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NOTE ON ABSTRACT STOCHASTIC SEMILINEAR EVOLUTION EQUATIONS

Tôn Việt Tạ

Abstract. This paper is devoted to studying abstract stochastic semilinear evolution equations with additive noise in Hilbert spaces. First, we prove the existence of unique local mild solutions and show their regularity. Second, we show the regular dependence of the solutions on initial data. Finally, some applications to stochastic partial differential equations are presented.

1. Introduction

Many interesting phenomena in the real world can be described by a system of nonlinear parabolic evolution equations. These equations generally not only generate a dynamical system but also a global attractor even a finite-dimensional attractor. Such an attractor then suggests that the phenomena enjoy some robustness in a certain abstract sense. Some may be the pattern formation and others may be the specific structure creation ([9, 10, 12]). In these cases, one of main issues is to study the robustness of the final states of system. It is therefore quite natural in order to investigate the robustness to consider an advanced version of stochastic parabolic evolution equations.

In this paper, we study the Cauchy problem for an abstract stochastic semilinear evolution equation:

$$
\begin{cases} dX + AXdt = [F_1(X) + F_2(t)]dt + G(t)dW(t), \quad 0 < t \le T, \\ X(0) = \xi \end{cases} \tag{1}
$$

in a separable Hilbert space H. Here, $A\colon \mathcal{D}(A) \subset H \to H$ is a sectorial operator. The process W is a cylindrical Wiener process on a separable Hilbert space U, and is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. The function F_1 is measurable from $(\Omega \times H, \mathcal{F}_T \times \mathcal{B}(H))$ into $(H, \mathcal{B}(H))$. Meanwhile, F_2 and G are measurable functions from $([0,T], \mathcal{B}([0,T]))$ into $(H, \mathcal{B}(H))$ and $(L_2(U;H), \mathcal{B}(L_2(U;H)))$, respectively (here, $L_2(U;H)$ denotes the space of all

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Hilbert-Schmidt operators from U to H). The initial value ξ is an H-valued $\mathcal{F}_0$-measurable random variable.

This kind of evolution equations has been investigated by several authors (see [1, 2, 4, 5, 8, 11, 13, 14, 15, 16], and references therein). Under the Lipschitz continuity and linear growth conditions on F_1, Ichikawa [8] and Da Prato-Zabczyk [5] proved the existence of unique global mild solutions in $L_p([0,T];H)$, Neerven-Veraar-Weis [16] showed the existence of unique strong solutions in $L_p([0,T];\mathcal{D}(A))$. In Banach space setting, Brzezniak [2] (see also Tạ-Yagi [13]) showed the existence of maximal local mild solutions. The space-time regularity of solutions to (1) has, however, not been developed well for the case where the domain of F_1 is a subset of H, and F_1does not satisfy the linear growth condition. Such a case occurs very often in many phenomena described by partial differential equations (PDEs) (see e.g., Yagi [18]).

In the present paper, we study the equation (1), where F_1 is defined on a subset of H and satisfies a Lipschitz condition (see (H3) in Section 3). We prove the existence and uniqueness of local mild solutions. We also show the space-time regularity and dependence on initial data of the solutions. Here, the local solutions are constructed on nonrandom intervals. Note that previous results in [2, 13] show that local solutions are defined on random intervals.

For the study, we use the semigroup approach. Let us explain this approach in the deterministic case. Consider the Cauchy problem for a linear evolution equation:

$$\begin{cases} \frac{dX}{dt} + AX = F(t), \qquad 0 < t \leq T, \\ X(0) = X_0. \end{cases}$$

Hille [7] and Yosida [19] invented the semigroup: $S(t) = e^{-tA}$, $0 \leq t < \infty$, generated by the linear operator $(-A)$, which directly provides a fundamental solution to the Cauchy problem:

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(s)ds.$$

Similarly, a solution to the Cauchy problem for a nonlinear evolution equation

$$\begin{cases} \frac{dX}{dt} + AX = F(t,X), \qquad 0 < t \leq T, \\ X(0) = X_0, \end{cases}$$

can be obtained as a solution of an integral equation:

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(s,X(s))ds.$$

By these formulas, one can get important information on solutions such as uniqueness, regularity, smoothing effect and so forth. Especially, for nonlinear problems one can derive Lipschitz continuity of solutions with respect to the initial values, even their Fréchet differentiability.

The organization of this paper is as follows. Section 2 is preliminary. We review some notions such as weighted Hölder continuous function spaces, analytical semigroups generated by sectorial operators, and cylindrical Wiener processes. Section 3 presents our main results. We assume that the function F_1 is defined only on a subset of the space H, $\mathcal{D}(F_1) = \mathcal{D}(A^\eta)$ for some $0 < \eta < 1$, and satisfies a Lipschitz condition (see (H3)). We suppose further that the initial value ξ takes values in a smaller space, namely, $\mathcal{D}(A^\beta)$. Theorem 3.2 gives the existence and uniqueness of local solutions as well as its temporal and spatial regularity. Theorem 3.3 gives the regularity of the expectation of local solutions. Theorem 3.4 shows the regular dependence of solutions on initial data. Finally, Section 4 gives some applications to stochastic PDEs.

2. Preliminary

2.1. Weighted Hölder continuous function spaces

Let us review the notion of weighted Hölder continuous function spaces $\mathcal{F}^{\beta,\sigma}((0,T];H)$ for two exponents $0 < \sigma < \beta < 1$. This kind of spaces is introduced by Yagi [18].

The space $\mathcal{F}^{\beta,\sigma}((0,T];H)$ consists of H-valued continuous functions F on $(0,T]$ with the following three properties:

(i)

$$t^{1-\beta}F(t) \text{ has a limit as } t \to 0. \tag{2}$$

(ii) F is Hölder continuous with exponent σ and weight $s^{1-\beta+\sigma}$, i.e.,

$$\sup_{0\le s<t\le T} \tfrac{s^{1-\beta+\sigma}\|F(t)-F(s)\|}{(t-s)^\sigma} = \sup_{0\le t\le T}\sup_{0\le s<t} \tfrac{s^{1-\beta+\sigma}\|F(t)-F(s)\|}{(t-s)^\sigma} < \infty. \tag{3}$$

(iii)

$$\lim_{t\to 0} w_F(t) = 0, \tag{4}$$

where $w_F(t) = \sup_{0\le s<t} \frac{s^{1-\beta+\sigma}\|F(t)-F(s)\|}{(t-s)^\sigma}$.

It is easily seen that $\mathcal{F}^{\beta,\sigma}((0,T];H)$ is a Banach space with norm

$$\|F\|_{\mathcal{F}^{\beta,\sigma}} = \sup_{0\le t\le T} t^{1-\beta}\|F(t)\| + \sup_{0\le s<t\le T} \tfrac{s^{1-\beta+\sigma}\|F(t)-F(s)\|}{(t-s)^\sigma}.$$

Clearly, for $F \in \mathcal{F}^{\beta,\sigma}((0,T];H)$,

$$\begin{cases} \|F(t)\| \le \|F\|_{\mathcal{F}^{\beta,\sigma}} t^{\beta-1}, \quad 0 < t \le T, \\ \|F(t)-F(s)\| \le w_F(t)(t-s)^\sigma s^{\beta-\sigma-1} \\ \qquad\qquad \le \|F\|_{\mathcal{F}^{\beta,\sigma}}(t-s)^\sigma s^{\beta-\sigma-1}, \quad 0 < s \le t \le T. \end{cases} \tag{5}$$

Remark 2.1. (a) The space $\mathcal{F}^{\beta,\sigma}((0,T];H)$ is not a trivial space. The function F defined by $F(t) = t^{\beta-1}f(t), 0 < t \le T$, belongs to this space, where f is any H-valued function such that $f \in \mathcal{C}^\sigma([0,T];H)$ and $f(0) = 0$.

(b) The space $\mathcal{F}^{\beta,\sigma}((a,b];H)$, $0 \le a < b < \infty$, is defined in a similar way. For more details, see [18].

2.2. Sectorial operators and analytical semigroups

A densely defined, closed linear operator A is said to be sectorial if it satisfies two conditions:

(H1) The spectrum $\sigma(A)$ of A is contained in an open sectorial domain Σ_ϖ:

$$\sigma(A) \subset \Sigma_\varpi = \{\lambda \in \mathbb{C} : |\arg\lambda| < \varpi\}, \qquad 0 < \varpi < \frac{\pi}{2}.$$

(H2) The resolvent of A satisfies the estimate

$$\|(\lambda - A)^{-1}\| \le \frac{M_\varpi}{|\lambda|}, \qquad \lambda \notin \Sigma_\varpi$$

with some constant $M_\varpi > 0$ depending only on the angle ϖ.

Let A be a sectorial operator. The fractional powers $A^\theta, -\infty < \theta < \infty$, are then defined as follows. For each complex number z such that $\operatorname{Re} z > 0$, A^{-z} is defined by using the Dunford integral in $\mathcal{L}(H)$:

$$A^{-z} = \frac{1}{2\pi i}\int_\gamma \lambda^{-z}(\lambda - A)^{-1} d\lambda.$$

Here, $\gamma = \gamma_- \cup \gamma_0 \cup \gamma_+$ is an integral contour surrounding the spectrum $\sigma(A)$ counterclockwise in the domain $\mathbb{C} \setminus (-\infty, 0] \cap \mathbb{C} \setminus \sigma(A)$ of the complex plane:

$$\gamma_\pm \colon \lambda = \rho e^{\pm i\varpi}, \quad \frac{\|A^{-1}\|^{-1}}{2} \le \rho < \infty,$$

and

$$\gamma_0 \colon \lambda = \frac{\|A^{-1}\|^{-1}}{2} e^{i\varphi}, \quad -\varpi \le \varphi \le \varpi.$$

It is known that A^{-z} is one to one for $\operatorname{Re} z > 0$. The following definition is thus meaningful:

$$A^z = (A^{-z})^{-1} \quad \text{for } \operatorname{Re} z > 0.$$

In addition, it is natural to define $A^0 = 1$. In this way, for every real number $-\infty < \theta < \infty$, A^θ has been defined. For more detail on fractional powers, see [18].

The following lemma shows useful estimates for fractional powers and the semigroup generated by a sectorial operator.

Lemma 2.2. *Let* (H1) *and* (H2) *be satisfied. Then,*

(i) $(-A)$ *generates an analytical semigroup*: $S(t) = e^{-tA}$, $0 \le t < \infty$.

(ii) *For* $0 \le \theta < \infty$,

$$\|A^\theta S(t)\| \le \iota_\theta t^{-\theta}, \quad 0 < t < \infty, \tag{6}$$

where $\iota_\theta = \sup_{0\le t<\infty} t^\theta \|A^\theta S(t)\| < \infty$. *In particular, there exists* $\nu > 0$ *such that*

$$\|S(t)\| \le \iota_0 e^{-\nu t} \le \iota_0, \quad 0 \le t < \infty. \tag{7}$$

(iii) *For* $0 < \theta \le 1$,

$$\|[S(t) - I]A^{-\theta}\| \le \frac{\iota_{1-\theta}}{\theta} t^\theta, \quad 0 \le t < \infty. \tag{8}$$

For the proof, see [18].

To end this subsection, let us recall a result presented in [17, 18].

Theorem 2.3. *Let* (H1) *and* (H2) *be satisfied. Let*

$$x \in \mathcal{D}(A^\beta) \ \ \text{and} \ \ F \in \mathcal{F}^{\beta,\sigma}((0,T];H)$$

for some $0 < \sigma < \beta < 1$. *Set*

$$I(t) = S(t)x + \int_0^t S(t-s)F(s)ds, \quad 0 \le t \le T,$$

where $S(\cdot)$ *is the analytical semigroup generated by* $(-A)$. *Then,* I *possesses the properties*:

$$A^\beta I \in \mathcal{C}([0,T];H),$$
$$\frac{dI}{dt}, AI \in \mathcal{F}^{\beta,\sigma}((0,T];H)$$

with the estimate

$$\Big\|\frac{dI}{dt}\Big\|_{\mathcal{F}^{\beta,\sigma}} + \|A^\beta I\|_{\mathcal{C}} + \|AI\|_{\mathcal{F}^{\beta,\sigma}} \le C[\|A^\beta x\| + \|F\|_{\mathcal{F}^{\beta,\sigma}}],$$

where $C > 0$ *is some constant depending only on* β *and* σ.

2.3. Cylindrical Wiener process

Let us review a central notion to the theory of stochastic evolution equations, namely, cylindrical Wiener processes on the Hilbert space U. First, we recall the definition of Q-Wiener processes on Hilbert spaces (see [3]).

Definition 1. An U-valued stochastic process W defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a Q-Wiener process if

- $W(0) = 0$ a.s.,
- W has continuous sample paths,
- W has independent increments,
- The law of $W(t) - W(s), 0 < s \le t$, is a Gaussian measure on U with mean 0 and covariance $(t-s)Q$, where Q is a symmetric nonnegative nuclear operator in $\mathcal{L}(U)$.

Remark 2.4. (i) When U is the real line $\mathbb{R}$, the operator Q is just a positive number q. The Q-Wiener process is then a Brownian motion on $\mathbb{R}$. When $U = \mathbb{R}^n (n = 2, 3, 4, \dots)$, Q is an $n \times n$ positive definite matrix. In this case, the Q-Wiener process is a Brownian motion in $\mathbb{R}^n$.

(ii) The operator Q is not only a bounded linear operator but also a nuclear operator, i.e., its trace is finite:

$$\operatorname{Tr}(Q) = \sum_{i=1}^{\infty} \langle Qe_i, e_i \rangle < \infty,$$

here $\{e_i\}_{i=1}^{\infty}$ is a complete orthonormal basis in U.

Let us now fix a larger Hilbert space U_1 such that U is embedded continuously into U_1 and the embedding $J\colon U \to U_1$ is Hilbert-Schmidt (i.e., $\sum_{i=1}^{\infty} \|Je_i\|_{U_1}^2 < \infty$). For example (see [6]), we take U_1 to be the closure of U under the norm

$$\|h\|_{U_1} = \left[\sum_{n=1}^{\infty} \frac{\langle h, e_n \rangle_U^2}{n^2} \right]^{\frac{1}{2}}.$$

For every $u \in U_1$, we have

$$\begin{aligned}
&\langle JJ^* e_m, u \rangle_{U_1} \\
&= \langle J^* e_m, J^* u \rangle_U \\
&= \langle \sum_{k=1}^{\infty} e_k \langle J^* e_m, e_k \rangle_U, \sum_{k=1}^{\infty} e_k \langle J^* u, e_k \rangle_U \rangle_U \\
&= \sum_{k=1}^{\infty} \langle J^* e_m, e_k \rangle_U \langle J^* u, e_k \rangle_U = \sum_{k=1}^{\infty} \langle e_m, Je_k \rangle_{U_1} \langle u, Je_k \rangle_{U_1} \\
&= \sum_{k=1}^{\infty} \langle e_m, e_k \rangle_{U_1} \langle u, e_k \rangle_{U_1} = \|e_m\|_{U_1}^2 \langle u, e_m \rangle_{U_1} \\
&= \frac{1}{m^2} \langle u, e_m \rangle_{U_1}, \quad m = 1, 2, 3, \dots .
\end{aligned}$$

Therefore, $JJ^* e_m = \frac{1}{m^2} e_m$ for $m = 1, 2, 3, \dots$ As a consequence,

$$\operatorname{Tr}(JJ^*) = \sum_{m=1}^{\infty} \langle JJ^* e_m, e_m \rangle = \sum_{m=1}^{\infty} \frac{1}{m^2} < \infty.$$

Thus, JJ^* is a nuclear operator.

