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BEHAVIOR OF POSITIVE SOLUTIONS OF A DIFFERENCE EQUATION

D.T. TOLLU, Y. YAZLIK* AND N. TAŞKARA

ABSTRACT. In this paper we deal with the difference equation

$$y_{n+1} = \frac{ay_{n-1}}{by_n y_{n-1} + cy_{n-1}y_{n-2} + d},\ n \in \mathbb{N}_0,$$

where the coefficients a, b, c, d are positive real numbers and the initial conditions y_{-2}, y_{-1}, y_0 are nonnegative real numbers. Here, we investigate global asymptotic stability, periodicity, boundedness and oscillation of positive solutions of the above equation.

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1. Introduction

Rational difference equations which is an important class of nonlinear difference equations arise in many branches of science. Therefore, these equations have been widely studied by mathematicians for the last decade. For example, in [5], Cinar gave forms of the solutions of the rational difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}},\ n \in \mathbb{N}_0,$$

with the nonnegative initial conditions x_{-1}, x_0. In [2], Andruch-Sobilo et al. investigated the behavior of the solutions of the rational difference equation

$$x_{n+1} = \frac{ax_{n-1}}{b + cx_n x_{n-1}},\ n \in \mathbb{N}_0,$$

with the positive real parameters a, b, c and the nonnegative initial conditions x_{-1}, x_0. Shojaei et al. [18] investigated the stability and periodic character of

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*Corresponding author.
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the rational third-order difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_n x_{n-1} x_{n-2}},\ n \in \mathbb{N}_0,$$

where the parameters α, β, γ and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers. Dehghan and Rastegar [6] investigated the stability, the periodic character and the boundedness nature of solutions of the third order difference equation

$$y_{n+1} = \frac{\alpha y_{n-1}}{\beta + \gamma y_n^k y_{n-1}^k y_{n-2}^k},\ n \in \mathbb{N}_0,$$

where the initial conditions y_{-2}, y_{-1}, y_0 and the parameters α, β and γ are positive real numbers and $k \geq 2$ is a fixed integer. For some related studies, see [1, 3, 8, 9, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 24, 25, 26, 22, 23, 27, 28, 29, 30, 31, 33, 32, 34, 35, 36, 37, 38].

In this study we consider the rational difference equation

$$y_{n+1} = \frac{a y_{n-1}}{b y_n y_{n-1} + c y_{n-1} y_{n-2} + d},\ n \in \mathbb{N}_0, \tag{1}$$

where the coefficients a, b, c, d are positive real numbers and the initial conditions y_{-2}, y_{-1}, y_0 are nonnegative numbers. We investigate global asymptotic stability, periodicity, boundedness and oscillation of positive solutions of Eq. (1). It is easy to see that the change of variables $y_n = \sqrt{\frac{d}{c}} x_n$ reduces Eq. (1) to the next equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta x_n x_{n-1} + x_{n-1} x_{n-2} + 1},\ n \in \mathbb{N}_0, \tag{2}$$

with $\alpha = \frac{a}{d}$, $\beta = \frac{b}{c}$ and the initial conditions x_{-2}, x_{-1}, x_0. Hence, from now on, we will consider Eq. (2).

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Then, for every set of initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f\left(x_n, x_{n-1}, \ldots, x_{n-k}\right),\ n \in \mathbb{N}_0, \tag{3}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.1. An equilibrium point for Eq. (3) is a point $\overline{x} \in I$ such that $\overline{x} = f\left(\overline{x}, \overline{x}, \ldots, \overline{x}\right)$.

Definition 1.2. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 1.3 (Stability)**.** Let $\overline{x}$ be an equilibrium point of Eq. (3).

(i) The equilibrium point $\overline{x}$ of Eq. (3) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x_{-k}, $x_{-(k+1)}, \ldots, x_0 \in I$ with $|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \delta$, we have $|x_n - \overline{x}| < \varepsilon$ for all $n \geq -k$.

