb{Z}^+$, then Eq.(1.2) possesses unbounded solutions. Especially, every solution of Eq.(1.2) which oscillates about the positive equilibrium point $\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$ with semicycles of length one is unbounded.*

*Proof.* From Theorem 3.6, we can assume without loss of generality that the solution $\{y_n\}_{n=-(k+2)}^{\infty}$ of Eq.(1.2) is such that

$$y_{2n+1} > \overline{y}_1 = (r-1)^{\frac{1}{p+q}} \text{ and } y_{2n+2} < \overline{y}_1 = (r-1)^{\frac{1}{p+q}}, \text{ for all } n \geq 0.$$

Then,

$$y_{2n+2} = \frac{ry_{2n-k}}{1 + y^p_{2n-k+1}y^q_{2n-k-1}} < \overline{y}_1 = (r-1)^{\frac{1}{p+q}}$$

and

$$y_{2n+3} = \frac{ry_{2n-k+1}}{1 + y^p_{2n-k+2}y^q_{2n-k}} > \overline{y}_1 = (r-1)^{\frac{1}{p+q}}.$$

Thus, we obtain that

$$\lim_{n\to\infty} y_{2n+1} = \infty$$

and

$$\lim_{n\to\infty} y_{2n+2} = 0$$

which completes the proof. □

**Corollary 3.9.** *Assume $r < 1$, $\frac{1}{p+q} \in 2\mathbb{Z}^+$ and $k \in 2\mathbb{Z}^+$, then Eq.(1.2) possesses unbounded solutions. Especially, every solution of Eq.(1.2) which oscillates about the positive equilibrium point $\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$ with semicycles of length one is unbounded.*

**Open Problem** Investigate dynamical behavior of Eq.(1.1) where the parameters $\alpha, \beta, \gamma, p, q$ are non-negative numbers, $k$ is an odd number and the initial values $x_{-(k+2)}$, $x_{-(k+1)}$, ..., $x_{-1}$, $x_0$ are non-negative numbers.

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