Based on the operator JJ^*, one can define a cylindrical Wiener process. The following definition is taken from [5, 6].

Definition 2. The U_1-valued Wiener process in Definition 1 with covariance $Q = JJ^*$ is called a cylindrical Wiener process on U.

The H-valued stochastic integrals against a cylindrical Wiener process on U is then constructed in the same way as what is usually done in finite dimensions. In [5], the stochastic integrals $\int_0^T \Phi(s) dW(s)$ are constructed for integrand Φ

in $\mathcal{N}^2(0,T;L_2(U;H))$, the space of all $L_2(U;H)$-valued predictable stochastic processes Φ on $[0,T]$ such that

$$\mathbb{E}\int_0^T \|\Phi(s)\|^2_{L_2(U;H)}ds < \infty.$$

It is known that the class $\mathcal{N}^2(0,T;L_2(U;H))$ is independent of the space U_1 chosen. Furthermore, stochastic integrals can be extended to $L_2(U;H)$-valued predictable stochastic processes Φ satisfying a weaker condition:

$$\mathbb{P}\left\{\int_0^T \|\Phi(s)\|^2_{L_2(U;H)}ds < \infty\right\} = 1.$$

The set of all such processes is denoted by $\mathcal{N}(0,T;L_2(U;H))$. The readers can find properties of stochastic integrals against cylindrical Wiener processes in [5]. Those are similar to ones of the usual stochastic integrals.

The following known result is used very often.

Theorem 2.5. *Let $G \in \mathcal{N}^2(0,T;L_2(U;H))$. Let B be a closed linear operator on H such that*

$$\mathbb{E}\int_0^T ||BG(s)||^2_{L_2(U;H)}ds < \infty.$$

Then,

$$B\int_0^T G(t)dW(t) = \int_0^T BG(t)dW(t) \quad a.s.$$

For the proof, see [5].

2.4. Mild solutions

Let us introduce a definition of mild solutions to (1) (see [5, 8]).

Definition 3. Let F_2 and G be H-valued and $L_2(U;H)$-valued functions satisfying the conditions:

$$\int_0^t \|S(t-s)F_2(s)\|ds < \infty, \quad 0 \le t \le T,$$

and

$$\int_0^t \|S(t-s)G(s)\|^2_{L_2(U;H)}ds < \infty, \quad 0 \le t \le T.$$

A predictable H-valued process X on $[0,T]$ is called a mild solution of (1) if

$$\int_0^T \|S(t-s)F_1(X(s))\|ds < \infty \quad \text{a.s.},$$

and

$$X(t) = S(t)\xi + \int_0^t S(t-s)[F_1(X(s)) + F_2(s)]ds$$

$$+\int_0^t S(t-s)G(s)dW(s) \quad \text{a.s., } 0 < t \le T.$$

In order to study the Hölder continuity of solutions, the Kolmogorov test is useful.

Theorem 2.6. *Let ζ be an H-valued stochastic process on $[0,T]$. Assume that for some $c > 0$ and $\epsilon_i > 0\,(i = 1,2)$,*

$$\mathbb{E}\|\zeta(t) - \zeta(s)\|^{\epsilon_1} \le c|t-s|^{1+\epsilon_2}, \quad 0 \le s,t \le T.$$

Then, ζ has a version whose $\mathbb{P}$-almost all trajectories are Hölder continuous functions with an arbitrarily smaller exponent than $\frac{\epsilon_2}{\epsilon_1}$.

For the proof, see e.g., [5].

3. Main results

In this section, we prove the existence and uniqueness of local mild solutions to (1) and show their regularity (Subsection 3.1). We then show regular dependence of solutions on initial data (Subsection 3.2).

Let fix constants η, β, σ such that

$$\begin{cases} 0 < \eta < \frac{1}{2}, \\ \max\{0, 2\eta - \frac{1}{2}\} < \beta < \eta, \\ 0 < \sigma < \beta. \end{cases}$$

Assume that

(H3) $F_1 \colon \mathcal{D}(A^\eta) \subset H \to H$ satisfies a Lipschitz condition of the form

$$\|F_1(x) - F_1(y)\| \le c_{F_1}\|A^\eta(x-y)\| \quad \text{a.s., } x,y \in \mathcal{D}(A^\eta),$$

where c_{F_1} is some positive constant.

(H4) $F_2 \in \mathcal{F}^{\beta,\sigma}((0,T];H)$.

(H5) $G \in B([0,T];L_2(U;H))$.

Here, $B([0,T];L_2(U;H))$ is the space of uniformly bounded $L_2(U;H)$-valued functions on $[0,T]$ with the supremum norm:

$$\|G\|_{B([0,T];L_2(U;H))} = \sup_{0\le t\le T}\|G(t)\|_{L_2(U;H)}.$$

Lemma 3.1. *Let* (H1), (H2), (H3), (H4) *and* (H5) *be satisfied. Then,*

(i) *For $0 \le \theta < 1$,*

$$\int_0^t \|A^\theta S(t-s)F_2(s)\|ds \le \iota_\theta \|F_2\|_{\mathcal{F}^{\beta,\sigma}} B(\beta, 1-\theta)t^{\beta-\theta}, \quad 0 < t \le T,$$

where $B(\cdot,\cdot)$ denotes the beta function. As a consequence,

$$A^\theta \int_0^t S(t-s)F_2(s)ds = \int_0^t A^\theta S(t-s)F_2(s)ds$$

and $A^\theta \int_0^\cdot S(\cdot - s)F_2(s)ds$ *is continuous on* $(0, T]$. *Furthermore, if* $\theta < \beta$, *then* $A^\theta \int_0^\cdot S(\cdot - s)F_2(s)ds$ *is also continuous at* $t = 0$.

(ii) *For* $0 \le \theta < \frac{1}{2}$,

$$\int_0^t \|A^\theta S(t-s)G(s)\|^2_{L_2(U;H)} ds \le \frac{\iota_\theta^2 t^{1-2\theta} \|G\|^2_{B([0,T];L_2(U;H))}}{1-2\theta}, \quad 0 \le t \le T.$$

As a consequence, the stochastic convolution W_G *defined by*

$$W_G(t) = \int_0^t S(t-s)G(s)dW(s), \quad 0 \le t \le T$$

satisfies

$$A^\theta W_G(t) = \int_0^t A^\theta S(t-s)G(s)dW(s) \quad a.s., \ 0 \le t \le T.$$

Furthermore, $A^\theta W_G$ *is continuous on* $[0, T]$.

(iii) *For any* $0 < \gamma < \frac{1+2\beta}{4} - \eta$, W_G *has the regularity*:

$$A^\eta W_G \in \mathcal{C}^\gamma((0, T]; H) \quad a.s.$$

Proof. First, let us prove (i). It follows from (5) and (6) that

$$\begin{aligned}
\int_0^t \|A^\theta S(t-s)F_2(s)\|ds &\le \int_0^t \|A^\theta S(t-s)\|\|F_2(s)\|ds \\
&\le \|F_2\|_{\mathcal{F}^{\beta,\sigma}} \iota_\theta \int_0^t (t-s)^{-\theta} s^{\beta-1} ds \\
&= \|F_2\|_{\mathcal{F}^{\beta,\sigma}} \iota_\theta t^{\beta-\theta} \int_0^1 u^{\beta-1}(1-u)^{-\theta} du \\
&= \iota_\theta \|F_2\|_{\mathcal{F}^{\beta,\sigma}} B(\beta, 1-\theta) t^{\beta-\theta}, \quad 0 < t \le T.
\end{aligned}$$

Hence, $\int_0^\cdot A^\theta S(\cdot - s)F_2(s)ds$ is continuous on $(0, T]$. It is also continuous at $t = 0$ if $\theta < \beta$. Since A^θ is closed, the statements in (i) follow.

Let us next verify (ii). Thanks to (6),

$$\begin{aligned}
\int_0^t \|A^\theta S(t-s)G(s)\|^2_{L_2(U;H)} ds &\le \int_0^t \|A^\theta S(t-s)\|^2 \|G(s)\|^2_{L_2(U;H)} ds \\
&\le \iota_\theta^2 \int_0^t (t-s)^{-2\theta} \|G\|^2_{B((0,T];L_2(U;H))} ds \\
&= \frac{\iota_\theta^2 t^{1-2\theta} \|G\|^2_{B([0,T];L_2(U;H))}}{1-2\theta} < \infty, \quad 0 \le t \le T.
\end{aligned}$$

The process $\int_0^\cdot A^\theta S(\cdot - s)G(s)dW(s)$ is therefore well-defined. The definition of stochastic integrals then provides that this process is a continuous martingale on $[0, T]$. Since A^β is closed, (ii) follows.

The proof for (iii) is similar to one in [5, 11]. So, we omit it. □

3.1. Existence and regularity of solutions

Let us first prove the existence of unique local mild solutions to (1) and show their space-time regularity.

Theorem 3.2. *Let* (H1), (H2), (H3), (H4) *and* (H5) *be satisfied. Let* $\xi \in \mathcal{D}(A^\beta)$ *such that* $\mathbb{E}\|A^\beta \xi\|^2 < \infty$. *Then,* (1) *possesses a unique local mild solution* X *in the function space*:

$$X \in \mathcal{C}([0, T_{loc}]; \mathcal{D}(A^\beta)), \quad A^\eta X \in \mathcal{C}^\gamma((0, T_{loc}]; H) \quad a.s. \tag{9}$$

for any $0 < \gamma < \frac{1+2\beta}{4} - \eta$. *Furthermore,* X *satisfies the estimate*:

$$\mathbb{E}\|A^\beta X(t)\|^2 + t^{2(\eta-\beta)}\mathbb{E}\|A^\eta X(t)\|^2 \le C_{F_1,F_2,G,\xi}, \quad 0 \le t \le T_{loc}. \tag{10}$$

Here, T_{loc} *and* $C_{F_1,F_2,G,\xi}$ *are non-random constants depending on the exponents and* $\mathbb{E}\|F_1(0)\|^2$, $\mathbb{E}\|A^\beta\xi\|^2$, $\|F_2\|^2_{\mathcal{F}^{\beta,\sigma}}$, $\|G\|^2_{B([0,T];L_2(U;H))}$.

Proof. We use the fixed point theorem for contractions to prove the existence and uniqueness of local solutions. For each $0 < S \le T$, set the underlying space:

$$\Xi(S) = \{Y \in \mathcal{C}((0, S]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, S]; \mathcal{D}(A^\beta)) \text{ such that}$$
$$\sup_{0<t\le S} t^{2(\eta-\beta)}\mathbb{E}\|A^\eta Y(t)\|^2 + \sup_{0\le t\le S} \mathbb{E}\|A^\beta Y(t)\|^2 < \infty\}.$$

Up to indistinguishability, $\Xi(S)$ is then a Banach space with norm:

$$\|Y\|_{\Xi(S)} = \Big[\sup_{0<t\le S} t^{2(\eta-\beta)}\mathbb{E}\|A^\eta Y(t)\|^2 + \sup_{0\le t\le S} \mathbb{E}\|A^\beta Y(t)\|^2\Big]^{\frac{1}{2}}. \tag{11}$$

Let fix a constant $\kappa > 0$ such that

$$\frac{\kappa^2}{2} > C_1 \vee C_2, \tag{12}$$

where two constants C_1 and C_2 will be fixed below. Consider a subset $\Upsilon(S)$ of $\Xi(S)$ which consists of functions $Y \in \Xi(S)$ such that

$$\max\left\{\sup_{0<t\le S} t^{2(\eta-\beta)}\mathbb{E}\|A^\eta Y(t)\|^2, \sup_{0\le t\le S} \mathbb{E}\|A^\beta Y(t)\|^2\right\} \le \kappa^2. \tag{13}$$

Obviously, $\Upsilon(S)$ is a nonempty closed subset of $\Xi(S)$.

For $Y \in \Upsilon(S)$, we define a function on $[0, S]$:

$$\Phi Y(t) = S(t)\xi + \int_0^t S(t-s)[F_1(Y(s)) + F_2(s)]ds + \int_0^t S(t-s)G(s)dW(s). \tag{14}$$

Our goal is then to verify that Φ is a contraction mapping from $\Upsilon(S)$ into itself, provided that S is sufficiently small, and that the fixed point of Φ is the desired solution of (1). For this purpose, we divide the proof into four steps.

Step 1. Let us show that $\Phi Y \in \Upsilon(S)$ for $Y \in \Upsilon(S)$.