(ii) The equilibrium point $\overline{x}$ of Eq. (3) is locally asymptotically stable if $\overline{x}$ is a locally stable and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$ with $|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \gamma$, we have $\lim\limits_{n\to\infty} x_n = \overline{x}$.

(iii) The equilibrium point $\overline{x}$ of Eq. (3) is a global attractor if for all x_{-k}, x_{-k+1}, $\ldots, x_0 \in I$, we have $\lim\limits_{n\to\infty} x_n = \overline{x}$.

(iv) The equilibrium point $\overline{x}$ of Eq. (3) is global asymptotically stable if $\overline{x}$ is locally stable and $\overline{x}$ is also a global attractor of Eq. (3).

(v) The equilibrium point $\overline{x}$ of Eq. (3) is unstable if $\overline{x}$ is not locally stable.

The lineralized equation associated with Eq. (3) is the equation

$$z_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_i} (\overline{x}, \overline{x}, \ldots, \overline{x})\, z_{n-i}, \quad n \in \mathbb{N}_0. \tag{4}$$

The characteristic equation associated with Eq. (3) is the equation

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_i} (\overline{x}, \overline{x}, \ldots, \overline{x})\, \lambda^{k-i} = 0. \tag{5}$$

Theorem 1.4 ([2], Linearized Stability Theorem)**.** *Assume that f is a function in C^1 and let $\overline{x}$ be an equilibrium point of Eq.* (3)*. Then the following statements are true:*

(i) If all roots of Eq.(5) *lie in the open disk $|\lambda| < 1$, then $\overline{x}$ is locally asyptotically stable.*

(ii) If at least one root of Eq.(5) *has absolute value greater than one, then $\overline{x}$ is unstable.*

Definition 1.5. An equilibrium point $\overline{x}$ of Eq. (3) is called a hyperbolic equilibrium point if Eq.(5) has no roots with absolute value equal to one. An equilibrium point $\overline{x}$ of Eq. (3) is called a nonhyperbolic equilibrium point if Eq.(5) has at least one root with absolute value equal to one.

Theorem 1.6 ([1], Theorem 1.6.3)**.** *Assume that the following conditions hold: Let $[a, b]$ be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(i) $f \in C\,[(0, \infty) \times (0, \infty), (0, \infty)]$.

(ii) $f(x, y)$ is decreasing in x and strictly decreasing in y.

(iii) $xf(x, x)$ is strictly increasing in x.

(iv) The equation

$$x_{n+1} = x_n f(x_n, x_{n-1}), \; n \in \mathbb{N}_0, \tag{6}$$

has a unique positive equilibrium $\overline{x}$. Then $\overline{x}$ is a global attractor of all positive solutions of Eq.(6)*.*

2. Stability of Eq. (2)

In this section we describe local and global behaviors of the solutions of Eq. (2). For the equilibrium points of Eq. (2), we write the equation

$$\overline{x} = \frac{\alpha\overline{x}}{(\beta+1)\,\overline{x}^2+1},$$

from which it follows that

$$\overline{x}\left((\beta+1)\,\overline{x}^2+1-\alpha\right)=0.$$

From the last equality, it is easily seen that when $\alpha \le 1$, the unique equilibrium point of Eq. (2) is $\overline{x}_1 = 0$ and when $\alpha > 1$, Eq. (2) has three equilibrium points such that $\overline{x}_1 = 0$, $\overline{x}_2 = \sqrt{\frac{\alpha-1}{\beta+1}}$ and $\overline{x}_3 = -\sqrt{\frac{\alpha-1}{\beta+1}}$.

