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STRONG STABILITY OF A TYPE OF JAMISON WEIGHTED SUMS FOR END RANDOM VARIABLES

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Abstract. In this paper, we consider the strong stability of a type of Jamison weighted sums, which not only extend the corresponding result of Jamison etc. [13] from i.i.d. case to END random variables, but also obtain the necessary and sufficient results. As an important consequence, we present the result of SLLN as that of i.i.d. case.

1. Introduction

Throughout this paper, we consider a sequence $\{X_n, n \geq 1\}$ of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$.

In the study of probability limit theorem, while the assumption of independence is not reasonable in real practice, many people considered the corresponding results for all kinds of dependent random variables, such as NA, PA, PQD, NQD, ND, etc. One of the important dependence structure is the extended negatively dependent structure, which was introduced by Liu [14] as the following.

Definition 1.1. Random variables $X_k, k = 1, \dots, n$ are said to be Lower Extended Negatively Dependent ($LEND$) if there is some $M > 0$ such that, for all real numbers $x_k, k = 1, \dots, n$,

$$P\left\{\bigcap_{k=1}^{n}(X_k \leq x_k)\right\} \leq M \prod_{k=1}^{n} P\{X_k \leq x_k\}; \tag{1.1}$$

they are said to be Upper Extended Negatively Dependent ($UEND$) if there is some $M > 0$ such that, for all real numbers $x_k, k = 1, \dots, n$,

$$P\left\{\bigcap_{k=1}^{n}(X_k > x_k)\right\} \leq M \prod_{k=1}^{n} P\{X_k > x_k\}; \tag{1.2}$$

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and they are said to be Extended Negatively Dependent (END) if they are both $LEND$ and $UEND$. A sequence of infinitely many random variables $\{X_k, k = 1, 2, \ldots\}$ is said to be $LEND/UEND/END$ if for each positive integer n, the random variables $X_1, X_2, \ldots, X_n$ are $LEND/UEND/END$, respectively.

The END structure covers a lot of negative dependence structures and, more interestingly, Liu [14] provided an example showing that the END structure can even reflect certain positive dependence structures. Some applications for END sequence have been found. See, for example, Liu [14] obtained the precise large deviations for dependent random variables with heavy tails. Liu [15] studied the sufficient and necessary conditions of moderate deviations for dependent random variables with heavy tails. Chen et al. [5] considered the SLLN for END random variables. Yan et al. [27] established the three series theorem for END random variables. Since END random variables are much weaker than independent random variables, NA random variables and NOD random variables, studying the limit behavior of END sequence is received more and more attentions.

In this paper, we will consider the strong stability of a type of Jamison weighted sums, which extend the corresponding result of Jamison etc. [13] from i.i.d. case to END random variables and obtain the necessary and sufficient results. At the same time, we can give the results of SLLN as those of i.i.d. case. The plan of the paper is as follows. The main results and some lemmas are presented in Section 2, and in Section 3 we give the proofs of the main results.

Throughout this paper, we note that C will be positive constants whose values are without importance, and, in addition, may change between appearances. we let x^+ and x^- denote $\max\{x, 0\}$ and $\max\{-x, 0\}$ respectively, and $a_n \asymp b_n$ means $Cb_n \le a_n \le Cb_n$. For every $c > 0$, we denote

$$X_k(c) = -cI(X_k \le -c) + X_k I(|X_k| < c) + cI(X_k \ge c);$$
$$Y_k(c) = X_k - X_k(c) = (X_k + c)I(X_k \le -c) + (X_k - c)I(X_k \ge c).$$

2. Main results and some lemmas

We first enumerate some necessary lemmas.