Let $Y \in \Upsilon(S)$. Due to (H3) and (13), we observe that

$$\begin{aligned}\mathbb{E}\|F_1(Y(t))\|^2 &\le \mathbb{E}[c_{F_1}\|A^\eta Y(t)\| + \|F_1(0)\|]^2\\ &\le 2[c_{F_1}^2\mathbb{E}\|A^\eta Y(t)\|^2 + \mathbb{E}\|F_1(0)\|^2]\\ &\le 2[c_{F_1}^2\kappa^2 t^{2(\beta-\eta)} + \mathbb{E}\|F_1(0)\|^2], \quad 0 < t \le S. \end{aligned} \tag{15}$$

First, we verify that ΦY satisfies (13). For $\beta \le \theta < \frac{1}{2}$, (14) gives

$$\begin{aligned}&t^{2(\theta-\beta)}\mathbb{E}\|A^\theta\{\Phi Y\}(t)\|^2\\ &\le 3t^{2(\theta-\beta)}\mathbb{E}\Big[\|A^\theta S(t)\xi\|^2 + \Big\|\int_0^t A^\theta S(t-s)[F_1(Y(s)) + F_2(s)]ds\Big\|^2\\ &\quad + \Big\|\int_0^t A^\theta S(t-s)G(s)dW(s)\Big\|^2\Big]\\ &\le 3t^{2(\theta-\beta)}\|A^{\theta-\beta}S(t)\|^2\mathbb{E}\|A^\beta\xi\|^2 + 6t^{2(\theta-\beta)}\mathbb{E}\Big\|\int_0^t A^\theta S(t-s)F_1(Y(s))ds\Big\|^2\\ &\quad + 6t^{2(\theta-\beta)}\mathbb{E}\Big\|\int_0^t A^\theta S(t-s)F_2(s)ds\Big\|^2\\ &\quad + 3t^{2(\theta-\beta)}\int_0^t \|A^\theta S(t-s)G(s)\|^2_{L_2(U;H)}ds.\end{aligned}$$

On the account of (5), (6) and Lemma 3.1, we have

$$\begin{aligned}&t^{2(\theta-\beta)}\mathbb{E}\|A^\theta\{\Phi Y\}(t)\|^2\\ &\le 3\iota_{\theta-\beta}^2\mathbb{E}\|A^\beta\xi\|^2 + 6t^{2(\theta-\beta)}\iota_\theta^2\mathbb{E}\Big[\int_0^t (t-s)^{-\theta}\|F_1(Y(s))\|ds\Big]^2\\ &\quad + 6\iota_\theta^2\|F_2\|^2_{\mathcal{F}^{\beta,\sigma}}B(\beta,1-\theta)^2 + \frac{3\iota_\theta^2\|G\|^2_{B([0,T];L_2(U;H))}t^{1-2\beta}}{1-2\theta}\\ &\le 3\iota_{\theta-\beta}^2\mathbb{E}\|A^\beta\xi\|^2 + 6t^{1+2(\theta-\beta)}\iota_\theta^2\int_0^t (t-s)^{-2\theta}\mathbb{E}\|F_1(Y(s))\|^2ds\\ &\quad + 6\iota_\theta^2\|F_2\|^2_{\mathcal{F}^{\beta,\sigma}}B(\beta,1-\theta)^2 + \frac{3\iota_\theta^2\|G\|^2_{B([0,T];L_2(U;H))}t^{1-2\beta}}{1-2\theta}.\end{aligned}$$

The second term in the right-hand side of the latter inequality is estimated by using (15):

$$\begin{aligned}&6t^{1+2(\theta-\beta)}\iota_\theta^2\int_0^t (t-s)^{-2\theta}\mathbb{E}\|F_1(Y(s))\|^2ds\\ &\le 12t^{1+2(\theta-\beta)}\iota_\theta^2\int_0^t (t-s)^{-2\theta}[c_{F_1}^2\kappa^2 s^{2(\beta-\eta)} + \mathbb{E}\|F_1(0)\|^2]ds\\ &\le 12\iota_\theta^2 c_{F_1}^2\kappa^2 t^{1+2(\theta-\eta)}\int_0^t (t-s)^{-2\theta}s^{2(\beta-\eta)}ds + \frac{12\iota_\theta^2\mathbb{E}\|F_1(0)\|^2}{1-2\theta}t^{2(1-\beta)}\end{aligned}$$

$$= 12\iota_\theta^2 c_{F_1}^2 \kappa^2 B(1+2\beta-2\eta, 1-2\theta) t^{2(1+\beta-2\eta)} + \frac{12\iota_\theta^2 \mathbb{E}\|F_1(0)\|^2}{1-2\theta} t^{2(1-\beta)}.$$

Thus,

$$\begin{aligned}
&t^{2(\theta-\beta)}\mathbb{E}\|A^\theta\{\Phi Y\}(t)\|^2 \\
&\leq 3\iota_{\theta-\beta}^2 \mathbb{E}\|A^\beta \xi\|^2 + 6\iota_\theta^2 \|F_2\|_{\mathcal{F}^{\beta,\sigma}}^2 B(\beta, 1-\theta)^2 + \frac{3\iota_\theta^2 \|G\|_{B([0,T];L_2(U;H))}^2 t^{1-2\beta}}{1-2\theta} \\
&\quad + 12\iota_\theta^2 c_{F_1}^2 \kappa^2 B(1+2\beta-2\eta, 1-2\theta) t^{2(1+\beta-2\eta)} + \frac{12\iota_\theta^2 \mathbb{E}\|F_1(0)\|^2}{1-2\theta} t^{2(1-\beta)}.
\end{aligned}$$

We apply these estimates with $\theta = \eta$ and $\theta = \beta$. It is therefore observed that if C_1 and C_2 are fixed in such a way that

$$\begin{aligned}
C_1 &> 3\iota_{\eta-\beta}^2 \mathbb{E}\|A^\beta \xi\|^2 + 6\iota_\eta^2 \|F_2\|_{\mathcal{F}^{\beta,\sigma}}^2 B(\beta, 1-\eta)^2, \\
C_2 &> 3\iota_0^2 \mathbb{E}\|A^\beta \xi\|^2 + 6\iota_\beta^2 \|F_2\|_{\mathcal{F}^{\beta,\sigma}}^2 B(\beta, 1-\beta)^2,
\end{aligned} \tag{16}$$

and if S is sufficiently small, then

$$\begin{aligned}
&t^{2(\eta-\beta)}\mathbb{E}\|A^\eta\{\Phi Y\}(t)\|^2 \\
&\leq C_1 + \frac{3\iota_\eta^2 \|G\|_{B([0,T];L_2(U;H))}^2 t^{1-2\beta}}{1-2\eta} \\
&\quad + 12\iota_\eta^2 c_{F_1}^2 \kappa^2 B(1+2\beta-2\eta, 1-2\eta) t^{2(1+\beta-2\eta)} + \frac{12\iota_\eta^2 \mathbb{E}\|F_1(0)\|^2}{1-2\eta} t^{2(1-\beta)} \\
&\leq \frac{\kappa^2}{2} + \frac{3\iota_\eta^2 \|G\|_{B([0,T];L_2(U;H))}^2 t^{1-2\beta}}{1-2\eta} \\
&\quad + 12\iota_\eta^2 c_{F_1}^2 \kappa^2 B(1+2\beta-2\eta, 1-2\eta) t^{2(1+\beta-2\eta)} + \frac{12\iota_\eta^2 \mathbb{E}\|F_1(0)\|^2}{1-2\eta} t^{2(1-\beta)} \\
&< \kappa^2, \quad 0 < t \leq S,
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
\mathbb{E}\|A^\beta\{\Phi Y\}(t)\|^2 &\leq C_2 + \frac{3\iota_\beta^2 \|G\|_{B([0,T];L_2(U;H))}^2 t^{1-2\beta}}{1-2\beta} \\
&\quad + 12\iota_\beta^2 c_{F_1}^2 \kappa^2 B(1+2\beta-2\eta, 1-2\beta) t^{2(1+\beta-2\eta)} \\
&\quad + \frac{12\iota_\beta^2 \mathbb{E}\|F_1(0)\|^2}{1-2\beta} t^{2(1-\beta)} \\
&\leq \frac{\kappa^2}{2} + \frac{3\iota_\beta^2 \|G\|_{B([0,T];L_2(U;H))}^2 t^{1-2\beta}}{1-2\beta} \\
&\quad + 12\iota_\beta^2 c_{F_1}^2 \kappa^2 B(1+2\beta-2\eta, 1-2\beta) t^{2(1+\beta-2\eta)} \\
&\quad + \frac{12\iota_\beta^2 \mathbb{E}\|F_1(0)\|^2}{1-2\beta} t^{2(1-\beta)} \\
&< \kappa^2, \quad 0 < t \leq S.
\end{aligned} \tag{18}$$

We have thus shown that

$$\max\left\{\sup_{0<t\le S} t^{2(\eta-\beta)}\mathbb{E}\|A^\eta \Phi Y(t)\|^2, \sup_{0\le t\le S}\mathbb{E}\|A^\beta \Phi Y(t)\|^2\right\} \le \kappa^2.$$

This means that ΦY satisfies (13).

Next, we prove that

$$\Phi Y \in \mathcal{C}((0,S];\mathcal{D}(A^\eta)) \cap \mathcal{C}([0,S];\mathcal{D}(A^\beta)) \qquad \text{a.s.}$$

Divide ΦY into two parts: $\Phi Y(t) = \Psi Y(t) + W_G(t)$, where

$$\Psi Y(t) = S(t)\xi + \int_0^t S(t-s)[F_1(Y(s)) + F_2(s)]ds, \tag{19}$$

and W_G is the stochastic convolution defined in Lemma 3.1. Lemma 3.1-(ii) for $\theta = \eta$ provides that

$$W_G \in \mathcal{C}([0,S];\mathcal{D}(A^\eta)) \subset \mathcal{C}([0,S];\mathcal{D}(A^\beta)) \qquad \text{a.s.}$$

Therefore, it suffices to verify that

$$\Psi Y \in \mathcal{C}((0,S];\mathcal{D}(A^\eta)) \cap \mathcal{C}([0,S];\mathcal{D}(A^\beta)) \qquad \text{a.s.} \tag{20}$$

In order to prove (20), we use the Kolmogorov test. For $0 < s < t \le S$, the semigroup property gives

$$\begin{aligned}
\Psi Y(t) - \Psi Y(s) &= S(t-s)S(s)\xi + S(t-s)\int_0^s S(s-r)[F_1(Y(r)) + F_2(r)]dr \\
&\quad + \int_s^t S(t-r)[F_1(Y(r)) + F_2(r)]dr - \Psi Y(s) \\
&= [S(t-s) - I]\Psi Y(s) + \int_s^t S(t-r)[F_1(Y(r)) + F_2(r)]dr.
\end{aligned}$$

Let $\frac{1}{2} < \rho < 1 - \eta$. Thanks to (5), (6) and (8), we have

$$\begin{aligned}
&\|A^\eta[\Psi Y(t) - \Psi Y(s)]\| \\
&\le \|[S(t-s) - I]A^{-\rho}\|\|A^{\eta+\rho}\Psi Y(s)\| \\
&\quad + \int_s^t \|A^\eta S(t-r)\|[\|F_1(Y(r))\| + \|F_2(r)\|]dr \\
&\le \frac{\iota_{1-\rho}(t-s)^\rho}{\rho}\left\|A^{\eta+\rho}\Big[S(s)\xi + \int_0^s S(s-r)[F_1(Y(r)) + F_2(r)]dr\Big]\right\| \\
&\quad + \iota_\eta \int_s^t (t-r)^{-\eta}[\|F_1(Y(r))\| + \|F_2(r)\|]dr \\
&\le \frac{\iota_{1-\rho}(t-s)^\rho}{\rho}\|A^{\eta+\rho-\beta}S(s)\|\|A^\beta\xi\| \\
&\quad + \frac{\iota_{1-\rho}(t-s)^\rho}{\rho}\int_0^s \|A^{\eta+\rho}S(s-r)\|\|F_1(Y(r))\|dr
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{\iota_{1-\rho}(t-s)^\rho}{\rho} \int_0^s \|A^{\eta+\rho} S(s-r)\| \|F_2(r)\| dr \\
&\quad + \iota_\eta \int_s^t (t-r)^{-\eta} \|F_1(Y(r))\| dr + \iota_\eta \int_s^t (t-r)^{-\eta} \|F_2(r)\| dr \\
&\leq \frac{\iota_{1-\rho}\iota_{\eta+\rho-\beta}(t-s)^\rho}{\rho} s^{-\eta-\rho+\beta} \|A^\beta \xi\| \\
&\quad + \frac{\iota_{1-\rho}\iota_{\eta+\rho}(t-s)^\rho}{\rho} \int_0^s (s-r)^{-\eta-\rho} \|F_1(Y(r))\| dr \\
&\quad + \frac{\iota_{1-\rho}\iota_{\eta+\rho}\|F_2\|_{\mathcal{F}^{\beta,\sigma}}(t-s)^\rho}{\rho} \int_0^s (s-r)^{-\eta-\rho} r^{\beta-1} dr \\
&\quad + \iota_\eta \int_s^t (t-r)^{-\eta} \|F_1(Y(r))\| dr + \iota_\eta \|F_2\|_{\mathcal{F}^{\beta,\sigma}} \int_s^t (t-r)^{-\eta} r^{\beta-1} dr \\
&= \frac{\iota_{1-\rho}\iota_{\eta+\rho-\beta}}{\rho} \|A^\beta \xi\| s^{\beta-\eta-\rho}(t-s)^\rho \\
&\quad + \frac{\iota_{1-\rho}\iota_{\eta+\rho}\|F_2\|_{\mathcal{F}^{\beta,\sigma}} B(\beta, 1-\eta-\rho)}{\rho} s^{\beta-\eta-\rho}(t-s)^\rho \\
&\quad + \iota_\eta \|F_2\|_{\mathcal{F}^{\beta,\sigma}} \int_s^t (t-r)^{-\eta} r^{\beta-1} dr \\
&\quad + \frac{\iota_{1-\rho}\iota_{\eta+\rho}(t-s)^\rho}{\rho} \int_0^s (s-r)^{-\eta-\rho} \|F_1(Y(r))\| dr \\
&\quad + \iota_\eta \int_s^t (t-r)^{-\eta} \|F_1(Y(r))\| dr.
\end{aligned}
$$

Dividing $\beta - 1$ as $\beta - 1 = (\eta + \rho - 1) + (\beta - \eta - \rho)$, it follows that

$$
\begin{aligned}
&\int_s^t (t-r)^{-\eta} r^{\beta-1} dr \\
&\leq \int_s^t (t-r)^{-\eta} (r-s)^{\eta+\rho-1} s^{\beta-\eta-\rho} dr \\
&= B(\eta+\rho, 1-\eta) s^{\beta-\eta-\rho} (t-s)^\rho.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|A^\eta[\Psi Y(t) - \Psi Y(s)]\| \\
&\leq \frac{\iota_{1-\rho}\iota_{\eta+\rho-\beta}}{\rho} \|A^\beta \xi\| s^{\beta-\eta-\rho}(t-s)^\rho \\
&\quad + \Big[\frac{\iota_{1-\rho}\iota_{\eta+\rho} B(\beta, 1-\eta-\rho)}{\rho} + \iota_\eta B(\eta+\rho, 1-\eta)\Big] \|F_2\|_{\mathcal{F}^{\beta,\sigma}} s^{\beta-\eta-\rho}(t-s)^\rho \\
&\quad + \frac{\iota_{1-\rho}\iota_{\eta+\rho}}{\rho}(t-s)^\rho \int_0^s (s-r)^{-\eta-\rho} \|F_1(Y(r))\| dr \\
&\quad + \iota_\eta \int_s^t (t-r)^{-\eta} \|F_1(Y(r))\| dr.
\end{aligned}
$$

