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COMMON FIXED POINT THEOREMS FOR $\mathcal{L}$-FUZZY MAPPINGS IN b-METRIC SPACES

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Abstract. In this paper, we prove common fixed point theorems for $\mathcal{L}$-fuzzy mappings under implicit relation in b-metric spaces. Further, results obtained for an integral type contractive condition. These theorems generalize and improve previous corresponding results.

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1. Introduction and preliminaries

In 1981, Heilpern [10] introduced the concept of fuzzy mapping and proved fixed point theorem for fuzzy contractive mappings in metric linear spaces as a generalization of Nadler [12] contraction principle. In 1967, Goguen [9] introduced the notion of $\mathcal{L}$-Fuzzy sets as a generalization of fuzzy sets. Recently, Rashid et al. [19] established the existence of common $\mathcal{L}$-fuzzy fixed point in complete metric spaces.

As a generalization of metric spaces, Bakhtin [1] introduced the concept of b-metric spaces and Czerwik [6, 7] used this concept to give some generalizations Banach's fixed point theorem.

In this paper, we define the notion of $\mathcal{L}$-fuzzy sets in b-metric spaces. Also, we prove common fixed point theorems for $\mathcal{L}$-fuzzy mappings under implicit relation in b-metric spaces. The object of our paper is to reduce the completeness of the whole space by completeness of subspace (joint orbitally complete) in b-metric spaces. Our results generalize and improve corresponding results of [3, 4, 14, 19] and others.

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Definition 1.1 ([1]). Let X be a nonempty set. A mapping $d : X \times X \to [0, \infty)$ is called b-metric if there exists a real number $b \geq 1$ such that for every $x, y, z \in X$, we have:

(d_1) $d(x, y) = 0 \Leftrightarrow x = y$,
(d_2) $d(x, y) = d(y, x)$,
(d_4) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

Put $b = 1$ in above definition then b-metric spaces give metric spaces.

Example 1.2 ([17]). Let $X = \{a, b, c\}$ and define $d(a, b) = d(b, a) = d(b, c) = d(c, b) = 1$ and $d(a, c) = d(c, a) = m \geq 2$, then

$$d(a, b) = \frac{m}{2}[d(a, c) + d(c, b)]$$

for all $a, b, c \in X$. If $m < 2$, the ordinary triangle inequality does not hold.

Definition 1.3 ([9]). A partially ordered set $(L, \leq_L, \vee, \wedge)$ is called

(I) a lattice, if $a \vee b \in L$ and $a \wedge b \in L$ for any $a, b \in L$,
(II) a complete lattice, if $\vee A \in L$ and $\wedge A \in L$ for any $A \subseteq L$,
(III) distributive if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$.

Definition 1.4 ([9]). An $\mathcal{L}$-fuzzy set A on a nonempty set X is a function $A : X \to L$, where L is complete distributive lattice with $1_{\mathcal{L}}$ and $0_{\mathcal{L}}$. In $\mathcal{L}$-fuzzy sets if $L = [0, 1]$, then we obtained fuzzy sets.

The $\alpha_{\mathcal{L}}$-level set of $\mathcal{L}$-fuzzy set A is denoted by $A_{\alpha_{\mathcal{L}}}$ and is defined as follows

$$A_{\alpha_{\mathcal{L}}} = \{x : \alpha_{\mathcal{L}} \preceq_L A(x)\} \quad \text{if} \quad \alpha_{\mathcal{L}} \in L \backslash \{0_{\mathcal{L}}\}, \quad A_{0_{\mathcal{L}}} = \overline{\{x : 0_{\mathcal{L}} \preceq_L A(x)\}},$$

where $\overline{B}$ denotes the closure of the set B. The characteristic function $\chi_{\mathcal{L}_A}$ of an $\mathcal{L}$-fuzzy set A as follows

$$\chi_{\mathcal{L}_A}(x) = \begin{cases} 0_{\mathcal{L}}, & \text{if} \quad x \notin A, \\ 1_{\mathcal{L}}, & \text{if} \quad x \in A. \end{cases}$$

Definition 1.5 ([19]). Let X be an arbitrary set and Y a metric space. A mapping T is called an $\mathcal{L}$-fuzzy mapping if T is a mapping from X into $\Im_{\mathcal{L}}(Y)$. An $\mathcal{L}$-fuzzy mapping T is an $\mathcal{L}$-fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$.