Lemma 2.1 (c.f. [14])**.** *Let random variables $X_1, X_2, \ldots, X_n$ be END.*

(1) *If $f_1, f_2, \ldots, f_n$ are all nondecreasing* (*or nonincreasing*) *functions, then random variables $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are END.*

(2) *For each $n \ge 1$, there exists a constant $M > 0$ such that*

$$E\left(\prod_{k=1}^{n} X_k^+\right) \le M \prod_{k=1}^{n} EX_k^+.$$

Lemma 2.2 (c.f. [27])**.** *Let $\{X_n, n \ge 1\}$ be a sequence of END random variables, $EX_n = 0$, $EX_n^2 < \infty$, $n \ge 1$ and for every $j \ge 0, k \ge 1$, $T_{j,k} =$*

$\sum_{i=j+1}^{j+k} X_i$, *then*

$$E\left(T_{j,k}\right)^2 \le C \sum_{i=j+1}^{j+k} EX_i^2; \tag{2.1}$$

$$E\left(\max_{1\le k\le n} \left(T_{j,k}\right)^2\right) \le C(\log n)^2 \sum_{i=j+1}^{j+n} EX_i^2. \tag{2.2}$$

Lemma 2.3 ([17])**.** *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space and* $A_1, A_2, \ldots$ *be a sequence of events. If* $\sum_{n=1}^{\infty} P(A_n) < \infty$*, then* $P(\limsup A_n) = 0$*. If*

$$\sum_{n=1}^{\infty} P(A_n) = \infty \tag{2.3}$$

and

$$P(A_k A_j) \le CP(A_k)P(A_j) \tag{2.4}$$

for all $k, j \ge L$ *such that* $k \ne j$ *and for some constants* $C \ge 1$ *and* L*. Then*

$$P(\limsup A_n) \ge 1/C. \tag{2.5}$$

Lemma 2.4. *Let* $\{X_n, n \ge 1\}$ *be a sequence of END random variables,* $\{w_n, n \ge 1\}$ *and* $\{W_n, n \ge 1\}$ *two sequences of positive real numbers. If*

$$W_n^{-1} \sum_{i=1}^{n} w_i X_i \to 0,\ n \to \infty,\ a.s. \tag{2.6}$$

and

$$\sup_n W_{n-1}/W_n \le G < \infty \quad \text{for some } G \in \mathbb{R}^+, \tag{2.7}$$

then for each $\epsilon > 0$*,*

$$\sum_{n=1}^{\infty} P(|X_n| \ge \epsilon w_n^{-1} W_n) < \infty. \tag{2.8}$$

Proof. By (2.6) and (2.7), it is easy to see that $w_n X_n / W_n \to 0$ a.s. as $n \to \infty$. Then

$$W_n^{-1} w_n X_n^{\pm} \to 0, \quad n \to \infty,\ a.s. \tag{2.9}$$

On the other hand, for every $\epsilon > 0$

$$\begin{aligned} &\sum_{n=1}^{\infty} P\{W_n^{-1} w_n |X_n| \ge \epsilon\} \\ \le\ &\sum_{n=1}^{\infty} P\left\{W_n^{-1} w_n X_n^{+} > \frac{1}{3}\epsilon\right\} + \sum_{n=1}^{\infty} P\left\{W_n^{-1} w_n X_n^{-} > \frac{1}{3}\epsilon\right\} \\ =:\ &\sum_{n=1}^{\infty} P\left\{A_n^{(1)}\right\} + \sum_{n=1}^{\infty} P\left\{A_n^{(2)}\right\}. \end{aligned} \tag{2.10}$$

By Lemma 2.1, it is easy to see that $\left\{W_i^{-1} w_i X_i^{\pm}\right\}$ are still to be a sequence of END random variables. Then, by the definition of END, there exists an $M > 0$ such that

$$P\left\{A_i^{(k)} A_j^{(k)}\right\} \leq MP\left\{A_i^{(k)}\right\} P\left\{A_j^{(k)}\right\}, \quad i \neq j,\ i, j \geq 1,\ k = 1, 2. \tag{2.11}$$

Without loss of generality, we assume that $M \geq 1$. If

$$\sum_{n=1}^{\infty} P\left\{A_n^{(k)}\right\} = \infty,\ k = 1, 2,$$

then, by Lemma 2.3, we have

$$P(\limsup A_n^{(k)}) \geq 1/M,\ k = 1, 2,$$

which contradicts (2.9). Therefore,

$$\sum_{n=1}^{\infty} P\left\{A_n^{(k)}\right\} < \infty,\ k = 1, 2.$$

Thus, taking into account this with (2.10), we obtained (2.8) and the proof is complete. □