Taking the expectation of the squares of the both hand sides of the above inequality, we obtain that

$$
\begin{aligned}
&\mathbb{E}\|A^\eta[\Psi Y(t)-\Psi Y(s)]\|^2\\
&\le \frac{4\iota_{1-\rho}^2\iota_{\eta+\rho-\beta}^2}{\rho^2}\mathbb{E}\|A^\beta\xi\|^2 s^{2(\beta-\eta-\rho)}(t-s)^{2\rho}\\
&\quad+4\Big[\frac{\iota_{1-\rho}\iota_{\eta+\rho}B(\beta,1-\eta-\rho)}{\rho}+\iota_\eta B(\eta+\rho,1-\eta)\Big]^2\\
&\quad\times\|F_2\|^2_{\mathcal{F}^{\beta,\sigma}}s^{2(\beta-\eta-\rho)}(t-s)^{2\rho}\\
&\quad+\frac{4\iota_{1-\rho}^2\iota_{\eta+\rho}^2}{\rho^2}(t-s)^{2\rho}\mathbb{E}\Big[\int_0^s(s-r)^{-\eta-\rho}\|F_1(Y(r))\|dr\Big]^2\\
&\quad+4\iota_\eta^2\mathbb{E}\Big[\int_s^t(t-r)^{-\eta}\|F_1(Y(r))\|dr\Big]^2.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\Big[\int_0^s(s-r)^{-\eta-\rho}\|F_1(Y(r))\|dr\Big]^2\\
&=\Big[\int_0^s(s-r)^{\frac{-\eta-\rho}{2}}(s-r)^{\frac{-\eta-\rho}{2}}\|F_1(Y(r))\|dr\Big]^2\\
&\le\int_0^s(s-r)^{-\eta-\rho}dr\int_0^s(s-r)^{-\eta-\rho}\|F_1(Y(r))\|^2dr\\
&=\frac{s^{1-\eta-\rho}}{1-\eta-\rho}\int_0^s(s-r)^{-\eta-\rho}\|F_1(Y(r))\|^2dr,
\end{aligned}
$$

we arrive at

$$
\begin{aligned}
(21)\qquad&\mathbb{E}\|A^\eta[\Psi Y(t)-\Psi Y(s)]\|^2\\
&\le \frac{4\iota_{1-\rho}^2\iota_{\eta+\rho-\beta}^2}{\rho^2}\mathbb{E}\|A^\beta\xi\|^2 s^{2(\beta-\eta-\rho)}(t-s)^{2\rho}\\
&\quad+4\Big[\frac{\iota_{1-\rho}\iota_{\eta+\rho}B(\beta,1-\eta-\rho)}{\rho}+\iota_\eta B(\eta+\rho,1-\eta)\Big]^2\\
&\quad\times\|F_2\|^2_{\mathcal{F}^{\beta,\sigma}}s^{2(\beta-\eta-\rho)}(t-s)^{2\rho}\\
&\quad+\frac{4\iota_{1-\rho}^2\iota_{\eta+\rho}^2}{\rho^2(1-\eta-\rho)}(t-s)^{2\rho}s^{1-\eta-\rho}\int_0^s(s-r)^{-\eta-\rho}\mathbb{E}\|F_1(Y(r))\|^2dr\\
&\quad+4\iota_\eta^2(t-s)\int_s^t(t-r)^{-2\eta}\mathbb{E}\|F_1(Y(r))\|^2dr.
\end{aligned}
$$

Both the integrals in (21) can be estimated by using (15):

$$
\begin{aligned}
&\int_0^s (s-r)^{-\eta-\rho}\mathbb{E}\|F_1(Y(r))\|^2 dr \\
&\leq 2c_{F_1}^2\kappa^2 \int_0^s (s-r)^{-\eta-\rho} r^{2(\beta-\eta)} dr + 2\mathbb{E}\|F_1(0)\|^2 \int_0^s (s-r)^{-\eta-\rho} dr \\
&= 2c_{F_1}^2\kappa^2 B(1+2\beta-2\eta, 1-\eta-\rho) s^{1+2\beta-3\eta-\rho} \\
&\quad + \frac{2\mathbb{E}\|F_1(0)\|^2 s^{1-\eta-\rho}}{1-\eta-\rho},
\end{aligned} \tag{22}
$$

and

$$
\begin{aligned}
&\int_s^t (t-r)^{-2\eta}\mathbb{E}\|F_1(Y(r))\|^2 dr \\
&\leq 2\int_s^t (t-r)^{-2\eta}[c_{F_1}^2\kappa^2 r^{2(\beta-\eta)} + \mathbb{E}\|F_1(0)\|^2] dr \\
&= 2c_{F_1}^2\kappa^2 \int_s^t (t-r)^{-2\eta} r^{2(\beta-\eta)} dr + \frac{2\mathbb{E}\|F_1(0)\|^2}{1-2\eta}(t-s)^{1-2\eta}.
\end{aligned} \tag{23}
$$

Divide $2(\beta-\eta)$ as $2(\beta-\eta) = (\beta-\frac{1}{2}) + (\frac{1}{2}+\beta-2\eta)$. Then

$$
\begin{aligned}
&\int_s^t (t-r)^{-2\eta} r^{2(\beta-\eta)} dr \\
&\leq \int_s^t (t-r)^{-2\eta}(r-s)^{\beta-\frac{1}{2}} t^{\frac{1}{2}+\beta-2\eta} dr \\
&= B(\frac{1}{2}+\beta, 1-2\eta) t^{\frac{1}{2}+\beta-2\eta}(t-s)^{\frac{1}{2}+\beta-2\eta}.
\end{aligned} \tag{24}
$$

Combining (21), (22), (23) and (24), we obtain an estimate:

$$
\begin{aligned}
&\mathbb{E}\|A^\eta[\Psi Y(t) - \Psi Y(s)]\|^2 \\
&\leq \frac{4\iota_{1-\rho}^2\iota_{\eta+\rho-\beta}^2}{\rho^2}\mathbb{E}\|A^\beta\xi\|^2 s^{2(\beta-\eta-\rho)}(t-s)^{2\rho} \\
&\quad + 4\Big[\frac{\iota_{1-\rho}\iota_{\eta+\rho}B(\beta, 1-\eta-\rho)}{\rho} + \iota_\eta B(\eta+\rho, 1-\eta)\Big]^2 \\
&\quad \times \|F_2\|_{\mathcal{F}^{\beta,\sigma}}^2 s^{2(\beta-\eta-\rho)}(t-s)^{2\rho} \\
&\quad + \frac{8\iota_{1-\rho}^2\iota_{\eta+\rho}^2 c_{F_1}^2\kappa^2 B(1+2\beta-2\eta, 1-\eta-\rho)}{\rho^2(1-\eta-\rho)}(t-s)^{2\rho} s^{2(1+\beta-2\eta-\rho)} \\
&\quad + \frac{8\iota_{1-\rho}^2\iota_{\eta+\rho}^2\mathbb{E}\|F_1(0)\|^2}{\rho^2(1-\eta-\rho)^2}(t-s)^{2\rho} s^{2(1-\eta-\rho)} \\
&\quad + 8\iota_\eta^2 c_{F_1}^2\kappa^2 B(\frac{1}{2}+\beta, 1-2\eta) t^{\frac{1}{2}+\beta-2\eta}(t-s)^{\frac{3}{2}+\beta-2\eta} \\
&\quad + \frac{8\iota_\eta^2\mathbb{E}\|F_1(0)\|^2}{1-2\eta}(t-s)^{2(1-\eta)}, \quad 0 < s < t \leq S.
\end{aligned} \tag{25}
$$

Since this estimate holds true for any $\frac{1}{2} < \rho < 1-\eta$, and since $1 < \frac{3}{2}+\beta-2\eta < 2(1-\eta)$, Theorem 2.6 then provides that $A^\eta \Psi Y$ is Hölder continuous on $(0,S]$ with an arbitrarily smaller exponent than $\frac{1+2\beta}{4}-\eta$. As a consequence, for any $0<\gamma<\frac{1+2\beta}{4}-\eta$,

$$\begin{cases} \Psi Y \in \mathcal{C}((0,S];\mathcal{D}(A^\eta)) \subset \mathcal{C}((0,S];\mathcal{D}(A^\beta)) \qquad \text{a.s.}, \\ A^\eta \Psi Y \in \mathcal{C}^\gamma((0,S];H) \qquad \text{a.s.} \end{cases} \tag{26}$$

In view of (20) and (26), it remains to show that $A^\beta \Psi Y$ is continuous at $t=0$. This function is separated into three terms:

$$A^\beta \Psi Y(t) = A^\beta S(t)\xi + A^\beta \int_0^t S(t-s)F_2(s)ds + A^\beta \int_0^t S(t-s)F_1(Y(s))ds.$$

Obviously, the first term $A^\beta S(\cdot)\xi$ is continuous at $t=0$, since

$$\lim_{t\to 0} \|A^\beta S(t)\xi - A^\beta \xi\| = \lim_{t\to 0} \|[S(t)-I]A^\beta \xi\| = 0.$$

The continuity of the second term $A^\beta \int_0^\cdot S(\cdot - s)F_2(s)ds$ at $t=0$ is verified in the following way. By the property of the space $\mathcal{F}^{\beta,\sigma}((0,T];H)$, we may put $z=\lim_{t\to 0} t^{1-\beta}F_2(t)$. Then,

$$\begin{aligned}
&\Big\|A^\beta \int_0^t S(t-s)F_2(s)ds\Big\| \\
&\le \Big\|\int_0^t A^\beta S(t-s)[F_2(s)-F_2(t)]ds\Big\| + \Big\|\int_0^t A^\beta S(t-s)F_2(t)ds\Big\| \\
&= \Big\|\int_0^t A^\beta S(t-s)[F_2(s)-F_2(t)]ds\Big\| + \Big\|[I-S(t)]A^{\beta-1}F_2(t)\Big\| \\
&\le \int_0^t \|A^\beta S(t-s)\|\|F_2(t)-F_2(s)\|ds \\
&\quad + \|t^{\beta-1}[I-S(t)]A^{\beta-1}[t^{1-\beta}F_2(t)-z]\| + \|t^{\beta-1}[I-S(t)]A^{\beta-1}z\|.
\end{aligned}$$

Thereby, (4), (6) and (8) give

$$\begin{aligned}
&\limsup_{t\to 0} \Big\|A^\beta \int_0^t S(t-s)F_2(s)ds\Big\| \\
&\le \iota_\beta \limsup_{t\to 0} \int_0^t \iota_\beta (t-s)^{-\beta}\|F_2(t)-F_2(s)\|ds \\
&\quad + \frac{\iota_\beta}{1-\beta}\limsup_{t\to 0}\|t^{1-\beta}F_2(t)-z\| \\
&\quad + \limsup_{t\to 0}\|t^{\beta-1}[I-S(t)]A^{\beta-1}z\| \\
&= \iota_\beta \limsup_{t\to 0}\int_0^t (t-s)^{\sigma-\beta}s^{-1+\beta-\sigma}\frac{s^{1-\beta+\sigma}\|F_2(t)-F_2(s)\|}{(t-s)^\sigma}ds \\
&\quad + \frac{\iota_\beta}{1-\beta}\limsup_{t\to 0}\|t^{1-\beta}F_2(t)-z\| + \limsup_{t\to 0}\|t^{\beta-1}[I-S(t)]A^{\beta-1}z\|
\end{aligned}$$

$$
\begin{aligned}
&\le \iota_\beta B(\beta-\sigma,1-\beta+\sigma)\limsup_{t\to 0}\sup_{s\in[0,t)}\frac{s^{1-\beta+\sigma}\|F_2(t)-F_2(s)\|}{(t-s)^\sigma}\\
&\quad+\limsup_{t\to 0}\|t^{\beta-1}[I-S(t)]A^{\beta-1}z\|\\
&=\limsup_{t\to 0}\|t^{\beta-1}[I-S(t)]A^{\beta-1}z\|.
\end{aligned}
$$

Since $\mathcal{D}(A^\beta)$ is dense in H, there exists a sequence $\{z_n\}_n$ in $\mathcal{D}(A^\beta)$ that converges to z as $n\to\infty$. Hence, (8) gives

$$
\begin{aligned}
\limsup_{t\to 0}\Big\|A^\beta\int_0^t S(t-s)F_2(s)ds\Big\| &\le \limsup_{t\to 0}\|t^{\beta-1}[I-S(t)]A^{\beta-1}(z-z_n)\|\\
&\quad+\limsup_{t\to 0}\|t^{\beta-1}[I-S(t)]A^{-1}A^\beta z_n\|\\
&\le \frac{\iota_\beta}{1-\beta}\|z-z_n\|+\iota_0\limsup_{t\to 0}t^\beta\|A^\beta z_n\|\\
&=\frac{\iota_\beta}{1-\beta}\|z-z_n\|,\qquad n=1,2,\dots.
\end{aligned}
$$

Letting n to ∞, we obtain that

$$
\lim_{t\to 0}A^\beta\int_0^t S(t-s)F_2(s)ds=0.
$$

This means that $A^\beta\int_0^\cdot S(\cdot-s)F_2(s)ds$ is continuous at $t=0$.

To see the continuity of the last term $A^\beta\int_0^\cdot S(\cdot-s)F_1(Y(s))ds$ at $t=0$, using (6) and (15), we have

$$
\begin{aligned}
&\mathbb{E}\Big\|A^\beta\int_0^t S(t-s)F_1(Y(s))ds\Big\|^2\\
&\le \mathbb{E}\Big[\int_0^t\|A^\beta S(t-s)\|\|F_1(Y(s))\|ds\Big]^2\\
&\le \iota_\beta^2\mathbb{E}\Big[\int_0^t(t-s)^{-\beta}\|F_1(Y(s))\|ds\Big]^2\\
&\le \iota_\beta^2 t\int_0^t(t-s)^{-2\beta}\mathbb{E}\|F_1(Y(s))\|^2ds\\
&\le 2\iota_\beta^2 t\int_0^t(t-s)^{-2\beta}[c_{F_1}^2\kappa^2 s^{2(\beta-\eta)}+\mathbb{E}\|F_1(0)\|^2]ds\\
&=2\iota_\beta^2\Big[c_{F_1}^2\kappa^2B(1+2\beta-2\eta,1-2\beta)t^{2(1-\eta)}+\frac{\mathbb{E}\|F_1(0)\|^2}{1-2\beta}t^{2(1-\beta)}\Big] \qquad (27)\\
&\to 0\qquad \text{as } t\to 0.
\end{aligned}
$$

Therefore, there exists a decreasing sequence $\{t_n\}_{n=1}^\infty$ converging to 0 such that

$$
\lim_{n\to\infty}A^\beta\int_0^{t_n}S(t_n-s)F_1(Y(s))ds=0.
$$

Since $A^\beta \int_0^\cdot S(\cdot - s)F_1(Y(s))ds$ is continuous on $(0, S]$, we conclude that

$$\lim_{t\to 0} A^\beta \int_0^t S(t-s)F_1(Y(s))ds = 0,$$

i.e., $A^\beta \int_0^\cdot S(\cdot - s)F_1(Y(s))ds$ is continuous at $t = 0$.

Step 2. Let us show that Φ is a contraction mapping of $\Xi(S)$, provided that $S > 0$ is sufficiently small.