Let

$$f : [0,\infty)^3 \longrightarrow [0,\infty)$$

be a function defined by

$$f(x,y,z) = \frac{\alpha y}{\beta xy + yz + 1}. \tag{7}$$

From (7), the partial derivatives of $f(x,y,z)$ evaluated at an equilibrium $\overline{x}$ of Eq. (2) are

$$f_x(\overline{x},\overline{x},\overline{x}) = \frac{-\alpha\beta\overline{x}^2}{\left((\beta+1)\,\overline{x}^2+1\right)^2}, \tag{8}$$

$$f_y(\overline{x},\overline{x},\overline{x}) = \frac{\alpha}{\left((\beta+1)\,\overline{x}^2+1\right)^2}, \tag{9}$$

$$f_z(\overline{x},\overline{x},\overline{x}) = \frac{-\alpha\overline{x}^2}{\left((\beta+1)\,\overline{x}^2+1\right)^2}. \tag{10}$$

By the following theorem, we determine local behavior of the solutions of Eq. (2).

Theorem 2.1. *The following statements are true:*

(i) When $\alpha < 1$ the unique equilibrium point $\overline{x}_1 = 0$ of Eq. (2) *is locally asymptotically stable.*

(ii) When $\alpha > 1$ the equilibrium point $\overline{x}_1 = 0$ of Eq. (2) *is unstable.*

(iii) When $\alpha > 1$ the positive equilibrium point $\overline{x}_2 = \sqrt{\frac{\alpha-1}{\beta+1}}$ of Eq. (2) *is unstable.*

(iv) When $\alpha > 1$ the positive equilibrium point $\overline{x}_2 = \sqrt{\frac{\alpha-1}{\beta+1}}$ of Eq. (2) *is nonhyperbolic.*

Proof. (i)-(ii) By using (8)-(10), we get the lineralized equation associated with Eq. (2) about the equilibrium point $\overline{x}_1 = 0$ as

$$z_{n+1} = \alpha z_{n-1},\ n \in \mathbb{N}_0. \tag{11}$$

The characteristic equation of Eq. (11) is

$$\lambda^3 - \alpha\lambda = 0$$

with the roots $\lambda_1 = 0$, $\lambda_2 = \sqrt{\alpha}$ and $\lambda_3 = -\sqrt{\alpha}$. It is clear that if $\alpha < 1$ then $|\lambda_i| < 1$ for $i = 1, 2, 3$ and if $\alpha \geq 1$ then $|\lambda_2| = |\lambda_3| \geq 1$.

(iii)-(iv) By using (8)-(10), we can write the lineralized equation associated with Eq. (2) about the equilibrium point $\overline{x}_2 = \sqrt{\frac{\alpha-1}{\beta+1}}$ as follows:

$$z_{n+1} + \frac{\beta(\alpha-1)}{\alpha(\beta+1)} z_n - \frac{1}{\alpha} z_{n-1} + \frac{(\alpha-1)}{\alpha(\beta+1)} z_{n-2} = 0, \ n \in \mathbb{N}_0,$$

which has the characteristic equation

$$P(\lambda) := \lambda^3 + \frac{\beta(\alpha-1)}{\alpha(\beta+1)} \lambda^2 - \frac{1}{\alpha}\lambda + \frac{(\alpha-1)}{\alpha(\beta+1)} = 0. \tag{12}$$

From Eq. (12), it is easily seen that $P(-1) = 0$, that is, $|\lambda_i| = 1$ for at least i, $(i = 1, 2, 3)$. So, the positive equilibrium point $\overline{x}_2 = \sqrt{\frac{\alpha-1}{\beta+1}}$ of Eq. (2) is unstable. Also, since $|\lambda_i| = 1$ for at least i, $(i = 1, 2, 3)$, the positive equilibrium point $\overline{x}_2 = \sqrt{\frac{\alpha-1}{\beta+1}}$ of Eq. (2) is nonhyperbolic. □

Theorem 2.2. *Assume that $\alpha < 1$. Then, the unique equilibrium point $\overline{x}_1 = 0$ of Eq. (2) is globally asymptotically stable.*