Lemma 2.5 (Theorem 1.5.4 of [1])**.** *A* (*positive, measurable*) *function f is slowly varying if and only if, for every $\varepsilon > 0$, there exist a nondecreasing function ϕ and a nonincreasing function ψ with*

$$x^{\varepsilon} f(x) \sim \phi(x), \quad x^{-\varepsilon} f(x) \sim \psi(x)\ (x \to \infty).$$

In what follows, we will consider the strong stability of a type of Jamison weighted sums for END random variables. We first introduce some notations as following:

Let $\{w_n\}$ be a sequence of positive real numbers, $W_n = \sum_{i=1}^{n} w_i$, $N(n) = \#\{i : w_i^{-1} W_i \leq n\}$, $n = 1, 2, \ldots$, $N(0) = 0$. And let $\{X_n, n \geq 1\}$ be a sequence of random variables, let $T_n = W_n^{-1} \sum_{i=1}^{n} w_i X_i$, $n = 1, 2, \ldots$.

Theorem A (Jamison et al. [13])**.** *Let $\{X_n, n \geq 1\}$ be i.i.d. random variables and*

$$E|X_1| < \infty, \quad EX_1 = 0. \tag{2.12}$$

In addition,

$$W_n \to \infty, \quad W_n^{-1} w_n \to 0,\ n \to \infty, \tag{2.13}$$

$$N(n) \leq Cn, \quad n = 1, 2, \ldots. \tag{2.14}$$

Then

$$T_n \to 0,\ n \to \infty,\ a.s. \tag{2.15}$$

In the following, we will consider the corresponding results of END r.v.s with the following weights:

$$w_n = n^\gamma l(n), \quad n \geq 1, \tag{2.16}$$

where $\gamma \geq -1$, $l(x), x \in \mathbb{R}^+$ be slowly varying function, and $l(x), x \in \mathbb{R}^+$ nondecreasing for $\gamma = -1$. Obviously, for such weights, condition (2.13) and (2.14) are fulfilled. By Proposition 1.5.7 of [1] and Potter's Theorem, for each $n \geq 1$, we easily have

$$W_n \asymp n^{1+\gamma}\, l(n), \quad \gamma > -1, \tag{2.17}$$

$$W_n \leq Cl(n) \log n, \quad \gamma = -1. \tag{2.18}$$

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with identical distribution, and the weights as in* (2.16). *If* (2.12) *satisfied, then* (2.15) *holds.*

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with identical distribution satisfying* (2.15)*, where the weights satisfy* (2.13) *and*

$$N(n) \geq Cn, \quad n \geq 1. \tag{2.19}$$

Then

$$E|X_1| < \infty. \tag{2.20}$$

Remark 2.1. In the absence of some essential methods, we could not extend Theorem A to END case completely. Though weights in Theorem 2.1 are regularly varying, they have a wide scope and could contain some common form adequately. In addition, we take into account the necessity of Theorem A for END random variables.

Corollary 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with identical distribution. Then for the weights $a_n = n^\gamma l(n)$, $\gamma > -1$,*

$$(2.15) \Longleftrightarrow (2.12).$$

Corollary 2.2. *Set $\gamma = 0, l(x) \equiv 1$ in Corollary* 2.1. *Then*

$$\lim_{n\to\infty} \frac{S_n}{n} = EX_1, \quad a.s. \Longleftrightarrow E|X_1| < \infty.$$

Remark 2.2. Corollary 2.2 present the same results about the strong law of large numbers for END random variables with identical distribution as those of i.i.d. case. Moreover, a lot of other similar results can be obtained, we just list another example as the following Corollary 2.3.

Corollary 2.3. *Set $\gamma = -\frac{1}{2}, l(x) = (\log x)^{1/2}$ in Corollary* 2.1. *Then* (2.12) *is equivalent with*

$$(n \log n)^{-\frac{1}{2}} \sum_{i=1}^{n} i^{-\frac{1}{2}} (\log i)^{\frac{1}{2}} X_i \to 0, \quad n \to \infty, \ a.s.$$