Let $Y_1, Y_2 \in \Xi(S)$ and $0 \le \theta < \frac{1}{2}$. It follows from (14) that

$$\begin{aligned} & t^{2(\theta-\beta)}\mathbb{E}\|A^\theta[\Phi Y_1(t) - \Phi Y_2(t)]\|^2 \\ &= t^{2(\theta-\beta)}\mathbb{E}\Big\|\int_0^t A^\theta S(t-s)[F_1(Y_1(s)) - F_1(Y_2(s))]ds\Big\|^2 \\ &\le t^{2(\theta-\beta)}\mathbb{E}\Big[\int_0^t \|A^\theta S(t-s)\|\|F_1(Y_1(s)) - F_1(Y_2(s))\|ds\Big]^2. \end{aligned}$$

Hence, (6), (H3) and (11) give

$$\begin{aligned} & t^{2(\theta-\beta)}\mathbb{E}\|A^\theta[\Phi Y_1(t) - \Phi Y_2(t)]\|^2 \\ &\le c_{F_1}^2 \iota_\theta^2 t^{2(\theta-\beta)}\mathbb{E}\Big[\int_0^t (t-s)^{-\theta}\|A^\eta(Y_1(s) - Y_2(s))\|ds\Big]^2 \\ &\le c_{F_1}^2 \iota_\theta^2 t^{1+2(\theta-\beta)}\mathbb{E}\int_0^t (t-s)^{-2\theta}\|A^\eta(Y_1(s) - Y_2(s))\|^2 ds \\ &\le c_{F_1}^2 \iota_\theta^2 t^{1+2(\theta-\beta)}\int_0^t (t-s)^{-2\theta}\mathbb{E}\|A^\eta(Y_1(s) - Y_2(s))\|^2 ds \\ &\le c_{F_1}^2 \iota_\theta^2 t^{1+2(\theta-\beta)}\int_0^t (t-s)^{-2\theta}s^{2(\beta-\eta)}\|Y_1 - Y_2\|_{\Xi(S)}^2 ds \\ &= c_{F_1}^2 \iota_\theta^2 B(1+2\beta-2\eta, 1-2\theta)t^{2(1-\eta)}\|Y_1 - Y_2\|_{\Xi(S)}^2. \end{aligned}$$

Applying these estimates with $\theta = \eta$ and $\theta = \beta$, we conclude that

$$\begin{aligned} & \|\Phi Y_1 - \Phi Y_2\|_{\Xi(S)}^2 \\ &= \sup_{0<t\le S} t^{2(\eta-\beta)}\mathbb{E}\|A^\eta[\Phi Y_1(t) - \Phi Y_2(t)]\|^2 \\ &\quad + \sup_{0\le t\le S} \mathbb{E}\|A^\beta[\Phi Y_1(t) - \Phi Y_2(t)]\|^2 \\ &\le c_{F_1}^2[\iota_\eta^2 B(1+2\beta-2\eta, 1-2\eta) + \iota_\beta^2 B(1+2\beta-2\eta, 1-2\beta)] \\ &\quad \times S^{2(1-\eta)}\|Y_1 - Y_2\|_{\Xi(S)}^2. \end{aligned} \tag{28}$$

Clearly, (28) shows that Φ is contractive in $\Xi(S)$, provided that $S > 0$ is sufficiently small.

Step 3. Let us prove:

- the existence of a local mild solution in the function space in (9)
- the estimate (10)

Let $S > 0$ be sufficiently small in such a way that Φ maps $\Upsilon(S)$ into itself and is contraction with respect to the norm of $\Xi(S)$. Due to Step 1 and Step 2, $S = T_{loc}$ can be determined by $\mathbb{E}\|F_1(0)\|^2$, $\|F_2\|^2_{\mathcal{F}^{\beta,\sigma}}$, $\|G\|^2_{B([0,T];L_2(U;H))}$ and $\mathbb{E}\|A^\beta \xi\|^2$. Thanks to the fixed point theorem, there exists a unique function $X \in \Upsilon(T_{loc})$ such that $X = \Phi X$. This means that X is a local mild solution of (1) in the function space:

$$X \in \mathcal{C}((0, T_{loc}]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, T_{loc}]; \mathcal{D}(A^\beta)) \qquad \text{a.s.}$$

In addition, thanks to Lemma 3.1-(iii) and (26), for any $0 < \gamma < \frac{1+2\beta}{4} - \eta$,

$$A^\eta X = A^\eta \Phi X = A^\eta \Psi X + A^\eta W_G \in \mathcal{C}^\gamma((0, T_{loc}]; H) \qquad \text{a.s.}$$

Furthermore, (10) is obtained from the definition of $\Upsilon(T_{loc})$ (see (13)).

Step 4. Let us finally show the uniqueness of local mild solutions.

Let $\bar{X}$ be any other local mild solution to (1) on the interval $[0, T_{loc}]$, which belongs to the space $\mathcal{C}((0, T_{loc}]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, T_{loc}]; \mathcal{D}(A^\beta))$.

The formulae

$$\begin{aligned} X(t) = S(t)\xi &+ \int_0^t S(t-s)F_2(s)ds + \int_0^t S(t-s)G(s)dW(s) \\ &+ \int_0^t S(t-s)F_1(X(s))ds, \end{aligned} \tag{29}$$

and

$$\begin{aligned} \bar{X}(t) = S(t)\xi &+ \int_0^t S(t-s)F_2(s)ds + \int_0^t S(t-s)G(s)dW(s) \\ &+ \int_0^t S(t-s)F_1(\bar{X}(s))ds \end{aligned}$$

imply that

$$X(t) - \bar{X}(t) = \int_0^t S(t-s)[F_1(X(s)) - F_1(\bar{X}(s))]ds, \qquad 0 \le t \le T_{loc}.$$

We can then repeat the same arguments as in Step 2 to deduce that

$$\begin{aligned} &\|X - \bar{X}\|^2_{\Xi(\bar{T})} \\ &\le c_{F_1}^2[\iota_\eta^2 B(1+2\beta-2\eta, 1-2\eta) + \iota_\beta^2 B(1+2\beta-2\eta, 1-2\beta)] \\ &\quad \times \bar{T}^{2(1-\eta)}\|X - \bar{X}\|^2_{\Xi(\bar{T})} \end{aligned} \tag{30}$$

for any $0 < \bar{T} \le T_{loc}$. Let $\bar{T}$ be a positive constant such that

$$c_{F_1}^2[\iota_\eta^2 B(1+2\beta-2\eta, 1-2\eta) + \iota_\beta^2 B(1+2\beta-2\eta, 1-2\beta)]\bar{T}^{2(1-\eta)} < 1.$$

Thus, (30) gives

$$X(t) = \bar{X}(t) \quad \text{ a.s., } 0 \le t \le \bar{T}.$$

We repeat the same procedure with initial time $\bar{T}$ and initial value $X(\bar{T}) = \bar{X}(\bar{T})$ to derive that

$$X(\bar{T}+t) = \bar{X}(\bar{T}+t) \quad \text{a.s., } 0 \le t \le \bar{T}.$$

This means that $X(t) = \bar{X}(t)$ a.s. on a larger interval $[0, 2\bar{T}]$. We continue this procedure by finite times, the extended interval can cover the given interval $[0, T_{loc}]$. Therefore, for $0 \le t \le T_{loc}$, $X(t) = \bar{X}(t)$ a.s. □

Let us next show the differentiability of the expectation of local mild solutions. Put

$$Z(t) = \mathbb{E}X(t), \quad 0 \le t \le T_{loc}.$$

Theorem 3.3. *Let the assumptions in Theorem 3.2 be satisfied. Assume that* $\sigma + \eta \le \frac{1}{2}$. *Then,*

$$\begin{cases} Z \in \mathcal{C}((0, T_{loc}]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_{loc}]; \mathcal{D}(A^\beta)) \cap \mathcal{C}^1((0, T_{loc}]; H), \\ \frac{dZ}{dt}, AZ \in \mathcal{F}^{\beta,\sigma}((0, T_{loc}]; H). \end{cases} \tag{31}$$

Furthermore, Z *satisfies the estimate*

$$\|A^\beta Z\|_{\mathcal{C}} + \Big|\Big|\frac{dZ}{dt}\Big|\Big|_{\mathcal{F}^{\beta,\sigma}} + \|AZ\|_{\mathcal{F}^{\beta,\sigma}} \le C_{F_1,F_2,\xi}, \quad 0 \le t \le T_{loc} \tag{32}$$

with some constant $C_{F_1,F_2,\xi}$ *depending on* $\mathbb{E}\|F_1(0)\|^2$, $\mathbb{E}\|A^\beta\xi\|^2$, $\|F_2\|^2_{\mathcal{F}^{\beta,\sigma}}$ *and* T_{loc}.

Proof. Throughout the proof, we use a universal constant C, which depends on exponents and $\mathbb{E}\|F_1(0)\|^2$, $\mathbb{E}\|A^\beta\xi\|^2$, $\|F_2\|^2_{\mathcal{F}^{\beta,\sigma}}$ and T_{loc}.

Since $\mathbb{E}\int_0^t S(t-s)G(s)dW(s) = 0$, we have an expression:

$$Z(t) = \mathbb{E}S(t)\xi + \int_0^t S(t-s)[\mathbb{E}F_1(X(s)) + F_2(s)]ds.$$

First, let us show that

$$\mathbb{E}F_1(X(\cdot)) \in \mathcal{F}^{\beta,\sigma}((0, T_{loc}]; H). \tag{33}$$

In view of (15),

$$\|\mathbb{E}F_1(X(t))\|^2 \le \mathbb{E}\|F_1(X(t))\|^2 \le C[t^{2(\beta-\eta)} + 1], \quad 0 < t \le T_{loc}.$$

Thereby,

$$\|t^{1-\beta}\mathbb{E}F_1(X(t))\|^2 \le C[t^{2(1-\eta)} + t^{2(1-\beta)}] \to 0 \quad \text{as } t \to 0.$$

The function $\mathbb{E}F_1(X(\cdot))$ therefore satisfies (2).

On the other hand, (H3), (19), (25) and (29) give

$$\begin{aligned} &\|\mathbb{E}F_1(X(t)) - \mathbb{E}F_1(X(s))\|^2 \\ &\le \mathbb{E}\|F_1(X(t)) - F_1(X(s))\|^2 \\ &\le c_{F_1}\mathbb{E}\|A^\eta[X(t) - X(s)]\|^2 \\ &\le C\{\mathbb{E}\|A^\eta[\Psi X(t) - \Psi X(s)]\|^2 + \mathbb{E}\|A^\eta[W_G(t) - W_G(s)]\|^2\} \end{aligned}$$

$$\leq Cs^{2(\beta-\eta-\rho)}(t-s)^{2\rho} + Cs^{2(1+\beta-2\eta-\rho)}(t-s)^{2\rho} + Cs^{2(1-\eta-\rho)}(t-s)^{2\rho} + C(t-s)^{\frac{3}{2}+\beta-2\eta} + C(t-s)^{2(1-\eta)} + \mathbb{E}\|A^\eta[W_G(t)-W_G(s)]\|^2, \quad 0<s<t<T_{loc}$$

for any $\frac{1}{2} < \rho < 1-\eta$. The last term in the latter inequality is evaluated by using (6) and (8):

$$\begin{aligned}
&\mathbb{E}\|A^\eta[W_G(t)-W_G(s)]\|^2 \\
&= \mathbb{E}\Big\|\int_0^s A^\eta[S(t-r)-S(s-r)]G(r)dW(r) + \int_s^t A^\eta S(t-r)G(r)dW(r)\Big\|^2 \\
&\leq \int_0^s \|A^{\eta+\sigma}S(s-r)[S(t-s)-I]A^{-\sigma}G(r)\|^2_{L_2(U;H)}dr \\
&\quad + \int_s^t \|A^\eta S(t-r)G(r)\|^2_{L_2(U;H)}dr \\
&\leq \iota^2_{\eta+\sigma}\|[S(t-s)-I]A^{-\sigma}\|^2\|G\|^2_{B([0,T];L_2(U;H))}\int_0^s (s-r)^{-2(\eta+\sigma)}dr \\
&\quad + \iota^2_\eta\|G\|^2_{B([0,T];L_2(U;H))}\int_s^t (t-r)^{-2\eta}dr \\
&\leq Cs^{1-2(\eta+\sigma)}(t-s)^{2\sigma} + C(t-s)^{1-2\eta}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{s^{2(1-\beta+\sigma)}\|\mathbb{E}F_1(X(t))-\mathbb{E}F_1(X(s))\|^2}{(t-s)^{2\sigma}} \\
&\leq Cs^{2(1-\eta-\rho+\sigma)}(t-s)^{2(\rho-\sigma)} + Cs^{2(2+\sigma-2\eta-\rho)}(t-s)^{2(\rho-\sigma)} \\
&\quad + Cs^{2(2-\beta+\sigma-\eta-\rho)}(t-s)^{2(\rho-\sigma)} + Cs^{2(1-\beta+\sigma)}(t-s)^{\frac{3}{2}+\beta-2\eta-2\sigma} \\
&\quad + Cs^{2(1-\beta+\sigma)}(t-s)^{2(1-\eta-\sigma)} + Cs^{3-2(\eta+\beta)} \\
&\quad + Cs^{2(1-\beta+\sigma)}(t-s)^{1-2(\eta+\sigma)}, \quad 0<s<t\leq T_{loc}.
\end{aligned}$$

This shows that (3) and (4) are also valid for the function $\mathbb{E}F_1(X(\cdot))$. Hence, (33) has been verified.