Definition 1.6 ([19]). Let (X, d) be a metric space and T_1, T_2 are $\mathcal{L}$-fuzzy mappings from X into $\Im_{\mathcal{L}}(Y)$. A point $z \in X$ is called an $\mathcal{L}$-fuzzy fixed point of T_1 if $z \in \{T_1 z\}_{\alpha_{\mathcal{L}}}$, where $\alpha_{\mathcal{L}} \in L \backslash \{0_{\mathcal{L}}\}$. The point $z \in X$ is called a common $\mathcal{L}$-fuzzy fixed point of T_1 and T_2 if $z \in \{T_1 z\}_{\alpha_{\mathcal{L}}} \cap \{T_2 z\}_{\alpha_{\mathcal{L}}}$.

Definition 1.7 ([5]). Let (X, d) be a b-metric space. A sequence $\{x_n\}$ in X is called:

(I) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.

(II) Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

A b-metric space is said to be complete if and only if each Cauchy sequence in this space is convergent.

Let (X, d) be a b-metric space, denote $CP(X)$ the collection of nonempty compact subsets of X and by $CL(X)$ the class of all nonempty closed subsets of X. For $x \in X$ and $A, B \in CL(X)$, we define $d(x, A) = \inf\{d(x, a) : a \in A\}$, $\delta(A, B) = \sup\{d(a, B) : a \in A\}$. Then the generalized Hausdorff b-metric H on $CL(X)$ inducted by d is defined as $H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$ for all $A, B \in CL(X)$.

Lemma 1.8 ([12]). *If $A, B \in CP(X)$ and $a \in A$, then for each $\epsilon > 0$, there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$.*

Lemma 1.9 ([7]). *Let (X, d) be a b-metric space, $A, B \in CL(X)$, then $d(a, B) \leq H(A, B)$ for all $a \in A$.*

Definition 1.10 ([18]). Let I, J be two mappings from a metric space X into itself and T_1, T_2 be fuzzy mappings from X into $W(X)$ (The set of all fuzzy sets of X which its α-level sets are nonempty compact subsets of X). If for some $x_0 \in X$, there exist $\{y_n\}$ in X such that

$$\{y_{2n+1}\} = \{Jx_{2n+1}\} \subset T_1x_{2n}, \quad \{y_{2n+2}\} = \{Ix_{2n+2}\} \subset T_2x_{2n+1}.$$

then $O(T_1, T_2, I, J, x_0)$ is called the orbit for the mappings (T_1, T_2, I, J)

Definition 1.11 ([18]). A metric space X is called x_0 joint orbitally complete, if every Cauchy sequence of each orbit at x_0 is convergent in X.

Now, one can introduce the following definition.

Definition 1.12. Let I, J be two mappings from a b-metric space X into itself and T_1, T_2 be $\mathcal{L}$-fuzzy mappings from X into $\Im_{\mathcal{L}}(X)$. If for some $x_0 \in X$, there exist $\{y_n\}$ in X such that

$$y_{2n+1} = Jx_{2n+1} \in \{T_1x_{2n}\}_{\alpha_{\mathcal{L}}}, \quad y_{2n+2} = Ix_{2n+2} \in \{T_2x_{2n+1}\}_{\alpha_{\mathcal{L}}}.$$

Then $O(T_1, T_2, I, J, x_0)$ is called the orbit for the mappings (T_1, T_2, I, J). b-metric space X is called x_0 joint orbitally complete, if every Cauchy sequence of each orbit at x_0 is convergent in X.