3. Proof of the main results

Proof of Theorem 2.1. By (2.12) and (2.14), we have

$$\begin{aligned}\sum_{i=1}^{\infty} P\{|X_i| \geq w_i^{-1}W_i\} &= \sum_{j=1}^{\infty} \sum_{j-1<w_i^{-1}W_i\leq j} P\{|X_1| \geq w_i^{-1}W_i\} \\ &\leq \sum_{j=1}^{\infty} P\{|X_1| > j-1\}\,(N(j)-N(j-1)) \\ &\leq C\sum_{j=1}^{\infty} N(j)P\{j-1<|X_1|\leq j\} \\ &\leq C\sum_{j=1}^{\infty} jP\{j-1<|X_1|\leq j\} \leq CE|X_1| < \infty.\end{aligned}$$

Thus

$$P\{|X_i| \geq w_i^{-1}W_i,\ \text{i.o.}\} = 0.$$

Therefore, in order to get (2.15), it needs only to prove

$$\widetilde{T}_n \triangleq W_n^{-1}\sum_{i=1}^{n} w_i X_i(w_i^{-1}W_i) \to 0,\ n\to\infty,\ \text{a.s.} \tag{3.1}$$

Since $EX_1 = 0$, it suffices to show that

$$\widetilde{T}_n^+ \triangleq W_n^{-1}\sum_{i=1}^{n} w_i X_i^+(w_i^{-1}W_i) \to EX_1^+,\ n\to\infty,\ \text{a.s.} \tag{3.2}$$

$$\widetilde{T}_n^- \triangleq W_n^{-1}\sum_{i=1}^{n} w_i X_i^-(w_i^{-1}W_i) \to EX_1^-,\ n\to\infty,\ \text{a.s.} \tag{3.3}$$

The derivations of these two results are similar, so we only present the proof of (3.2). For arbitrary $\alpha > 1$, set $k_n = [\alpha^n], n \geq 1$ ($[\cdot]$ stands for integer part of a number). We first show that

$$\widetilde{T}_{k_n}^+ \to EX_1^+, \quad n\to\infty,\ \text{a.s.} \tag{3.4}$$

By (2.17), (2.18) and Lemma 2.5, we see that $w_i^{-1}W_i$ is quasi-monotone non-decreasing and tends to ∞. Thus

$$\limsup_{n\to\infty} E\widetilde{T}_{k_n}^+ \leq \lim_{n\to\infty} EX_1^+(w_{k_n}^{-1}W_{k_n}) = EX_1^+. \tag{3.5}$$

On the other hand, for each $n \geq 1$,

$$\begin{aligned}E\widetilde{T}_{k_n}^+ &= W_{k_n}^{-1}\sum_{i=1}^{k_n} w_i E\left[X_i^+ I(X_i^+ \leq W_i w_i^{-1}) + W_i w_i^{-1} I(X_i > W_i w_i^{-1})\right] \\ &= W_{k_n}^{-1}\sum_{i=1}^{k_n} w_i EX_1^+ I(X_1^+ \leq W_i w_i^{-1}) + W_{k_n}^{-1}\sum_{i=1}^{k_n} W_i P\{X_1 > W_i w_i^{-1}\}\end{aligned}$$

$$\geq W_{k_n}^{-1}\left(\sum_{i=1}^{\varepsilon_n}+\sum_{i=\varepsilon_n+1}^{k_n}\right)w_i EX_1^+ I(X_1^+\leq W_i w_i^{-1})$$

$$\geq W_{k_n}^{-1}\sum_{i=\varepsilon_n+1}^{k_n} w_i EX_1^+ I(X_1^+\leq W_{\varepsilon_n} w_{\varepsilon_n}^{-1})$$

$$= (1-W_{\varepsilon_n}W_{k_n}^{-1})EX_1^+ I(X_1^+\leq W_{\varepsilon_n} w_{\varepsilon_n}^{-1}), \tag{3.6}$$

where $\varepsilon_n=\begin{cases}[k_n^{1/2}], & \gamma>-1,\\ [\log k_n], & \gamma=-1.\end{cases}$

In fact, when $\gamma>-1$, by (2.17) and Potter's Theorem

$$W_{\varepsilon_n}W_{k_n}^{-1}\leq Ck_n^{-(1+\gamma)/2}\,\frac{l(k_n^{1/2})}{l(k_n)}\to 0,\quad n\to\infty. \tag{3.7}$$

