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MAPS PRESERVING η-PRODUCT $A^*B + \eta BA^*$ ON C^*-ALGEBRAS

Vahid Darvish, Haji Mohammad Nazari, Hamid Rohi, and Ali Taghavi

Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras such that $\mathcal{A}$ is prime. In this paper, we investigate the additivity of maps Φ from $\mathcal{A}$ onto $\mathcal{B}$ that are bijective and satisfy

$$\Phi(A^*B + \eta BA^*) = \Phi(A)^*\Phi(B) + \eta\Phi(B)\Phi(A)^*$$

for all $A, B \in \mathcal{A}$ where η is a non-zero scalar such that $\eta \neq \pm 1$. Moreover, if $\Phi(I)$ is a projection, then Φ is a $*$-isomorphism.

1. Introduction

Let $\mathcal{R}$ and $\mathcal{R}^{'}$ be rings. We say the map $\Phi : \mathcal{R} \to \mathcal{R}^{'}$ preserves product or is multiplicative if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{R}$. The question of when a product preserving or multiplicative map is additive was discussed by several authors, see [13] and references therein. Motivated by this, many authors pay more attention to the map on rings (and algebras) preserving the Lie product $[A, B] = AB - BA$ or the Jordan product $A \circ B = AB + BA$ (for example, Refs. ref [1, 2, 3, 6, 7, 10, 11, 12, 17, 19]). These results show that, in some sense, the Jordan product or Lie product structure is enough to determine the ring or algebraic structure. Historically, many mathematicians devoted themselves to the study of additive or linear Jordan or Lie product preservers between rings or operator algebras. Such maps are always called Jordan homomorphism or Lie homomorphism [13, 14, 15].

Let $\mathcal{R}$ be a $*$-ring. For $A, B \in \mathcal{R}$, denoted by $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, which are two different kinds of new products. This product is found playing a more and more important role in some research topics, and its study has recently attracted many author's attention (for example, see [4, 16, 20]). A natural problem is to study whether the map Φ preserving the new product on ring or algebra $\mathcal{R}$ is a ring or algebraic isomorphism. In [5], J. Cui and C. K. Li proved a bijective map Φ on factor von Neumann algebras which preserves $[A, B]_*$ must be a $*$-isomorphism. Moreover, in [8] C.

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Li et al., discussed the non-linear bijective map preserving $A \bullet B$ is also $*$-ring isomorphism. Also, in [22], A. Taghavi et al., proved a bijective unital map (not necessarily linear) which preserves $AP \pm PA^*$ for projection operators P must be $*$-additive (i.e., additive and star-preserving) on prime C^*-algebra.

Recently, L. Liu and G. X. Ji [9] proved that a bijective map Φ on factor von Neumann algebras preserves, $A^*B + BA^*$ if and only if Φ is a $*$-isomorphism.

Moreover, in [21] A. Taghavi proved that if $\Phi : \mathcal{A} \to \mathcal{B}$ is an additive surjective mapping satisfying $\Phi(|A|) = |\Phi(A)|$ for every $A \in \mathcal{A}$ and $\Phi(I)$ is a projection, then the restriction of mapping Φ to both $\mathcal{A}_s$ and $\mathcal{A}_{sk}$ is a Jordan $*$-homomorphism onto corresponding set in $\mathcal{B}$, where $\mathcal{A}_s$ and $\mathcal{A}_{sk}$ denote the set of all self-adjoint and skew-adjoint elements, respectively. Furthermore, if $\mathcal{B}$ is a C^*-algebra of real-rank zero, then Φ is a $\mathbb{C}$-linear or $\mathbb{C}$-antilinear $*$-homomorphism.

In this paper, motivated by above results, we discuss such a bijective map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ on prime C^*-algebras which preserves $A^*B + \eta BA^*$, where η is a non-zero scalar such that $\eta \neq \pm 1$, is additive. Moreover, if $\Phi(I)$ is a projection, then Φ is a $*$-isomorphism.

It is well known that C^*-algebra $\mathcal{A}$ is prime, in the sense that $A\mathcal{A}B = 0$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$.

2. Main results

We need the following lemmas for proving our theorems.