Definition 1.13 ([18]). Let I be a mapping from a nonempty subset M of a metric space (X, d) into itself and T be fuzzy mappings from M into $W(M)$. A hybrid pair(I, T) is called D-compatible iff $\{It\} \subset Tt$ for some t in M implies $ITt = TIt$.

We can also define the following in setting of $\mathcal{L}$-fuzzy sets.

Definition 1.14. Let (X, d) be a b-metric space. The mappings $I : X \to X$ and $T : X \to \Im_{\mathcal{L}}(X)$ are called D-compatible iff $It \in \{Tt\}_{\alpha_{\mathcal{L}}}$ for some t in X implies $I\{Tt\}_{\alpha_{\mathcal{L}}} \subset \{TIt\}_{\alpha_{\mathcal{L}}}$.

Popa [16] (cf.[11]) utilized the idea of implicit function to unify the fixed point theorems. Imdad and Ali [11] employed this idea in fuzzy metric spaces. Now, we define the following class of implicit functions as follows:

Let Ψ be the family of all continuous mappings $F : [0,\infty)^6 \to [0,\infty)$ satisfying the following properties:

(Ψ_1) F is non-decreasing in the 1^{st} variable and non-increasing in the 3^{rd}, 4^{th}, 5^{th}, 6^{th} coordinate variables,
(Ψ_{21}) there exists $h \in (0,1)$ such that for every $u, v \geq 0$, $b \geq 1$ with $F(u, v, v, u, b(u+v), 0) \leq 0$ or
(Ψ_{22}) $F(u, v, u, v, 0, b(u+v)) \leq 0$ implies $u \leq hv$.
(Ψ_3) $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

Example 1.15. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - h\max\{t_2, (t_3 + t_4)t_5t_6\}$.

Example 1.16. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - h\min\{t_2, (t_3 + t_4), (t_5 + t_6)\}$.

Example 1.17. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - h\min\{t_2, b(t_3 + t_4), (t_5 + t_6)\}$.

Example 1.18. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - ht_2$.

Example 1.19. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - h\min\{t_2, (t_3 + t_4), \frac{(t_5+t_6)}{b}\}$.

2. Main results

Theorem 2.1. *Let I, J be two self mappings from a b-metric space (X, d) into itself and T_1, T_2 are $\mathcal{L}$-fuzzy mappings from X into $\Im_{\mathcal{L}}(X)$ such that $\{T_1x\}_{\alpha_{\mathcal{L}}}$ and $\{T_2x\}_{\alpha_{\mathcal{L}}}$ are nonempty closed subsets of X for all $x \in X$ and*

1. *$\{T_1(X)\}_{\alpha_{\mathcal{L}}} \subset J(X)$, $\{T_2(X)\}_{\alpha_{\mathcal{L}}} \subset I(X)$,*
2. *the pairs (T_1, I) and (T_2, J) are D-compatible mappings,*
3. *$I(X)$ is x_0 joint orbitally complete for some $x_0 \in X$.*

If there is a $F \in \Psi$ such that for all $x, y \in X$,

$$F\begin{pmatrix} H(\{T_1x\}_{\alpha_{\mathcal{L}}}, \{T_2y\}_{\alpha_{\mathcal{L}}}), d(Ix, Jy), d(Ix, \{T_1x\}_{\alpha_{\mathcal{L}}}), \\ d(Jy, \{T_2y\}_{\alpha_{\mathcal{L}}}), d(Ix, \{T_2y\}_{\alpha_{\mathcal{L}}}), d(Jy, \{T_1x\}_{\alpha_{\mathcal{L}}}) \end{pmatrix} \leq 0, \tag{1}$$

then there exists $z \in X$ such that $z = Iz = Jz$ and $z \in \{T_1z\}_{\alpha_{\mathcal{L}}} \cap \{T_2z\}_{\alpha_{\mathcal{L}}}$.