And for $\gamma=-1$, taking into account monotonicity of $l(x), x\in\mathbb{R}^+$, then by (2.18) we have

$$W_{\varepsilon_n}W_{k_n}^{-1}\leq C\frac{\log\log k_n}{\log k_n}\cdot\frac{l(\log k_n)}{l(k_n)}\to 0,\quad n\to\infty. \tag{3.8}$$

Combine (3.7), (3.8) and (3.6) we see that

$$\liminf_{n\to\infty} E\widetilde{T}_{k_n}^+\geq\liminf_{n\to\infty}(1-W_{\varepsilon_n}W_{k_n}^{-1})EX_1^+ I(X_1^+\leq W_{\varepsilon_n} w_{\varepsilon_n}^{-1})=EX_1^+. \tag{3.9}$$

By (3.5) and (3.9), we get

$$\lim_{n\to\infty} E\widetilde{T}_{k_n}^+=EX_1^+. \tag{3.10}$$

Therefore, to get (3.4), we need only prove

$$\overline{T}_{k_n}^+\triangleq W_{k_n}^{-1}\sum_{i=1}^{k_n} w_i\left(X_i^+(W_i w_i^{-1})-EX_i^+(W_i w_i^{-1})\right)\to 0,\ n\to\infty,\ \text{a.s.} \tag{3.11}$$

It suffices to show that, for every $\varepsilon>0$,

$$\sum_{n=1}^{\infty}P\left\{\left|\overline{T}_{k_n}^+\right|\geq\varepsilon\right\}<\infty. \tag{3.12}$$

By Lemma 2.1, we see that $\{X_i^+(W_i w_i^{-1})\}$ is still be a sequence of END random variables, then by Tchebyshev inequality and Lemma 2.2 we have

$$\begin{aligned}\sum_{n=1}^{\infty}P\left\{\left|\overline{T}_{k_n}^+\right|\geq\varepsilon\right\}&\leq C\sum_{n=1}^{\infty}E\left|\overline{T}_{k_n}^+\right|^2\\&\leq C\sum_{n=1}^{\infty}W_{k_n}^{-2}\sum_{i=1}^{k_n}w_i^2 E\left(X_i^+(W_i w_i^{-1})\right)^2\\&= C\sum_{n=1}^{\infty}W_{k_n}^{-2}\sum_{i=1}^{k_n}w_i^2 E(X_1^+)^2 I(X_1^+\leq W_i w_i^{-1})\end{aligned}$$

$$+ C\sum_{n=1}^{\infty} W_{k_n}^{-2} \sum_{i=1}^{k_n} W_i^2 P\{X_1^+ > W_i w_i^{-1}\}$$

$$=: I_1 + I_2. \tag{3.13}$$

Therefore, in order to get (3.12), it needs only to prove that $I_i < \infty$, $i = 1, 2$ for $\gamma > -1$ and $\gamma = -1$ respectively.

First for $\gamma > -1$. By (2.16), (2.17), (2.14) and (2.12),

$$\begin{aligned}
I_1 &= C\sum_{i=1}^{\infty} w_i^2 E(X_1^+)^2 I(X_1^+ \le W_i w_i^{-1}) \sum_{n:k_n \ge i} W_{k_n}^{-2} \\
&\le C\sum_{i=1}^{\infty} w_i^2 E(X_1^+)^2 I(X_1^+ \le W_i w_i^{-1}) W_i^{-2} \\
&= C\sum_{j=1}^{\infty} \sum_{j-1 < W_i w_i^{-1} \le j} w_i^2 W_i^{-2} E(X_1^+)^2 I(X_1^+ \le W_i w_i^{-1}) \\
&\le C\sum_{j=2}^{\infty} (j-1)^{-2} E(X_1^+)^2 I(X_1^+ \le j)(N(j) - N(j-1)) \\
&\le C\sum_{j=1}^{\infty} P\{X_1^+ \le j\}(N(j) - N(j-1)) \\
&\le C\sum_{j=1}^{\infty} N(j) P\{j-1 < X_1^+ \le j\} \\
&\le C\sum_{j=1}^{\infty} j P\{j-1 < X_1^+ \le j\} \le CEX_1^+ < \infty,
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
I_2 &= C\sum_{i=1}^{\infty} W_i^2 P\{X_1^+ > W_i w_i^{-1}\} \sum_{n:k_n \ge i} W_{k_n}^{-2} \\
&\le C\sum_{i=1}^{\infty} P\{X_1^+ > W_i w_i^{-1}\} \le EX_1^+ < \infty.
\end{aligned} \tag{3.15}$$

Then, by (3.14) and (3.15) we get (3.12) for $\gamma > -1$. Similar for the case of $\gamma = -1$. Thus, we get (3.4). Next for (3.2).