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq. (2). From Theorem 7, the equilibrium point $\overline{x}_1 = 0$ of Eq. (2) is locally asymptotically stable, when $\alpha < 1$. Hence, It is sufficient to show that

$$\lim_{n\to\infty} x_n = \overline{x} = 0.$$

From Eq. (2), we write

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta x_n x_{n-1} + x_{n-1}x_{n-2} + 1} \leq \alpha x_{n-1} \tag{13}$$

for $n \in \mathbb{N}_0$. From the inequality in (13), we get the inequalities

$$x_{2n+1} \leq \alpha x_{2n-1} \leq \alpha^2 x_{2n-3} \leq \cdots \leq \alpha^{n+1} x_{-1} \tag{14}$$

and

$$x_{2n+2} \leq \alpha x_{2n} \leq \alpha^2 x_{2n-2} \leq \cdots \leq \alpha^{n+1} x_0 \tag{15}$$

for $n \in \mathbb{N}_0$. From which (14) and (15) follows that

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} x_{2n+2} = 0,$$

if $\alpha < 1$. So, the proof is completed. □

3. Unbounded Solutions of Eq. (2)

In this section we show that Eq. (2) has the unbounded solutions.

Theorem 3.1. *Assume that* $\alpha > 1$ *and let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of Eq.* (2)*. Then, the following statements are true:*

(i) If $x_{-1} = 0$ *and* $x_{-2}x_0 \neq 0$ *(or if* $x_{-2} = x_{-1} = 0$ *and* $x_0 \neq 0$*), then* $x_{2n-1} = 0$ *and* $x_{2n} \to \infty$ *as* $n \to \infty$*.*

(ii) If $x_0 = 0$ *and* $x_{-2}x_{-1} \neq 0$ *(or if* $x_{-2} = x_0 = 0$ *and* $x_{-1} \neq 0$*), then* $x_{2n-1} \to \infty$ *as* $n \to \infty$ *and* $x_{2n} = 0$*.*

Proof. First, from Eq. (2), we write

$$x_{2n+1} = \frac{\alpha x_{2n-1}}{\beta x_{2n}x_{2n-1} + x_{2n-1}x_{2n-2} + 1}, \ n \in \mathbb{N}_0, \tag{16}$$

and

$$x_{2n+2} = \frac{\alpha x_{2n}}{\beta x_{2n+1}x_{2n} + x_{2n}x_{2n-1} + 1}, \ n \in \mathbb{N}_0. \tag{17}$$

It is clear that if $x_{-2} = x_{-1} = x_0 = 0$ then $x_{2n} = x_{2n+1} = 0$ which is the trivial solution $x_n = 0$ of Eq. (2) for $n \in \mathbb{N}_0$.

(i) If $x_{-1} = 0$, then we get that

$$x_{2n-1} = 0, \ n \in \mathbb{N}_0 \tag{18}$$

from Eq. (16). By substituting (18) in Eq. (17), we have the equation

$$x_{2n+2} = \alpha x_{2n}, \ n \in \mathbb{N}_0$$

from which it follows that

$$x_{2n} = \alpha^n x_0, \ n \in \mathbb{N}_0. \tag{19}$$

The result follows from (19) for $\alpha > 1$.

(ii) If $x_0 = 0$, then we get

$$x_{2n-2} = 0, \ n \in \mathbb{N} \tag{20}$$

from Eq. (17). By substituting (20) in Eq. (16), we have the equation

$$x_{2n+1} = \alpha x_{2n-1}, \ n \in \mathbb{N}$$

from which it follows that

$$x_{2n-1} = \alpha^{n-1} x_1, \ n \in \mathbb{N}. \tag{21}$$

We also observe that

$$x_1 = \frac{\alpha x_{-1}}{x_{-1}x_{-2} + 1}. \tag{22}$$

Consequently, from Eq. (21) and Eq. (22), we get

$$x_{2n-1} = \frac{\alpha^n x_{-1}}{x_{-1}x_{-2} + 1}, \ x_{2n} = 0, \ n \in \mathbb{N}. \tag{23}$$

The result follows from (23) for $\alpha > 1$. □

4. Periodic Solutions of Eq. (2)

In this section we show that Eq. (2) has the periodic solutions. The following theorem studies period 2 solutions of Eq. (2) under the condition $\alpha = 1$.