Proof. Let $x_0 \in X$, there exist $y_1 = Jx_1 \in \{T_1x_0\}_{\alpha_{\mathcal{L}}}$, but $\{T_1x_0\}_{\alpha_{\mathcal{L}}} \in CP(X)$ and $\{T_2x_1\}_{\alpha_{\mathcal{L}}} \in CP(X)$, then there exist $y_2 = Ix_2 \in \{T_2x_1\}_{\alpha_{\mathcal{L}}}$ such that $d(y_1, y_2) \leq H(\{T_1x_0\}_{\alpha_{\mathcal{L}}}, \{T_2x_1\}_{\alpha_{\mathcal{L}}})$. Since

$$\begin{aligned} & F\begin{pmatrix} d(y_1, y_2), d(y_0, y_1), d(y_0, y_1) \\ d(y_1, y_2), b(d(y_0, y_1) + d(y_1, y_2)), 0 \end{pmatrix} \\ \leq \quad & F\begin{pmatrix} H(\{T_1x_0\}_{\alpha_{\mathcal{L}}}, \{T_2x_1\}_{\alpha_{\mathcal{L}}}), d(Ix_0, Jx_1), d(Ix_0, \{T_1x_0\}_{\alpha_{\mathcal{L}}}), \\ d(Jx_1, \{T_2x_1\}_{\alpha_{\mathcal{L}}}), d(Ix_0, \{T_2x_1\}_{\alpha_{\mathcal{L}}}), d(Jx_1, \{T_1x_0\}_{\alpha_{\mathcal{L}}}) \end{pmatrix} \\ \leq \quad & 0. \end{aligned}$$

From the property (Ψ_{21}), there exists $h \in (0,1)$ such that $d(y_1, y_2) \leq hd(y_0, y_1)$. Similarly, one can deduce from the property (Ψ_{22}) that there exists $h \in (0,1)$ such that $d(y_2, y_3) \leq hd(y_1, y_2)$. Then, we have an orbit $O(T_1, T_2, I, J, x_0)$ such that

$$\begin{aligned} y_{2n+1} &= Jx_{2n+1} \in \{T_1 x_{2n}\}_{\alpha_{\mathcal{L}}}, \\ y_{2n+2} &= Ix_{2n+2} \in \{T_2 x_{2n+1}\}_{\alpha_{\mathcal{L}}}. \end{aligned}$$

By induction we obtain $d(y_n, y_{n+1}) \leq h^n d(y_0, y_1)$. Since

$$\begin{aligned} d(y_n, y_m) &\leq bd(y_n, y_{n+1}) + b^2 d(y_{n+1}, y_{n+2}) + ... + b^{m-n-1} d(y_{m-1}, y_m) \\ &\leq bd(y_n, y_{n+1}) + b^2 d(y_{n+1}, y_{n+2}) + ... + b^{m-n} d(y_{m-1}, y_m) \\ &\leq bh^n d(y_0, y_1)t + b^2 h^{n+1} d(y_0, y_1) + ... + b^{m-n} h^{m-1} d(y_0, y_1) \\ &= \frac{bh^n}{1-bh} d(y_0, y_1). \end{aligned}$$

Therefore $\lim_{n,m\to\infty} d(y_n, y_m) = 0$. Hence $\{y_n\}$ is a Cauchy sequence. As $\{y_{2n+2}\}$ is a Cauchy sequence in $I(X)$, and $I(X)$ is joint orbitally complete, therefore there exists $z \in X$ such that $y_{2n+2} \to z = Iu$, for some $u \in X$. Next, we show that $z \in \{T_1 u\}_{\alpha_L}$. Since