For each $n \ge 1$, there exists positive number $f(n)$ satisfied $f(n) \to \infty$ and

$$k_{f(n)-1} = \left[\alpha^{f(n)-1}\right] < n \le \left[\alpha^{f(n)}\right] = k_{f(n)}. \tag{3.16}$$

Moreover,

$$W_n^{-1} \sum_{i=1}^{n} w_i X_i^+(W_i w_i^{-1}) \le W_{k_{f(n)-1}}^{-1} \sum_{i=1}^{k_{f(n)}} w_i X_i^+(W_i w_i^{-1})$$

$$= W_{k_{f(n)}} W_{k_{f(n)-1}}^{-1} W_{k_{f(n)}}^{-1} \sum_{i=1}^{k_{f(n)}} w_i X_i^+ (W_i w_i^{-1}). \tag{3.17}$$

Case I: $\gamma > -1$. By (3.4) and O'Stolz Theorem

$$\begin{aligned} &\limsup_{n\to\infty} W_n^{-1} \sum_{i=1}^{n} w_i X_i^+ (W_i w_i^{-1}) \\ &\le \limsup_{n\to\infty} W_{k_{f(n)}} W_{k_{f(n)-1}}^{-1} E X_1^+ \\ &\le \alpha^{1+\gamma} E X_1^+ \quad \text{a.s.} \end{aligned} \tag{3.18}$$

Case II: $\gamma = -1$. Similarly we can have

$$\begin{aligned} &\limsup_{n\to\infty} W_n^{-1} \sum_{i=1}^{n} w_i X_i^+ (W_i w_i^{-1}) \\ &\le \limsup_{n\to\infty} \log \alpha^{f(n)} (l(\alpha^{f(n)-1}))^{-1} l(\alpha^{f(n)}) (\log \alpha^{f(n)-1})^{-1} E X_1^+ \\ &= E X_1^+ \quad \text{a.s.} \end{aligned} \tag{3.19}$$

Since α may be arbitrarily close to 1, then by (3.18) and (3.19), we get that

$$\limsup_{n\to\infty} W_n^{-1} \sum_{i=1}^{n} w_i X_i^+ (W_i w_i^{-1}) \le E X_1^+ \quad \text{a.s.}$$

Similarly we have

$$\liminf_{n\to\infty} W_n^{-1} \sum_{i=1}^{n} w_i X_i^+ (W_i w_i^{-1}) \ge E X_1^+ \quad \text{a.s.}$$

Thus we get (3.2), and then (3.1), which complete the proof. $\square$

Proof of Theorem 2.2. By Lemma 2.4, we have

$$\sum_{n=1}^{\infty} P(|X_n| \ge \epsilon w_n^{-1} W_n) < \infty, \quad \forall\ \epsilon > 0.$$

Combine this with (2.19), we get

$$\begin{aligned} \infty &> \sum_{i=1}^{\infty} P\{W_i^{-1} w_i |X_i| \ge \epsilon\} = \sum_{j=1}^{\infty} \sum_{j-1 < W_i w_i^{-1} \le j} P\{|X_1| \ge \epsilon W_i w_i^{-1}\} \\ &\ge \sum_{j=1}^{\infty} (N(j) - N(j-1)) P\{|X_1| > \epsilon j\} \\ &\ge C \sum_{j=1}^{\infty} N(j) (P\{\epsilon(j-1) \le |X_1| < \epsilon j\}) \end{aligned}$$

$$\geq C\sum_{j=1}^{\infty} j(P\{\epsilon(j-1) \leq |X_1| < \epsilon j\}) \geq CE|X_1|,$$

(2.20) obtained and the proof completed. □

The proofs of Corollary 2.1, Corollary 2.2 and Corollary 2.3 are obvious, we omitted.

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