As a result of (H4) and (33),

$$\mathbb{E}F_1(X(\cdot)) + F_2(\cdot) \in \mathcal{F}^{\beta,\sigma}((0,T_{loc}];H).$$

Since $\mathbb{E}S(t)\xi = S(t)\mathbb{E}\xi$ and $\mathbb{E}\xi \in \mathcal{D}(A^\beta)$, Theorem 2.3 applied to the function Z provides (31) and (32). The proof is completed. $\square$

3.2. Regular dependence of solutions on initial data

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bounded balls:

$$\mathcal{B}_1 = \{f \in \mathcal{F}^{\beta,\sigma}((0,T];H); \|f\|_{\mathcal{F}^{\beta,\sigma}} \leq R_1\}, \quad 0<R_1<\infty,$$

$$\mathcal{B}_2 = \{g \in B([0,T];L_2(U;H)); \|G\|_{B([0,T];L_2(U;H))} \leq R_2\}, \quad 0<R_2<\infty,$$

of the spaces $\mathcal{F}^{\beta,\sigma}((0,T];H)$ and $B([0,T];L_2(U;H))$, respectively. Let B_A be a set of random variables:

$$B_A = \{\zeta; \zeta \in \mathcal{D}(A^\beta) \text{ a.s. and } \mathbb{E}\|A^\beta\zeta\|^2 \le R_3^2\}, \quad 0 < R_3 < \infty.$$

According to Theorem 3.2, for every $F_2 \in \mathcal{B}_1, G \in \mathcal{B}_2$ and $\xi \in B_A$, there exists a unique local solution of (1) on some interval $[0, T_{loc}]$. Furthermore, in view of Step 1 and Step 2 in the proof for Theorem 3.2, we have

$$\begin{aligned}&\text{there is a time } T_{\mathcal{B}_1,\mathcal{B}_2,B_A} > 0 \text{ such that}\\ &[0, T_{\mathcal{B}_1,\mathcal{B}_2,B_A}] \subset [0, T_{loc}] \text{ for all } (F_2, G, \xi) \in \mathcal{B}_1 \times \mathcal{B}_2 \times B_A.\end{aligned} \tag{34}$$

Indeed, by (17), (18) and (28), T_{loc} can be chosen to be any time S satisfying the conditions:

$$\begin{aligned}\frac{\kappa^2}{2} \ge & \frac{3\iota_\eta^2\|G\|^2_{B([0,T];L_2(U;H))}S^{1-2\beta}}{1-2\eta}\\ &+ 12\iota_\eta^2 c_{F_1}^2\kappa^2 B(1+2\beta-2\eta, 1-2\eta)S^{2(1+\beta-2\eta)}\\ &+ \frac{12\iota_\eta^2\mathbb{E}\|F_1(0)\|^2}{1-2\eta}S^{2(1-\beta)},\end{aligned}$$

$$\begin{aligned}\frac{\kappa^2}{2} \ge & \frac{3\iota_\beta^2\|G\|^2_{B([0,T];L_2(U;H))}S^{1-2\beta}}{1-2\beta}\\ &+ 12\iota_\beta^2 c_{F_1}^2\kappa^2 B(1+2\beta-2\eta, 1-2\beta)S^{2(1+\beta-2\eta)}\\ &+ \frac{12\iota_\beta^2\mathbb{E}\|F_1(0)\|^2}{1-2\beta}S^{2(1-\beta)},\end{aligned}$$

and

$$1 > c_{F_1}^2[\iota_\eta^2 B(1+2\beta-2\eta, 1-2\eta) + \iota_\beta^2 B(1+2\beta-2\eta, 1-2\beta)]S^{2(1-\eta)},$$

where κ is defined by (12) and (16). As a consequence, we can choose T_{loc} such that it depends continuously on $\mathbb{E}\|F_1(0)\|^2$, $\mathbb{E}\|A^\beta\xi\|^2$, $\|G\|^2_{B([0,T];L_2(U;H))}$ and $\|F_2\|^2_{\mathcal{F}^{\beta,\sigma}}$. Thus, (34) follows.

We are now ready to show the continuous dependence of solutions on (F_2, G, ξ) in the sense specified in the following theorem.

Theorem 3.4. *Let* (H1), (H2), (H3), (H4) *and* (H5) *be satisfied. Let X and $\bar{X}$ be the solutions of* (1) *for the data (F_2, G, ξ) and $(\bar{F}_2, \bar{G}, \bar{\xi})$ in $\mathcal{B}_1 \times \mathcal{B}_2 \times B_A$, respectively. Then, there exists a constant $C_{\mathcal{B}_1,\mathcal{B}_2,B_A}$ depending only on $\mathcal{B}_1, \mathcal{B}_2$ and B_A such that*

$$\begin{aligned}& t^{2\eta}\mathbb{E}\|A^\eta[X(t)-\bar{X}(t)]\|^2 + t^{2\eta}\mathbb{E}\|A^\beta[X(t)-\bar{X}(t)]\|^2\\ &+ \mathbb{E}\|X(t)-\bar{X}(t)\|^2 \le C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|\xi-\bar{\xi}\|^2 + t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}\\ &+ t\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))}], \quad 0 < t < T_{\mathcal{B}_1,\mathcal{B}_2,B_A},\end{aligned} \tag{35}$$

and

$$t^{2(\eta-\beta)}[\mathbb{E}\|A^{\eta}[X(t)-\bar{X}(t)]\|^2+\mathbb{E}\|A^{\beta}[X(t)-\bar{X}(t)]\|^2] \le C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|A^{\beta}(\xi-\bar{\xi})\|^2+\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} + \|G-\bar{G}\|^2_{B([0,T];L_2(U;H))}], \quad 0<t<T_{\mathcal{B}_1,\mathcal{B}_2,B_A}. \tag{36}$$

In order to prove this theorem, we use a generalized inequality of Gronwall type.

Lemma 3.5. *Let $0<a\le b, \mu>0$ and $\nu>0$ be constants. Let f be a continuous and increasing function on $[0,\infty)$ and φ be a nonnegative bounded function on $[a,b]$. If φ satisfies the integral inequality*

$$\varphi(t)\le f(t)+a^{-\mu}\int_a^t (t-r)^{\nu-1}\varphi(r)dr, \qquad a\le t\le b,$$

then there exists $c>0$ such that

$$\varphi(t)\le cf(t), \quad a\le s<t\le b.$$

Proof. Let Γ be the gamma function. By induction, we verify the estimate:

$$\varphi(t)\le \sum_{k=0}^{n} a^{-k\mu}f(t)t^{k\nu}\frac{\Gamma(\nu)^k}{\Gamma(1+k\nu)}+a^{-\mu(n+1)}\frac{\Gamma(\nu)^{n+1}}{\Gamma((n+1)\nu)} \times\int_a^t (t-s)^{(n+1)\nu-1}\varphi(s)ds, \quad a\le t\le b. \tag{37}$$

Indeed, the case $n=0$ is obvious. Assume that this inequality holds true for n. Then,

$$\varphi(t)\le \sum_{k=0}^{n} a^{-k\mu}f(t)t^{k\nu}\frac{\Gamma(\nu)^k}{\Gamma(1+k\nu)} + a^{-\mu(n+1)}\frac{\Gamma(\nu)^{n+1}}{\Gamma((n+1)\nu)}\int_a^t (t-s)^{(n+1)\nu-1} \times[f(s)+a^{-\mu}\int_a^s (s-r)^{\nu-1}\varphi(r)dr]ds. \tag{38}$$

Since f is increasing, we observe that

$$\int_a^t (t-s)^{(n+1)\nu-1}f(s)ds \le f(t)\frac{(t-a)^{(n+1)\nu}}{(n+1)\nu} = \frac{f(t)(t-a)^{(n+1)\nu}\Gamma((n+1)\nu)}{\Gamma(1+(n+1)\nu)}. \tag{39}$$

In addition,

$$\begin{aligned}
&\int_a^t \int_a^s (t-s)^{(n+1)\nu-1}(s-r)^{\nu-1}\varphi(r)drds\\
&\le \int_0^t \int_0^s (t-s)^{(n+1)\nu-1}(s-r)^{\nu-1}\varphi(r)drds\\
&= \int_0^t \int_r^t (t-s)^{(n+1)\nu-1}(s-r)^{\nu-1}ds\varphi(r)dr\\
&= \int_0^t (t-r)^{(n+2)\nu-1}\int_0^1 (1-u)^{(n+1)\nu-1}u^{\nu-1}du\varphi(r)dr\\
&= B(\nu,(n+1)\nu)\int_0^t (t-r)^{(n+2)\nu-1}\varphi(r)dr\\
&= \frac{\Gamma(n+1)\nu)\Gamma(\nu)}{\Gamma(n+2)\nu)}\int_0^t (t-r)^{(n+2)\nu-1}\varphi(r)dr. \qquad (40)
\end{aligned}$$

Thanks to (38), (39) and (40), the estimate (37) holds true for $n+1$.

Since φ is bounded on $[a,b]$, the second term in the right-hand side of (37) is estimated by:

$$\begin{aligned}
&a^{-\mu(n+1)}\frac{\Gamma(\nu)^{n+1}}{\Gamma((n+1)\nu)}\int_a^t (t-s)^{(n+1)\nu-1}\varphi(s)ds\\
&\le \frac{a^{-\mu(n+1)}\Gamma(\nu)^{n+1}(t-a)^{(n+1)\nu}\sup_{s\in[a,t]}\varphi(s)}{(n+1)\nu)\Gamma((n+1)\nu)}.
\end{aligned}$$

Due to the Stirling's formula, it is known that

$$\Gamma(x+1)\sim\sqrt{2\pi x}\Big(\frac{x}{e}\Big)^x \qquad \text{as } x\to\infty.$$

This term therefore converges to zero as $n\to\infty$. As a consequence,

$$\varphi(t)\le f(t)\sum_{k=0}^{\infty}\frac{[a^{-\mu}t^{\nu}\Gamma(\nu)]^k}{\Gamma(1+k\nu)}, \quad a\le t\le b.$$

It is known that (e.g., [18, Lemma 1.2])

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{[a^{-\mu}t^{\nu}\Gamma(\nu)]^k}{\Gamma(1+k\nu)}\\
&\le \frac{2}{\min_{0<s<\infty}\Gamma(s)\nu}(1+t[a^{-\mu}\Gamma(\nu)]^{\frac{1}{\nu}})e^{t[a^{-\mu}\Gamma(\nu)]^{\frac{1}{\nu}}+1}, \quad 0\le t<\infty.
\end{aligned}$$

Thus, the lemma has been proved. □

Proof of Theorem 3.4. This theorem is proved by using analogous arguments as in the proof of Theorem 3.2. We use a universal constant $C_{\mathcal{B}_1,\mathcal{B}_2,B_A}$, which depends only on the exponents and $\mathcal{B}_1,\mathcal{B}_2$ and B_A.

First, let us give an estimate for

$$t^{2\eta}\mathbb{E}[\|A^\eta[X(t)-\bar{X}(t)]\|^2+\|A^\beta[X(t)-\bar{X}(t)]\|^2].$$

For $0\le\theta<\frac{1}{2}$ and $0<t\le T_{\mathcal{B}_1,\mathcal{B}_2,B_A}$, (5), (6) and (H3) give

$$\begin{aligned}
&t^\theta\|A^\theta[X(t)-\bar{X}(t)]\|\\
&=\Big|\Big|t^\theta A^\theta S(t)(\xi-\bar{\xi})+\int_0^t t^\theta A^\theta S(t-s)[F_1(X(s))-F_1(\bar{X}(s))]ds\\
&\quad+\int_0^t t^\theta A^\theta S(t-s)[F_2(s)-\bar{F}_2(s)]ds\\
&\quad+\int_0^t t^\theta A^\theta S(t-s)[G(s)-\bar{G}(s)]dW(s)\Big|\Big|\\
&\le \iota_\theta\|\xi-\bar{\xi}\|+\iota_\theta c_{F_1}\int_0^t t^\theta(t-s)^{-\theta}\|A^\eta[X(s)-\bar{X}(s)]\|ds\\
&\quad+\iota_\theta\|F_2-\bar{F}_2\|_{\mathcal{F}^{\beta,\sigma}}\int_0^t t^\theta(t-s)^{-\theta}s^{\beta-1}ds\\
&\quad+\Big|\Big|\int_0^t t^\theta A^\theta S(t-s)[G(s)-\bar{G}(s)]dW(s)\Big|\Big|\\
&=\iota_\theta\|\xi-\bar{\xi}\|+\iota_\theta\|F_2-\bar{F}_2\|_{\mathcal{F}^{\beta,\sigma}}B(\beta,1-\theta)t^\beta\\
&\quad+\iota_\theta c_{F_1}\int_0^t t^\theta(t-s)^{-\theta}\|A^\eta[X(s)-\bar{X}(s)]\|ds\\
&\quad+\Big|\Big|\int_0^t t^\theta A^\theta S(t-s)[G(s)-\bar{G}(s)]dW(s)\Big|\Big|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\|t^\theta A^\theta[X(t)-\bar{X}(t)]\|^2\\
&\le 4\iota_\theta^2\mathbb{E}\|\xi-\bar{\xi}\|^2+4\iota_\theta^2\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}B(\beta,1-\theta)^2t^{2\beta}\\
&\quad+4\iota_\theta^2c_{F_1}^2t^{2\theta}\mathbb{E}\Big[\int_0^t(t-s)^{-\theta}\|A^\eta[X(s)-\bar{X}(s)]\|ds\Big]^2\\
&\quad+4\mathbb{E}\Big|\Big|\int_0^t t^\theta A^\theta S(t-s)[G(s)-\bar{G}(s)]dW(s)\Big|\Big|^2\\
&\le 4\iota_\theta^2\mathbb{E}\|\xi-\bar{\xi}\|^2+4\iota_\theta^2\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}B(\beta,1-\theta)^2t^{2\beta}\\
&\quad+4\iota_\theta^2c_{F_1}^2t^{2\theta}\mathbb{E}\Big[\int_0^t(t-s)^{-\theta}\|A^\eta[X(s)-\bar{X}(s)]\|ds\Big]^2\\
&\quad+4\int_0^t\|t^\theta A^\theta S(t-s)\|^2\|G(s)-\bar{G}(s)\|^2_{L_2(U;H)}ds\\
&\le 4\iota_\theta^2\mathbb{E}\|\xi-\bar{\xi}\|^2+4\iota_\theta^2\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}B(\beta,1-\theta)^2t^{2\beta}
\end{aligned}$$

$$
\begin{aligned}
&+ 4\iota_\theta^2 c_{F_1}^2 t^{2\theta+1} \int_0^t (t-s)^{-2\theta}\mathbb{E}\|A^\eta[X(s)-\bar{X}(s)]\|^2 ds \\
&+ 4\iota_\theta^2 \|G-\bar{G}\|^2_{B([0,T];L_2(U;H))} \int_0^t t^{2\theta}(t-s)^{-2\theta} ds \\
&\le 4\iota_\theta^2 \mathbb{E}\|\xi-\bar{\xi}\|^2 + 4\iota_\theta^2 B(\beta,1-\theta)^2 t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} \\
&+ \frac{4\iota_\theta^2 t}{1-2\theta}\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))} \\
&+ 4\iota_\theta^2 c_{F_1}^2 t^{2\theta+1} \int_0^t (t-s)^{-2\theta}\mathbb{E}\|A^\eta[X(s)-\bar{X}(s)]\|^2 ds.
\end{aligned} \tag{41}
$$

Applying these estimates with $\theta=\beta$ and $\theta=\eta$, we have

$$
\begin{aligned}
&\mathbb{E}\|A^\beta[X(t)-\bar{X}(t)]\|^2 \\
&\le 4\iota_\beta^2\mathbb{E}\|\xi-\bar{\xi}\|^2 t^{-2\beta} + 4\iota_\beta^2 B(\beta,1-\beta)^2\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} \\
&\quad + \frac{4\iota_\beta^2 t^{1-2\beta}}{1-2\beta}\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))} \\
&\quad + 4\iota_\beta^2 c_{F_1}^2 t \int_0^t (t-s)^{-2\beta}\mathbb{E}\|A^\eta[X(s)-\bar{X}(s)]\|^2 ds, \quad 0<t\le T_{\mathcal{B}_1,\mathcal{B}_2,B_A},
\end{aligned}
$$

and

$$
\begin{aligned}
&t^{2\eta}\mathbb{E}\|A^\eta[X(t)-\bar{X}(t)]\|^2 \\
&\le 4\iota_\eta^2\mathbb{E}\|\xi-\bar{\xi}\|^2 + 4\iota_\eta^2 B(\beta,1-\eta)^2 t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} \\
&\quad + \frac{4\iota_\eta^2 t}{1-2\eta}\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))} \\
&\quad + 4\iota_\eta^2 c_{F_1}^2 t^{2\eta+1} \int_0^t (t-s)^{-2\eta}\mathbb{E}\|A^\eta[X(s)-\bar{X}(s)]\|^2 ds, \quad 0<t\le T_{\mathcal{B}_1,\mathcal{B}_2,B_A}.
\end{aligned}
$$