$$\begin{aligned} & F\begin{pmatrix} d(y_{2n+2}, \{T_1 u\}_{\alpha_{\mathcal{L}}}), d(z, y_{2n+1}), d(z, \{T_1 u\}_{\alpha_{\mathcal{L}}}), \\ d(y_{2n+1}, y_{2n+2}), d(z, y_{2n+2}), bd(y_{2n+1}, \{T_1 u\}_{\alpha_{\mathcal{L}}}) \end{pmatrix} \\ \leq \quad & F\begin{pmatrix} H(\{T_1 u\}_{\alpha_{\mathcal{L}}}, \{T_2 x_{2n+1}\}_{\alpha_{\mathcal{L}}}), d(Iu, Jx_{2n+1}), d(Iu, \{T_1 u\}_{\alpha_{\mathcal{L}}}), \\ d(Jx_{2n+1}, \{T_2 x_{2n+1}\}_{\alpha_{\mathcal{L}}}), d(Iu, \{T_2 x_{2n+1}\}_{\alpha_{\mathcal{L}}}), d(Jx_{2n+1}, \{T_1 u\}_{\alpha_{\mathcal{L}}}) \end{pmatrix} \\ \leq \quad & 0. \end{aligned}$$

As $n \to \infty$

$$F(d(z, \{T_1 u\}_{\alpha_{\mathcal{L}}}), 0, d(z, \{T_1 u\}_{\alpha_{\mathcal{L}}}), 0, 0, bd(z, \{T_1 u\}_{\alpha_{\mathcal{L}}})) \leq 0$$

By (Ψ_{22}), we have $d(z, \{T_1 u\}_{\alpha_{\mathcal{L}}}) \leq h.0 = 0$. Thus $z \in \{T_1 u\}_{\alpha_{\mathcal{L}}}$. As $z = Iu \in \{T_1 u\}_{\alpha_{\mathcal{L}}} \in J(X)$ therefore there exists $\upsilon \in X$ such that $z = J\upsilon$. Similarly $z = J\upsilon \in \{T_2 \upsilon\}_{\alpha_{\mathcal{L}}}$. Since the pair (T_1, I) are D-compatible and $z = Iu \in \{T_1 u\}_{\alpha_{\mathcal{L}}}$ therefore $Iz = IIu \in \{IT_1 u\}_{\alpha_{\mathcal{L}}} \in \{T_1 Iu\}_{\alpha_{\mathcal{L}}} = \{T_1 z\}_{\alpha_{\mathcal{L}}}$. Also $Jz = JJ\upsilon \in \{JT_2 \upsilon\}_{\alpha_{\mathcal{L}}} \in \{T_2 J\upsilon\}_{\alpha_{\mathcal{L}}} = \{T_2 z\}_{\alpha_{\mathcal{L}}}$. Next, we show that $z = Iz$. If not, then suppose $d(z, Iz) > 0$, then

$$\begin{aligned} & F\begin{pmatrix} d(z, Iz), d(z, Iz), d(z, Iz), \\ d(z, Iz), d(z, Iz), d(z, Iz) \end{pmatrix} \\ \leq \quad & F\begin{pmatrix} H(\{T_1 z\}_{\alpha_{\mathcal{L}}}, \{T_2 \upsilon\}_{\alpha_{\mathcal{L}}}), d(Iz, J\upsilon), d(Iz, \{T_1 z\}_{\alpha_{\mathcal{L}}}), \\ d(J\upsilon, \{T_2 \upsilon\}_{\alpha_{\mathcal{L}}}), d(Iz, \{T_2 \upsilon\}_{\alpha_{\mathcal{L}}}), d(J\upsilon, \{T_1 z\}_{\alpha_{\mathcal{L}}}) \end{pmatrix} \\ \leq \quad & 0. \end{aligned}$$

It further implies

$$F(d(z,Iz),d(z,Iz),0,0,d(z,Iz),d(z,Iz))\leq 0.$$

It contradicts (Ψ_3). Thus $d(z,Iz)=0$. Therefore $z=Iz\in\{T_1z\}_{\alpha_{\mathcal{L}}}$. Similarly $z=Jz\in\{T_2z\}_{\alpha_{\mathcal{L}}}$. Hence $z\in\{T_1z\}_{\alpha_{\mathcal{L}}}\cap\{T_2z\}_{\alpha_{\mathcal{L}}}$. □