Theorem 4.1. *Assume that $\alpha = 1$. Then, the following statements are true:*
(i) Eq. (2) has the prime period 2 solutions in the form of

$$\dots, p, q, p, q, \dots.$$

(ii) Every solution of Eq. (2) converges to a period 2 solution of Eq. (2).

Proof. (i) Let

$$\dots, p, q, p, q, \dots$$

be a period 2 solution of Eq. (2) with $p \neq q$. Then, we write

$$p = \frac{\alpha p}{(\beta + 1)\, pq + 1},\ q = \frac{\alpha q}{(\beta + 1)\, pq + 1},$$

from which it follows that

$$(p - q)\left((\beta + 1)\, pq + 1 - \alpha\right) = 0. \tag{24}$$

The equality in (24) implies that if $\alpha = 1$, then $pq = 0$. So, there is a period 2 solution either in the form of

$$\dots, p, 0, p, 0, \dots$$

or in the form of

$$\dots, 0, q, 0, q, \dots.$$

(ii) Assume that $\alpha = 1$ and let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq. (2). Then, for $n \in \mathbb{N}_0$, we have that

$$x_{n+1} - x_{n-1} = \frac{-\beta x_n x_{n-1}^2 - x_{n-1}^2 x_{n-2}}{\beta x_n x_{n-1} + x_{n-1} x_{n-2} + 1} \leq 0.$$

from which it follows that

$$x_{2n+1} \leq x_{2n-1},\ x_{2n+2} \leq x_{2n}. \tag{25}$$

The inequalities in (25) imply that while the even-subscript terms of the solution $\{x_n\}_{n=-2}^{\infty}$ decreasingly converge one of the periodic points of the solution, the odd-subscript terms decreasingly converge another. □

The following theorem studies period 2 solutions of Eq. (2) under the condition $\alpha > 1$.

Theorem 4.2. *Assume that $\alpha > 1$ and $x_{-2}x_{-1} = x_{-1}x_0 = \frac{\alpha-1}{\beta+1}$. Then, Eq. (2) has the periodic solutions with period 2.*

Proof. First, we note that if $x_{-2}x_{-1} = x_{-1}x_0 = \frac{\alpha-1}{\beta+1}$, then $x_{-2}x_{-1}x_0 \neq 0$, and so $x_n > 0$ for $n \in \mathbb{N}_0$. Thus, we can multiply both sides of Eq. (2) by x_n as follows:

$$x_{n+1}x_n = \frac{\alpha x_n x_{n-1}}{\beta x_n x_{n-1} + x_{n-1}x_{n-2} + 1}, \ n \in \mathbb{N}_0. \tag{26}$$

By applying the change of variables

$$x_n x_{n-1} = u_n \tag{27}$$

to Eq. (26), we get the equation

$$u_{n+1} = \frac{\alpha u_n}{\beta u_n + u_{n-1} + 1}, \ n \in \mathbb{N}_0. \tag{28}$$

On the other hand, the change of variables (27) yields

$$x_n = \frac{u_n}{u_{n-1}} x_{n-2}, \ n \in \mathbb{N}_0. \tag{29}$$

Obviously, the equilibrium solutions of Eq. (28) satisfies Eq. (29). If $\alpha > 1$, then Eq. (28) has the zero equilibrium point $\overline{u}_1 = 0$ and the positive equilibrium point $\overline{u}_2 = \frac{\alpha-1}{\beta+1}$.