By putting

$$
q(t) = t^{2\eta}\mathbb{E}[\|A^\eta[X(t)-\bar{X}(t)]\|^2 + \|A^\beta[X(t)-\bar{X}(t)]\|^2],
$$

we then obtain an integral inequality

$$
\begin{aligned}
q(t) &\le 4[\iota_\beta^2 t^{2(\eta-\beta)} + \iota_\eta^2]\mathbb{E}\|\xi-\bar{\xi}\|^2 \\
&+ 4[\iota_\beta^2 B(\beta,1-\beta)^2 t^{2\eta} + \iota_\eta^2 B(\beta,1-\eta)^2 t^{2\beta}]\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} \\
&+ 4\Big[\frac{\iota_\beta^2 t^{1+2\eta-2\beta}}{1-2\beta} + \frac{\iota_\eta^2 t}{1-2\eta}\Big]\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))} \\
&+ 4c_{F_1}^2 t^{2\eta+1}\int_0^t [\iota_\beta^2(t-s)^{-2\beta} + \iota_\eta^2(t-s)^{-2\eta}] s^{-2\eta} q(s) ds \\
&\le C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|\xi-\bar{\xi}\|^2 + t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} \\
&+ t\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))}]
\end{aligned} \tag{42}
$$

$$+ 4c_{F_1}^2 t^{2\eta+1} \int_0^t [\iota_\beta^2 (t-s)^{-2\beta} + \iota_\eta^2 (t-s)^{-2\eta}] \times s^{-2\eta} q(s) ds, \quad 0 < t \le T_{\mathcal{B}_1,\mathcal{B}_2,B_A}.$$

We solve the integral inequality (42) as follows. Let $\epsilon > 0$ denote a small parameter. For $0 \le t \le \epsilon$,

$$\begin{aligned}
q(t) \le\ & C_{\mathcal{B}_1,\mathcal{B}_2,B_A} [\mathbb{E}\|\xi - \bar{\xi}\|^2 + t^{2\beta} \|F_2 - \bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} + t\|G - \bar{G}\|^2_{B([0,T];L_2(U;H))}] \\
& + 4c_{F_1}^2 t^{2\eta+1} \int_0^t [\iota_\beta^2 (t-s)^{-2\beta} + \iota_\eta^2 (t-s)^{-2\eta}] s^{-2\eta} ds \sup_{s\in[0,\epsilon]} q(s) \\
=\ & C_{\mathcal{B}_1,\mathcal{B}_2,B_A} [\mathbb{E}\|\xi - \bar{\xi}\|^2 + t^{2\beta} \|F_2 - \bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} + t\|G - \bar{G}\|^2_{B([0,T];L_2(U;H))}] \\
& + 4c_{F_1}^2 [\iota_\beta^2 B(1-2\eta, 1-2\beta) t^{2(1-\beta)} + \iota_\eta^2 B(1-2\eta, 1-2\eta) t^{2(1-\eta)}] \sup_{s\in[0,\epsilon]} q(s) \\
\le\ & C_{\mathcal{B}_1,\mathcal{B}_2,B_A} [\mathbb{E}\|\xi - \bar{\xi}\|^2 + \epsilon^{2\beta} \|F_2 - \bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} + \epsilon\|G - \bar{G}\|^2_{B([0,T];L_2(U;H))}] \\
& + 4c_{F_1}^2 [\iota_\beta^2 B(1-2\eta, 1-2\beta) \epsilon^{2(1-\beta)} + \iota_\eta^2 B(1-2\eta, 1-2\eta) \epsilon^{2(1-\eta)}] \sup_{s\in[0,\epsilon]} q(s).
\end{aligned}$$

Taking the supremum on both the hand sides of the above inequality, we observe that

$$\begin{aligned}
& \{1 - 4c_{F_1}^2 [\iota_\beta^2 B(1-2\eta, 1-2\beta) \epsilon^{2(1-\beta)} + \iota_\eta^2 B(1-2\eta, 1-2\eta) \epsilon^{2(1-\eta)}]\} \sup_{s\in[0,\epsilon]} q(s) \\
& \le C_{\mathcal{B}_1,\mathcal{B}_2,B_A} [\mathbb{E}\|\xi - \bar{\xi}\|^2 + \epsilon^{2\beta} \|F_2 - \bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} + \epsilon\|G - \bar{G}\|^2_{B([0,T];L_2(U;H))}].
\end{aligned}$$

If ϵ is taken sufficiently small so that

$$1 - 4c_{F_1}^2 [\iota_\beta^2 B(1-2\eta, 1-2\beta) \epsilon^{2(1-\beta)} + \iota_\eta^2 B(1-2\eta, 1-2\eta) \epsilon^{2(1-\eta)}] \ge \frac{1}{2}, \tag{43}$$

then

$$\sup_{s\in[0,\epsilon]} q(s) \le C_{\mathcal{B}_1,\mathcal{B}_2,B_A} [\mathbb{E}\|\xi - \bar{\xi}\|^2 + \epsilon^{2\beta} \|F_2 - \bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} + \epsilon\|G - \bar{G}\|^2_{B([0,T];L_2(U;H))}]. \tag{44}$$

As a consequence,

$$q(\epsilon) \le C_{\mathcal{B}_1,\mathcal{B}_2,B_A} [\mathbb{E}\|\xi - \bar{\xi}\|^2 + \epsilon^{2\beta} \|F_2 - \bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} + \epsilon\|G - \bar{G}\|^2_{B([0,T];L_2(U;H))}] \tag{45}$$

for any ϵ satisfying (43).

In the meantime, for $\epsilon < t \le T_{\mathcal{B}_1,\mathcal{B}_2,B_A}$,

$$\begin{aligned}
q(t) \le\ & C_{\mathcal{B}_1,\mathcal{B}_2,B_A} [\mathbb{E}\|\xi - \bar{\xi}\|^2 + t^{2\beta} \|F_2 - \bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} + t\|G - \bar{G}\|^2_{B([0,T];L_2(U;H))}] \\
& + 4c_{F_1}^2 t^{2\eta+1} \int_0^\epsilon [\iota_\beta^2 (t-s)^{-2\beta} + \iota_\eta^2 (t-s)^{-2\eta}] s^{-2\eta} ds \sup_{s\in[0,\epsilon]} q(s) \\
& + 4c_{F_1}^2 t^{2\eta+1} \int_\epsilon^t [\iota_\beta^2 (t-s)^{-2\beta} + \iota_\eta^2 (t-s)^{-2\eta}] s^{-2\eta} q(s) ds
\end{aligned}$$

$$
\begin{aligned}
&\leq C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|\xi-\bar{\xi}\|^2+t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}+t\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))}]\\
&\quad+4c_{F_1}^2[\iota_\beta^2 B(1-2\eta,1-2\beta)\epsilon^{2(1-\beta)}+\iota_\eta^2 B(1-2\eta,1-2\eta)\epsilon^{2(1-\eta)}]\\
&\quad\times\sup_{s\in[0,\epsilon]}q(s)\\
&\quad+4c_{F_1}^2 t^{2\eta+1}\int_\epsilon^t[\iota_\beta^2(t-s)^{2(\eta-\beta)}+\iota_\eta^2](t-s)^{-2\eta}\epsilon^{-2\eta}q(s)ds\\
&\leq C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|\xi-\bar{\xi}\|^2+t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}+t\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))}]\\
&\quad+4c_{F_1}^2[\iota_\beta^2 B(1-2\eta,1-2\beta)\epsilon^{2(1-\beta)}+\iota_\eta^2 B(1-2\eta,1-2\eta)\epsilon^{2(1-\eta)}]\\
&\quad\times\sup_{s\in[0,\epsilon]}q(s)+4c_{F_1}^2\epsilon^{-2\eta}T^{2\eta+1}[\iota_\beta^2 T^{2(\eta-\beta)}+\iota_\eta^2]\int_\epsilon^t(t-s)^{-2\eta}q(s)ds.
\end{aligned}
$$

Lemma 3.5 then provides that

$$
\begin{aligned}
q(t)&\leq C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|\xi-\bar{\xi}\|^2+t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}+t\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))}]\\
&\quad+4c_{F_1}^2[\iota_\beta^2 B(1-2\eta,1-2\beta)\epsilon^{2(1-\beta)}+\iota_\eta^2 B(1-2\eta,1-2\eta)\epsilon^{2(1-\eta)}]\\
&\quad\times\sup_{s\in[0,\epsilon]}q(s)\\
&\leq C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|\xi-\bar{\xi}\|^2+t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}+t\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))}]\\
&\quad+4c_{F_1}^2[\iota_\beta^2 B(1-2\eta,1-2\beta)T^{2(1-\beta)}+\iota_\eta^2 B(1-2\eta,1-2\eta)T^{2(1-\eta)}]\\
&\quad\times\sup_{s\in[0,\epsilon]}q(s),\quad \epsilon<t\leq T_{\mathcal{B}_1,\mathcal{B}_2,B_A}.
\end{aligned}
$$

Thanks to (44),

$$
\begin{aligned}
q(t)&\leq C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|\xi-\bar{\xi}\|^2+t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}+t\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))}]\\
&\quad+C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|\xi-\bar{\xi}\|^2+\epsilon^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}\\
&\quad+\epsilon\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))}],\quad \epsilon<t\leq T_{\mathcal{B}_1,\mathcal{B}_2,B_A}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
q(t)&\leq C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|\xi-\bar{\xi}\|^2+t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}\\
&\quad+t\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))}],\quad \epsilon<t\leq T_{\mathcal{B}_1,\mathcal{B}_2,B_A}.
\end{aligned}\tag{46}
$$

Combining (45) and (46), we conclude that

$$
\begin{aligned}
&t^{2\eta}\mathbb{E}[\|A^\eta[X(s)-\bar{X}(s)]\|^2+\|A^\beta[X(s)-\bar{X}(s)]\|^2]\\
&=q(t)\\
&\leq C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|\xi-\bar{\xi}\|^2+t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}\\
&\quad+t\|G-\bar{G}\|^2_{B([0,T];L_2(U;H))}],\quad 0<t\leq T_{\mathcal{B}_1,\mathcal{B}_2,B_A}.
\end{aligned}\tag{47}
$$

Second, let us give an estimate for $\mathbb{E}\|X(t)-\bar{X}(t)\|^2$. Taking $\theta=0$ in (41), we have

$$
\mathbb{E}\|X(t)-\bar{X}(t)\|^2\leq 4\iota_0\mathbb{E}\|\xi-\bar{\xi}\|^2+4\iota_0 B(\beta,1)^2t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}}
$$

$$+ 4\iota_0 t\|G-\bar G\|^2_{B([0,T];L_2(U;H))} + 4\iota_0^2 c_{F_1}^2 t\int_0^t s^{-2\eta}q(s)ds.$$

The term $t\int_0^t s^{-2\eta}q(s)ds$ is estimated by using (47):

$$\begin{aligned}
&t\int_0^t s^{-2\eta}q(s)ds\\
&\le C_{\mathcal{B}_1,\mathcal{B}_2,B_A} t\int_0^t s^{-2\eta}[\mathbb{E}\|\xi-\bar\xi\|^2 + s^{2\beta}\|F_2-\bar F_2\|^2_{\mathcal{F}^{\beta,\sigma}}\\
&\qquad\qquad + s\|G-\bar G\|^2_{B([0,T];L_2(U;H))}]ds\\
&\le \frac{C_{\mathcal{B}_1,\mathcal{B}_2,B_A} t^{2(1-\eta)}\mathbb{E}\|\xi-\bar\xi\|^2}{1-2\eta} + \frac{C_{\mathcal{B}_1,\mathcal{B}_2,B_A} t^{2(1+\beta-\eta)}\|F_2-\bar F_2\|^2_{\mathcal{F}^{\beta,\sigma}}}{1+2\beta-2\eta}\\
&\quad + \frac{C_{\mathcal{B}_1,\mathcal{B}_2,B_A} t^{3-2\eta}\|G-\bar G\|^2_{B([0,T];L_2(U;H))}}{2(1-\eta)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\|X(t)-\bar X(t)\|^2\\
&\le C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|\xi-\bar\xi\|^2 + t^{2\beta}\|F_2-\bar F_2\|^2_{\mathcal{F}^{\beta,\sigma}}\\
&\quad + t\|G-\bar G\|^2_{B([0,T];L_2(U;H))}], \quad 0<t\le T_{\mathcal{B}_1,\mathcal{B}_2,B_A}.
\end{aligned} \tag{48}$$

Thanks to (47) and (48), the estimate (35) has been verified.

Finally, let us prove the estimate (36). By substituting the estimate

$$\|A^\theta S(t)(\xi-\bar\xi)\| \le \iota_{\theta-\beta}t^{\beta-\theta}\|A^\beta(\xi-\bar\xi)\|$$

with $\theta=\beta$ and $\theta=\eta$ for $\|A^\theta S(t)(\xi-\bar\xi)\|\le \iota_\theta t^{-\theta}\|\xi-\bar\xi\|$, we obtain a similar result to (41):

$$\begin{aligned}
&\mathbb{E}\|t^\theta A^\theta[X(t)-\bar X(t)]\|^2\\
&\le 4\iota_{\theta-\beta}^2 t^{2(\beta-\theta)}\mathbb{E}\|A^\beta(\xi-\bar\xi)\|^2 + 4\iota_\theta^2 B(\beta,1-\theta)^2 t^{2\beta}\|F_2-\bar F_2\|^2_{\mathcal{F}^{\beta,\sigma}}\\
&\quad + \frac{4\iota_\theta^2 t}{1-2\theta}\|G-\bar G\|^2_{B([0,T];L_2(U;H))}\\
&\quad + 4\iota_\theta^2 c_{F_1}^2 t^{2\theta+1}\int_0^t (t-s)^{-2\theta}\mathbb{E}\|A^\eta[X(s)-\bar X(s)]\|^2]ds.
\end{aligned}$$

Using the same arguments as for (47), we conclude that (36) holds true. This completes the proof of the theorem. □

4. Applications to stochastic PDEs

We present some applications to stochastic PDEs. These equations are considered under Neumann or Dirichlet type boundary conditions.