Example 2.2. Let (X,d) be a b-metric space with $b=2$, $X=[0,1]$, $d(x,y)=|x-y|^2$ and $L=[0,1]$. Define the maps I,J,T_1,T_2 on X as $Ix=\frac{2x}{3}$, $Jx=\frac{x}{4}$ for all $x,y\in X$. Define also

$$(T_1x)(y)=\begin{cases}0, & \text{if} \quad 0\leq y\leq \frac{1}{5},\\ \frac{1}{3}, & \text{if} \quad \frac{1}{5}<y<\frac{2x}{3},\\ \frac{2}{3}, & \text{if} \quad \frac{2x}{3}\leq y\leq 1.\end{cases}$$

and

$$(T_2x)(y)=\begin{cases}0, & \text{if} \quad 0\leq y\leq \frac{1}{4},\\ \frac{1}{6}, & \text{if} \quad \frac{1}{4}<y<\frac{x}{4},\\ \frac{1}{4}, & \text{if} \quad \frac{x}{4}\leq y\leq 1.\end{cases}$$

Now for $\alpha=\frac{2}{3}$, $I\{T_1x\}_{\frac{2}{3}}=[\frac{4x}{9},\frac{2}{3}]\subset[\frac{4x}{9},1]=\{T_1Ix\}_{\frac{2}{3}}$ and for $\alpha=\frac{1}{4}$, $J\{T_2x\}_{\frac{1}{4}}=[\frac{x}{16},\frac{1}{4}]\subset[\frac{x}{16},1]=\{T_2Jx\}_{\frac{1}{4}}$. i.e., (I,T_1) and (J,T_2) are D-compatible. Further, let $F(t_1,...,t_6)=t_3=d(Ix,\{T_1x\}_\alpha)=0$. Then $0=I0=J0\subset[0,\frac{1}{5}]\cap[0,\frac{1}{4}]=\{T_10\}_\alpha\cap\{T_20\}_\alpha$ is a common fixed point.

Remark 2.1.
(1) Theorem 2.1 is a generalization of Theorem 2.2 [4], Theorem 14 [19] and Theorem 3.1 [14].
(2) If put $J=I$ and $T_1=T_2=T$ in Theorem 2.1, we obtain an orbit (x_0,I,T) of x_0 for I and T, in this case we have the following results:

Corollary 2.3. *Let I be a self mapping from a b-metric space (X,d) into itself and T be an $\mathcal{L}$-fuzzy mapping from X into $\Im_{\mathcal{L}}(X)$ such that $\{Tx\}_{\alpha_{\mathcal{L}}}$ nonempty closed subsets of X for all $x\in X$, $\{T(X)\}_{\alpha_{\mathcal{L}}}\subset I(X)$, the pair (T,I) is D-compatible mappings, and $I(X)$ is x_0 joint orbitally complete for some $x_0\in X$. If there is a $F\in\Psi$ such that for all $x,y\in X$,*

$$F\begin{pmatrix} H(\{Tx\}_{\alpha_{\mathcal{L}}},\{Ty\}_{\alpha_{\mathcal{L}}}),d(Ix,Iy),d(Ix,\{Tx\}_{\alpha_{\mathcal{L}}}),\\ d(Iy,\{Ty\}_{\alpha_{\mathcal{L}}}),d(Ix,\{Ty\}_{\alpha_{\mathcal{L}}}),d(Iy,\{Tx\}_{\alpha_{\mathcal{L}}})\end{pmatrix}\leq 0,$$

then there exists $z\in X$ such that $z=Iz\in\{Tz\}_{\alpha_{\mathcal{L}}}$.