The equilibrium solution $u_n = 0$ is a singular case for Eq. (29). In this case, from the assumption $x_n x_{n-1} = 0$ and Eq. (2), we get that

$$x_{n+1} = \alpha x_{n-1}, \ n \in \mathbb{N}_0. \tag{30}$$

It is clear that Eq. (30) does not have any periodic solution for $\alpha > 1$.

The equilibrium solution $u_n = \frac{\alpha-1}{\beta+1}$ satisfies Eq. (29). In this case, from the assumption $x_n x_{n-1} = \frac{\alpha-1}{\beta+1}$ and Eq. (2), we get that

$$x_{n+1} = x_{n-1}, \ n \in \mathbb{N}_0, \tag{31}$$

which implies that Eq. (2) has the periodic solutions with period 2. More precisely, if $\alpha > 1$ and $x_{-2}x_{-1} = x_{-1}x_0 = \frac{\alpha-1}{\beta+1}$, then the form of the period 2 solution is

$$\ldots, \frac{\alpha-1}{(\beta+1)\,p}, p, \frac{\alpha-1}{(\beta+1)\,p}, p, \ldots.$$

So, the proof is completed. □

We need the next lemmas.

Lemma 4.3. *The positive equilibrium point* $\overline{u}_2 = \frac{\alpha-1}{\beta+1}$ *of Eq.* (28) *is global attractor.*

Proof. Let

$$g : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty)$$

be a function defined by

$$g(u,v) = \frac{\alpha}{\beta u + v + 1}. \tag{32}$$

First, from (32), we evaluate the partial derivatives

$$g_u(u,v) = \frac{-\alpha\beta}{(\beta u + v + 1)^2} < 0, \ g_v(u,v) = \frac{-\alpha}{(\beta u + v + 1)^2} < 0. \tag{33}$$

Therefore, $g(u,v)$ is is decreasing in x and strictly decreasing in y. Second, we consider the function

$$h(u) = ug(u,u) = \frac{\alpha u}{(\beta+1)u+1}. \tag{34}$$

From (34), we have

$$h'(u) = \frac{\alpha}{((\beta+1)u+1)^2} > 0,$$

which shows that the function $h(u) = ug(u,u)$ is strictly increasing in u. So, the conditions of Theorem 1.6 hold. Consequently, we have that

$$\lim_{n\to\infty} u_n = \frac{\alpha-1}{\beta+1}, \tag{35}$$

which completes the proof. □

Lemma 4.4. *$f(x,y,z)$ is monotonically decreasing in x, $z \in [0,\infty)$ for each $y \in [0,\infty)$, and $f(x,y,z)$ is monotonically increasing in $y \in [0,\infty)$ for each x, $z \in [0,\infty)$.*

Proof. The result can be seen from the partial derivatives of $f(x,y,z)$ in (8)-(10). □

Theorem 4.5. *Assume that $\alpha > 1$ and $x_{-1}x_0 \neq 0$. Then, every non-equilibrium solution of Eq. (2) converges to a period 2 solution of Eq. (2).*

Proof. Assume that $\alpha > 1$ and $x_{-1}x_0 \neq 0$. Here, it is insignificant whether $x_{-2} = 0$ or $x_{-2} \neq 0$. Because, if $x_{-1}x_0 \neq 0$ and $x_{-2} = 0$, then we have by using Eq. (2) that $x_1 = \frac{\alpha x_{-1}}{\beta x_0 x_{-1}+1} \neq 0$. So, we may start with nonzero initial values x_{-2}, x_{-1}, x_0 which causes $x_n > 0$.