4.1. Example 1

Consider the stochastic PDE:

$$(49)\quad \begin{cases} du(x,t) = \{[a(x)u'(x)]' + \frac{u}{1+u^2} + t^{\beta-1}f(t)\varphi_1(x)\}dt + g(t)\varphi_2(x)dw_t \\ \quad \text{in } (0,1)\times(0,T), \\ u'(0) = u'(1) = 0, \\ u(x,0) = u_0(x) \quad \text{in } (0,1). \end{cases}$$

Here, w is a real-valued standard Wiener process. The functions $\varphi_i (i = 1, 2)$ are real-valued and square integrable on $(0, 1)$. Meanwhile, f is a real-valued σ-Hölder continuous function on $[0, T]$ with $0 < \sigma < \beta < 1$, g is a real-valued bounded function on $[0, T]$, and a is a real-value function on $(0, 1)$ satisfying the condition

$$a \in \mathcal{C}^1([0,1]) \quad \text{and} \quad a(x) \geq a_0, \quad 0 < x < 1$$

with some constant $a_0 > 0$.

We handle the equation (49) in the Hilbert space $L_2((0, 1))$. Let A be the realization of the differential operator

$$-\frac{d}{dx}[a(x)\frac{d}{dx}] + 1$$

in $L_2((0, 1))$ under the Neumann type boundary conditions:

$$u'(0) = u'(1) = 0.$$

According to [18, Theorem 2.12], A is a sectorial operator of $L_2((0, 1))$ with domain

$$\mathcal{D}(A) = \{u \in H^2((0,1)); u'(0) = u'(1) = 0\}.$$

Using A, (49) is formulated as an abstract problem of the form (1). Here, the nonlinear operator F_1 is given by

$$F_1(u) = u + \frac{u}{1+u^2},$$

and the functions $F_2\colon (0, T] \to L_2((0, 1))$ and $G\colon [0, T] \to L_2(\mathbb{R}; L_2((0, 1)))$ are defined by

$$F_2(t) = t^{\beta-1}f(t)\varphi(x), \quad G(t) = g(t)\varphi(x).$$

Obviously,

$$G \in B([0,T]; L_2(\mathbb{R}; L_2((0,1))))$$

and

$$F_2 \in \mathcal{F}^{\beta,\sigma}((0,T]; L_2((0,1))) \quad \text{(see Remark 2.1).}$$

Let fix η such that

$$\begin{cases} 0 < \eta < \frac{1}{2}, \\ \max\{0, 2\eta - \frac{1}{2}\} < \beta < \eta. \end{cases}$$

For $u, v \in \mathcal{D}(A^\eta)$, we have

$$\begin{aligned}\|F_1(u) - F_1(u)\|_{L_2((0,1))} &= \|u - v + \frac{u}{1+u^2} - \frac{v}{1+v^2}\|_{L_2((0,1))} \\ &\leq \|u-v\|_{L_2((0,1))} + \Big\|\frac{(uv-1)(u-v)}{(1+u^2)(1+v^2)}\Big\|_{L_2((0,1))} \\ &\leq C_1\|u-v\|_{L_2((0,1))} \leq C_2\|A^\eta(u-v)\|_{L_2((0,1))}\end{aligned}$$

with some constants $C_1, C_2 > 0$.

All the structural assumptions are therefore satisfied in $L_2((0,1))$. By using Theorems 3.2, 3.3 and 3.4, we have the following results.

Claim 1 (existence of unique solutions). Let $u_0 \in \mathcal{D}(A^\beta)$ a.s. with $\mathbb{E}\|A^\beta u_0\|^2 < \infty$. Then, (49) possesses a unique local mild solution u in the function space:

$$u \in \mathcal{C}([0, T_{loc}]; \mathcal{D}(A^\beta)), \quad A^\eta u \in \mathcal{C}^\gamma((0, T_{loc}]; L_2((0,1))) \quad \text{a.s.}$$

for any $0 < \gamma < \frac{1+2\beta}{4} - \eta$. Furthermore, u satisfies the estimate:

$$\mathbb{E}\|A^\beta u(t)\|^2 + t^{2(\eta-\beta)}\mathbb{E}\|A^\eta u(t)\|^2 \leq C_{F_2,G,u_0}, \quad 0 \leq t \leq T_{loc}.$$

Here, T_{loc} and C_{F_2,G,u_0} are some constants depending on the exponents and $\mathbb{E}\|A^\beta u_0\|^2$, $\|F_2\|^2_{\mathcal{F}^{\beta,\sigma}}$, $\|G\|^2_{B([0,T];L_2(\mathbb{R};L_2((0,1))))}$.

Claim 2 (differentiability of expectation of solutions). Let $\sigma + \eta \leq \frac{1}{2}$. Put $z(t) = \mathbb{E}u(t)$, then

$$\begin{cases} z \in \mathcal{C}((0, T_{loc}]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_{loc}]; \mathcal{D}(A^\beta)) \cap \mathcal{C}^1((0, T_{loc}]; L_2((0,1))), \\ \frac{dz}{dt}, Az \in \mathcal{F}^{\beta,\sigma}((0, T_{loc}]; L_2((0,1))). \end{cases}$$

Furthermore, z satisfies the estimate:

$$\|A^\beta z\|_{\mathcal{C}} + \Big\|\frac{dz}{dt}\Big\|_{\mathcal{F}^{\beta,\sigma}} + \|Az\|_{\mathcal{F}^{\beta,\sigma}} \leq C_{F_2,u_0}, \quad 0 \leq t \leq T_{loc}$$

with some constant C_{F_2,u_0} depending on $\mathbb{E}\|A^\beta u_0\|^2$, $\|F_2\|^2_{\mathcal{F}^{\beta,\sigma}}$ and T_{loc}.

Claim 3 (continuous dependence on initial data). Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bounded balls:

$$\mathcal{B}_1 = \{f \in \mathcal{F}^{\beta,\sigma}((0,T]; L_2((0,1))); \|f\|_{\mathcal{F}^{\beta,\sigma}} \leq R_1\}, \quad 0 < R_1 < \infty,$$

$$\begin{aligned}\mathcal{B}_2 = \{g \in B([0,T]; L_2(U; L_2((0,1))));& \\ \|G\|_{B([0,T];L_2(\mathbb{R};L_2((0,1))))} \leq R_2\},& \quad 0 < R_2 < \infty,\end{aligned}$$

of the spaces $\mathcal{F}^{\beta,\sigma}((0,T]; L_2((0,1)))$ and $B([0,T]; L_2(U; H))$, respectively. Let B_A be a set of random variable:

$$B_A = \{\zeta; \zeta \in \mathcal{D}(A^\beta) \text{ a.s. and } \mathbb{E}\|A^\beta \zeta\|^2 \leq R_3^2\}, \quad 0 < R_3 < \infty.$$

Let u and $\bar{u}$ be the solutions of (49) for the data (F_2, G, u_0) and $(\bar{F}_2, \bar{G}, \bar{u}_0)$ in $\mathcal{B}_1 \times \mathcal{B}_2 \times B_A$, respectively. Then, there exist constants $T_{\mathcal{B}_1,\mathcal{B}_2,B_A}$ and $C_{\mathcal{B}_1,\mathcal{B}_2,B_A}$ depending only on $\mathcal{B}_1, \mathcal{B}_2$ and B_A such that

$$t^{2\eta}\mathbb{E}\|A^\eta[u(t) - \bar{u}(t)]\|^2 + t^{2\eta}\mathbb{E}\|A^\beta[u(t) - \bar{u}(t)]\|^2 + \mathbb{E}\|u(t) - \bar{u}(t)\|^2$$

$$\leq C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|u_0-\bar{u}_0\|^2+t^{2\beta}\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} \\ +t\|G-\bar{G}\|^2_{B([0,T];L_2(\mathbb{R};L_2((0,1))))}], \quad 0<t\leq T_{\mathcal{B}_1,\mathcal{B}_2,B_A}$$

and

$$t^{2(\eta-\beta)}[\mathbb{E}\|A^\eta[u(t)-\bar{u}(t)]\|^2+\mathbb{E}\|A^\beta[u(t)-\bar{u}(t)]\|^2] \\ \leq C_{\mathcal{B}_1,\mathcal{B}_2,B_A}[\mathbb{E}\|A^\beta(u_0-\bar{u}_0)\|^2+\|F_2-\bar{F}_2\|^2_{\mathcal{F}^{\beta,\sigma}} \\ +\|G-\bar{G}\|^2_{B([0,T];L_2(\mathbb{R};L_2((0,1))))}], \quad 0<t\leq T_{\mathcal{B}_1,\mathcal{B}_2,B_A}.$$

4.2. Example 2

Let us consider the initial value problem

$$\begin{cases} du(x,t)=\Big[\sum_{i,j=1}^n \dfrac{\partial}{\partial x_j}\Big[a_{ij}(x)\dfrac{\partial}{\partial x_i}u\Big]+f_1(u)+t^{\beta-1}f_2(t)\varphi(x)\Big]dt \\ \qquad\qquad +g(t)dW(t) \quad \text{ in } \mathbb{R}^n\times(0,T), \\ u(x,0)=u_0(x) \quad \text{ in } \mathbb{R}^n. \end{cases} \tag{50}$$

Here, W is a cylindrical Wiener process on some separable Hilbert space U; φ is a function in $H^{-1}(\mathbb{R}^n)$; f_1 is a real-valued function on $\mathbb{R}$; f_2 is a σ-Hölder continuous function on $[0,T]$ with $0<\sigma<\beta<1$; g is a $L_2(U;H^{-1}(\mathbb{R}^n))$-valued bounded function on $[0,T]$; $a_{ij}(x), 1\leq i,j\leq n$, are real-valued functions in $\mathbb{R}^n$ satisfying the conditions:

- $\sum_{i,j=1}^n a_{ij}(x)z_iz_j \geq a_0\|z\|^2, \quad z=(z_1,\dots,z_n)\in\mathbb{R}^n, \quad$ a.e. $x\in\mathbb{R}^n$ with some constant $a_0>0$
- $a_{ij}\in L_\infty(\mathbb{R}^n) \quad$ a.e. $x\in\mathbb{R}^n$

We handle (50) in the Hilbert space $H^{-1}(\mathbb{R}^n)$. Let A be the realization of the differential operator

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_j}[a_{ij}(x)\frac{\partial}{\partial x_i}]+1$$

in $H^{-1}(\mathbb{R}^n)$. Thanks to [18, Theorem 2.2], A is a sectorial operator of $H^{-1}(\mathbb{R}^n)$ with domain $\mathcal{D}(A)=H^1(\mathbb{R}^n)$. As a consequence, $(-A)$ generates an analytical semigroup on $H^{-1}(\mathbb{R}^n)$. Using A, the equation (50) is formulated as an abstract problem of the form (1), where F_1,F_2 and G are defined as follows. The nonlinear operator F_1 is given by

$$F(u)=u+f_1(u).$$

Here, we assume that this function is defined on the domain of A^η and satisfies the condition:

$$\|F_1(u)-F_1(v)\|_{H^{-1}(\mathbb{R}^n)}\leq c\|A^\eta(u-v)\|_{H^{-1}(\mathbb{R}^n)}, \quad u,v\in\mathcal{D}(A^\eta)$$

with $c>0$, $0<\eta<\frac{1}{2}$ and $\max\{0,2\eta-\frac{1}{2}\}<\beta<\eta$. The term $F_2\colon(0,T]\to H^{-1}(\mathbb{R}^n)$ is defined by

$$F_2(t)=t^{\beta-1}f_2(t)\varphi(x).$$

Finally, the term $G\colon [0,T] \to L_2(U; H^{-1}(\mathbb{R}^n))$ is defined by $G(t) = g(t)$.

As in Example 1, it is easily seen that

$$G \in B([0,T]; L_2(U; H^{-1}(\mathbb{R}^n)) \quad \text{and} \quad F_2 \in \mathcal{F}^{\beta,\sigma}((0,T]; H^{-1}(\mathbb{R}^n)).$$

Therefore, all the structural assumptions are satisfied in $H^{-1}(\mathbb{R}^n)$. By using Theorems 3.2, 3.3 and 3.4, similar claims as in Example 1 are obtained.

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References

[1] Z. Brzeźniak, *Stochastic partial differential equations in M-type 2 Banach spaces*, Potential Anal. **4** (1995), no. 1, 1–45.

[2] ———, *On stochastic convolution in Banach spaces and applications*, Stochastics Stochastics Rep. **61** (1997), no. 3-4, 245–295.

[3] R. F. Curtain and P. L. Falb, *Stochastic differential equations in Hilbert space*, J. Differential Equations **10** (1971), 412–430.

[4] G. Da Prato, F. Flandoli, E. Priola, and M. Röckner, *Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift*, Ann. Probab. **41** (2013), no. 5, 3306–3344.

[5] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge, 1992.

[6] M. Hairer, *An Introduction to Stochastic PDEs*, [arXiv:0907.4178] ArXiv e-prints, 78 pages, 2009.

[7] E. Hille, *Functional analysis and semi-groups*, American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, New York, 1948.

[8] A. Ichikawa, *Stability of semilinear stochastic evolution equations*, J. Math. Anal. Appl. **90** (1982), no. 1, 12–44.

[9] L. T. H. Nguyen, T. V. Tạ, and A. Yagi, *A quantitative investigations for ODE model describing fish schooling*, Sci. Math. Jpn. **77** (2014), 403–413.

[10] ———, *Obstacle avoidance patterns and cohesiveness of fish schooling*, J. Theoret. Biol. **406** (2016), 116–123.

[11] T. V. Tạ, *Regularity of solutions of abstract linear evolution equations*, Lith. Math. J. **56** (2016), 268–290.

[12] T. V. Tạ, L. T. H. Nguyen, and A. Yagi, *A stochastic differential equation model for foraging swarms*, [arXiv:1509.00063v1] ArXiv e-prints, 15 pages, 2015.

[13] T. V. Tạ and A. Yagi, *Abstract stochastic evolution equations in M-type 2 Banach spaces*, [arXiv:1410.0144v2] ArXiv e-prints, 71 pages, 2014.

[14] T. V. Tạ, Y. Yamamoto, and A. Yagi, *Strict solutions to stochastic parabolic evolution equations in M-type 2 Banach spaces*, 24 pages, to appear in Funkcial. Ekvac.

[15] J. M. A. M. van Neerven, M. C. Veraar, and L. Weis, *Stochastic evolution equations in UMD Banach spaces*, J. Funct. Anal. **255** (2008), no. 4, 940–993.

[16] ———, *Maximal L^p-regularity for stochastic evolution equations*, SIAM J. Math. Anal. **44** (2012), no. 3, 1372–1414.

[17] A. Yagi, *Fractional powers of operators and evolution equations of parabolic type*, Proc. Japan Acad. Ser. A Math. Sci. **64** (1988), no. 7, 227–230.

[18] ———, *Abstract Parabolic Evolution Equations and their Applications*, Springer, Berlin, 2010.

[19] K. Yosida, *On the differentiability and the representation of one-parameter semigroups of linear operators*, J. Math. Soc. Japan **1** (1948), 15–21.

Tôn Việt Tạ
Promotive Center for International Education and Research of Agriculture
Faculty of Agriculture
Kyushu University
Higashi-ku, Fukuoka 812-8581, Japan
E-mail address: tavietton@agr.kyushu-u.ac.jp