Corollary 2.4. *Let (X,d) be a b-metric space, $I:X\to X$ and T be a fuzzy mapping from X into $\Im(X)$ such that $\{Tx\}_\alpha$ is nonempty closed subsets of X for all $x\in X$, $\{T(X)\}_\alpha\subset I(X)$, the pair (T,I) is D-compatible mappings and*

$I(X)$ is x_0 joint orbitally complete for some $x_0 \in X$. If there is a $F \in \Psi$ such that for all $x, y \in X$,

$$F\left(\begin{array}{c} H(\{Tx\}_\alpha, \{Ty\}_\alpha), d(Ix, Iy), d(Ix, \{Tx\}_\alpha), \\ d(Iy, \{Ty\}_\alpha), d(Ix, \{Ty\}_\alpha), d(Iy, \{Tx\}_\alpha) \end{array}\right) \leq 0,$$

then there exists $z \in X$ such that $z = Iz \in \{Tz\}_\alpha$.

If we put $I = J = i_d$(i_d :=the identity mapping on (X, d)) in Theorem 2.1 we have the following Corollary which generalize Theorem 2.6 [3].

Corollary 2.5. *Let T_1, T_2 be $\mathcal{L}$-fuzzy mappings from a complete b-metric space (X, d) into $\Im_\mathcal{L}(X)$. If there is a $F \in \Psi$ such that for all $x, y \in X$,*

$$F\left(\begin{array}{c} H(\{T_1x\}_{\alpha_\mathcal{L}}, \{T_2y\}_{\alpha_\mathcal{L}}), d(x, y), d(x, \{T_1x\}_{\alpha_\mathcal{L}}), \\ d(y, \{T_2y\}_{\alpha_\mathcal{L}}), d(x, \{T_2y\}_{\alpha_\mathcal{L}}), d(y, \{T_1x\}_{\alpha_\mathcal{L}}) \end{array}\right) \leq 0,$$

then there exists $z \in X$ such that $z \in \{T_1z\}_{\alpha_\mathcal{L}} \cap \{T_2z\}_{\alpha_\mathcal{L}}$.

Corollary 2.6. *Let T be an $\mathcal{L}$-fuzzy mapping from a complete b-metric space (X, d) into $\Im_\mathcal{L}(X)$. If there is a $F \in \Psi$ such that for all $x, y \in X$,*

$$F\left(\begin{array}{c} H(\{Tx\}_{\alpha_\mathcal{L}}, \{Ty\}_{\alpha_\mathcal{L}}), d(x, y), d(x, \{Tx\}_{\alpha_\mathcal{L}}), \\ d(y, \{Ty\}_{\alpha_\mathcal{L}}), d(x, \{Ty\}_{\alpha_\mathcal{L}}), d(y, \{Tx\}_{\alpha_\mathcal{L}}) \end{array}\right) \leq 0,$$

then there exists $z \in X$ such that $z \in \{Tz\}_{\alpha_\mathcal{L}}$.

Remark 2.2. Corollary 2.6 is a generalization of Theorem 3.1 [14] and Theorem 1 [15].

Now, we state the following result for family of $\mathcal{L}$-fuzzy mappings in b-metric spaces.

Theorem 2.7. *Let I, J be two self mappings from a b-metric space (X, d) into itself and $\{T_n\}_{\alpha_\mathcal{L}}, n \in \mathbb{N}$ be $\mathcal{L}$-fuzzy mappings from X into $\Im_\mathcal{L}(X)$ such that*

(1) *$\{T_i(X)\}_{\alpha_\mathcal{L}} \subset J(X)$, $\{T_j(X)\}_{\alpha_\mathcal{L}} \subset I(X)$, $i = 2n, j = 2n + 1$*
(2) *the pairs (T_i, I) and (T_j, J) are D-compatible mappings,*
(3) *$I(X)$ is x_0 joint orbitally complete for some $x_0 \in X$.*