The limit in (35) along with (27) implies that

$$\lim_{n\to\infty} x_{n+1}x_n = \frac{\alpha-1}{\beta+1}. \tag{36}$$

Let

$$0 < x_{n+1} < \sqrt{\frac{\alpha-1}{\beta+1}} \tag{37}$$

for $n \in \mathbb{N}_0$. From (37) and Lemma 4.4, we get that

$$\lim_{n\to\infty} x_{n+1} = l \tag{38}$$

such that $0 < l < \sqrt{\frac{\alpha-1}{\beta+1}}$. By using the limit in (38), we again write (36) as follows:

$$\lim_{n\to\infty} x_{n+1} \lim_{n\to\infty} x_n = l \lim_{n\to\infty} x_n = \frac{\alpha-1}{\beta+1}. \tag{39}$$

From (39), it follows that

$$L = \lim_{n\to\infty} x_n = \frac{\alpha - 1}{(\beta + 1)\, l}. \tag{40}$$

So, the proof is completed. □

Corollary 4.6. *Assume that $\alpha > 1$ and $x_{-1}x_0 \neq 0$. Then, every solution of Eq. (2) is bounded.*

Proof. The result follows from (38) and (40). □

Corollary 4.7. *Assume that $\alpha > 1$ and $x_{-1}x_0 \neq 0$. Then, every non-equilibrium solution of Eq. (2) eventually oscillates about $\overline{x}_2 = \sqrt{\frac{\alpha-1}{\beta+1}}$ with a semicycle of length one.*

Proof. Due to the assumption (37), there exists the inequality $0 < l < \sqrt{\frac{\alpha-1}{\beta+1}}$. So, from (40), we get the inequality

$$L = \frac{\alpha - 1}{(\beta + 1)\, l} > \sqrt{\frac{\alpha - 1}{\beta + 1}},$$

which shows that every solution of Eq. (2) eventually oscillates about $\overline{x}_2 = \sqrt{\frac{\alpha-1}{\beta+1}}$ with a semicycle of length one. □

5. Numerical Examples

In this section, we give numerical examples for the positive solutions of Eq. (2) in the cases $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$.

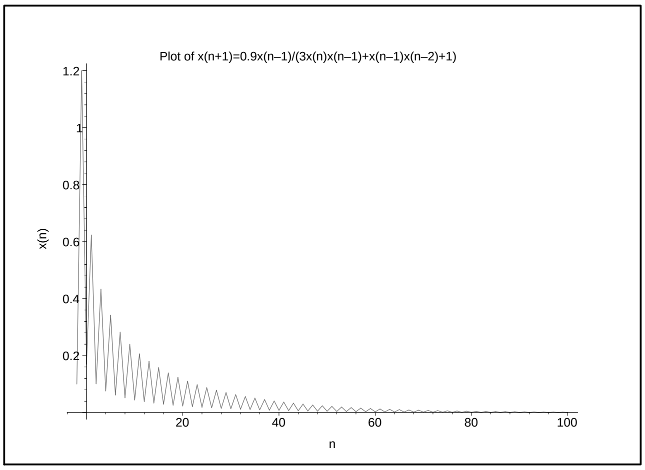

FIGURE 1. If we take $\alpha = 0.9$, $\beta = 3$, $x_{-2} = 0.1$, $x_{-1} = 1.2$, $x_0 = 0.17$, then the solution of Eq. (2) is as follows:

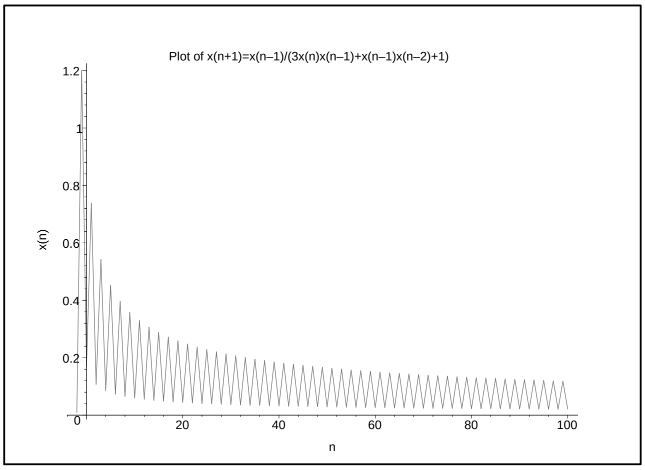

FIGURE 2. If we take $\alpha = 1$, $\beta = 3$, $x_{-2} = 0.01$, $x_{-1} = 1.2$, $x_0 = 0.17$, then the solution of Eq. (2) is as follows:

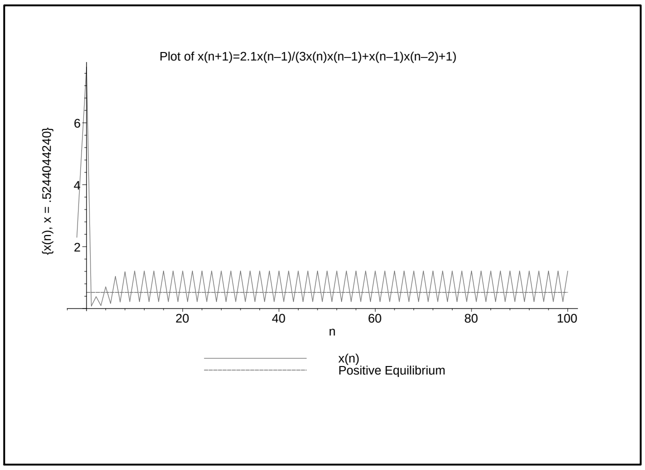

FIGURE 3. If we take $\alpha = 2.1$, $\beta = 3$, $x_{-2} = 2.3$, $x_{-1} = 5.1$, $x_0 = 7.8$, then the solution of Eq. (2) is as follows:

6. Conclusion

We summarize our results related to the positive solutions of Eq. (2) in the following table:

Cases	Results
$x_{-1} = x_0 = 0$	There is the trivial solution $x_n = 0$ for $n \in \mathbb{N}_0$.
$\alpha < 1$ and $x_{-1}x_0 \neq 0$	The equilibrium point $\overline{x}_1 = 0$ of Eq. (2) is globally asymptotically stable.
$\alpha \geq 1$ and $x_{-2}x_{-1}x_0 \neq 0$	Eq. (2) has the prime period 2 solutions.
$\alpha \geq 1$ and $x_{-1}x_0 \neq 0$	Eq. (2) has the prime period 2 solutions. Also, every non-equilibrium solution of Eq. (2) converges to a period 2 solution of the equation.
$x_{-1} = 0$ and $x_0 \neq 0$	There is a unbounded solution such that $x_{2n-1} = 0$ and $x_{2n} \to \infty$ as $n \to \infty$.
$x_{-1} \neq 0$ and $x_0 = 0$	There is a unbounded solution such that $x_{2n-1} \to \infty$ as $n \to \infty$ and $x_{2n} = 0$.

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D.T. Tollu received his M.Sc. and Ph.D at Selçuk University. He is currently an Assistant Professor at Necmettin Erbakan University. His research interests include rational difference equations.

Department of Mathematics–Computer Sciences, Faculty of Sciences, Necmettin Erbakan University Meram Campus , 42090 Meram, Konya, Turkey.
e-mail: dttollu@konya.edu.tr

Y. Yazlik received his M.Sc. and Ph.D at Selçuk University. He is currently an Associate Professor at Nevşehir Hacı Bektaş Veli University. His research interests are rational difference equations and number sequences.

Department of Mathematics, Faculty of Science and Art, Nevşehir Hacı Bektaş Veli University, 50300, Nevsehir, Turkey.
e-mail: yyazlik@nevsehir.edu.tr

N. Taşkara received his M.Sc. and Ph.D. at Selçuk University. He is currently an Associate Professor at Selçuk University. His research interests are rational difference equations and number sequences.

Department of Mathematics, Faculty of Science, Selçuk University, Campus, 42075, Konya, Turkey.
e-mail: ntaskara@selcuk.edu.tr