If there is a $F \in \Psi$ such that for all $x, y \in X$,

$$F\left(\begin{array}{c} H(\{T_ix\}_{\alpha_\mathcal{L}}, \{T_jy\}_{\alpha_\mathcal{L}}), d(Ix, Jy), d(Ix, \{T_ix\}_{\alpha_\mathcal{L}}), \\ d(Jy, \{T_jy\}_{\alpha_\mathcal{L}}), d(Ix, \{T_jy\}_{\alpha_\mathcal{L}}), d(Jy, \{T_ix\}_{\alpha_\mathcal{L}}) \end{array}\right) \leq 0,$$

then there exists $z \in X$ such that $z = Iz = Jz$ and $z \in \cap_{n=0}^{\infty}\{T_nz\}_{\alpha_\mathcal{L}}$.

Proof. Proof of this theorem is similar to Theorem 2.1. Therefore, proof skipped. □

Remark 2.3. Theorem 2.7 is a generalization of [4, Theorem 2.3] and [18, Theorem 1].

Remark 2.4. In view of Examples 1.15-1.19, one can derive several new fixed point results.

3. Results with integral type contraction

In 2002, Branciari [2] defined an integral type contraction and obtained a generalization of Banach contraction principle. Some results on fixed point theorems of integral type contraction have appeared (see, e.g. [8, 13]). In this section, we prove a fixed point result for integral type contractive condition with implicit relation for two pairs of $\mathcal{L}$-fuzzy and non self mappings in b-metric spaces.

Let $\hat{\Psi}$ be the family of all continuous mappings $F : [0,\infty)^6 \to [0,\infty)$ satisfying the following properties:

($\hat{\Psi}_1$) F is non-decreasing in the 1^{st} variable and non-increasing in the 3^{rd}, 4^{th}, 5^{th}, 6^{th} coordinate variables,

($\hat{\Psi}_{21}$) there exists $h \in (0,1)$ such that for every $u, v \geq 0$, $b \geq 1$ with $\int_0^{F(u,v,v,u,b(u+v),0)} \varphi(s)ds \leq 0$ or

($\hat{\Psi}_{22}$) $\int_0^{F(u,v,u,v,0,b(u+v))} \varphi(s)ds \leq 0$ implies $u \leq hv$.

($\hat{\Psi}_3$) $\int_0^{F(u,u,0,0,u,u)} \varphi(s)ds > 0$ for all $u > 0$.

where $\varphi : [0,\infty) \to [0,\infty)$ is a summable non negative Lebesgue integrable function such that for each $\epsilon \in [0,1]$, $\int_0^{\epsilon} \varphi(s)ds \geq 0$. Note that if $\varphi(s) = 1$, then $\hat{\Psi} \Rightarrow \Psi$.

Example 3.1. $\int_0^{F(t_1,t_2,t_3,t_4,t_5,t_6)} \varphi(s)ds = \int_0^{t_1-ht_2} \varphi(s)ds.$

Theorem 3.2. *In Theorem 2.1, if we replace the inequality (2.1) with the following: there is a $F \in \hat{\Psi}$ such that for all $x, y \in X$*

$$\int_0^{F\begin{pmatrix} H(\{T_1x\}_{\alpha_\mathcal{L}}, \{T_2y\}_{\alpha_\mathcal{L}}), d(Ix,Jy), d(Ix,\{T_1x\}_{\alpha_\mathcal{L}}), \\ d(Jy,\{T_2y\}_{\alpha_\mathcal{L}}), d(Ix,\{T_2y\}_{\alpha_\mathcal{L}}), d(Jy,\{T_1x\}_{\alpha_\mathcal{L}}) \end{pmatrix}} \varphi(s)ds \leq 0. \qquad (2)$$

then consequences of Theorem 2.1 remain true.

Remark 3.1. Define $\varphi(s) = 1$ in Theorem 3.2, then Theorem 3.2 implies Theorem 2.1.

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