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras with identities and $\Phi : \mathcal{A} \to \mathcal{B}$ be a map which satisfies $\Phi(A^*B + \eta BA^*) = \Phi(A)^*\Phi(B) + \eta\Phi(B)\Phi(A)^*$ for all $A, B \in \mathcal{A}$ and a scalar η such that 0 or $I \in \Phi(\mathcal{A})$. Then, we have $\Phi(0) = 0$.*

Proof. Let $\Phi(A^*B + \eta BA^*) = \Phi(A)^*\Phi(B) + \eta\Phi(B)\Phi(A)^*$ for all $A, B \in \mathcal{A}$. We assume $B = 0$, so we have $\Phi(0) = \Phi(A)^*\Phi(0) + \eta\Phi(0)\Phi(A)^*$. Since there is an operator A such that $\Phi(A) = 0$ or $\Phi(A) = I$, we obtain $\Phi(0) = 0$. $\square$

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras with identities and $\Phi : \mathcal{A} \to \mathcal{B}$ be a map which satisfies $\Phi(A^*B + \eta BA^*) = \Phi(A)^*\Phi(B) + \eta\Phi(B)\Phi(A)^*$ for all $A, B \in \mathcal{A}$ and a non-zero scalar η. Let $\Phi(T) = \Phi(A) + \Phi(B)$, where $T, A, B \in \mathcal{A}$. Then, we have*

$$\Phi(XT + \eta TX) = \Phi(XA + \eta AX) + \Phi(XB + \eta BX) \tag{2.1}$$

for all $X \in \mathcal{A}$.

Proof. We multiply $\Phi(T) = \Phi(A) + \Phi(B)$ by $\Phi(X)^*$ from the left and by $\eta\Phi(X)^*$ from the right, respectively and adding together we have

$$\begin{aligned}\Phi(X)^*\Phi(T) + \eta\Phi(T)\Phi(X)^* = {} & \Phi(X)^*\Phi(A) + \eta\Phi(A)\Phi(X)^* \\ & + \Phi(X)^*\Phi(B) + \eta\Phi(B)\Phi(X)^*.\end{aligned}$$

Now, we change X^* to X, so we have (2.1). $\square$

Here we prove additivity of Φ.

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras such that $\mathcal{A}$ is prime with $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ the identities of them, respectively. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a bijective map which satisfies $\Phi(A^*B + \eta BA^*) = \Phi(A)^*\Phi(B) + \eta\Phi(B)\Phi(A)^*$ for all $A, B \in \mathcal{A}$ and η is a non-zero scalar such that $\eta \neq \pm 1$, then Φ is additive.*

Proof of Main Theorem. Let P_1 be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j,\ i, j = 1, 2$, then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$ we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follow, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$.

For showing additivity of Φ on $\mathcal{A}$ we use above partition of $\mathcal{A}$ and give some claims that prove Φ is additive on each $\mathcal{A}_{ij},\ i, j = 1, 2$.

Claim 1. *For every $A_{ii} \in \mathcal{A}_{ii}$ and $D_{jj} \in \mathcal{A}_{jj}$ such that $i \neq j$, we have*
$$\Phi(A_{ii} + D_{jj}) = \Phi(A_{ii}) + \Phi(D_{jj}).$$

Since Φ is surjective, we can find an element $T = T_{ii} + T_{ij} + T_{ji} + T_{jj} \in \mathcal{A}$ such that
$$\Phi(T) = \Phi(A_{ii}) + \Phi(D_{jj}), \tag{2.2}$$
we should show $T = A_{ii} + D_{jj}$.

We apply Lemma 2.2 to (2.2) for $X = P_i$, then we can write
$$\Phi(P_iT + \eta TP_i) = \Phi(P_iA_{ii} + \eta A_{ii}P_i) + \Phi(P_iD_{jj} + \eta D_{jj}P_i),$$
equivalently,
$$\Phi(T_{ii} + T_{ij} + \eta T_{ii} + \eta T_{ji}) = \Phi(A_{ii} + \eta A_{ii}) + \Phi(0),$$
by injectivity of Φ and Lemma 2.1, we get $T_{ii} + T_{ij} + \eta T_{ii} + \eta T_{ji} = A_{ii} + \eta A_{ii}$. Since $\eta \neq 0$ we obtain from the latter equality $T_{ij} = T_{ji} = 0$. Also, from $(1 + \eta)(T_{ii} - A_{ii}) = 0$ we have $T_{ii} = A_{ii}$, since $\eta \neq -1$.

Similarly, we can apply Lemma 2.2 to (2.2) for $X = P_j$, then we can write
$$\Phi(P_jT + \eta TP_j) = \Phi(P_jA_{ii} + \eta A_{ii}P_j) + \Phi(P_jD_{jj} + \eta D_{jj}P_j),$$
or
$$\Phi(T_{jj} + T_{ji} + \eta T_{jj} + \eta T_{ij}) = \Phi(0) + \Phi(D_{jj} + \eta D_{jj})$$
by injectivity of Φ and Lemma 2.1, we have $T_{jj} + T_{ji} + \eta T_{jj} + \eta T_{ij} = D_{jj} + \eta D_{jj}$. It follows $(1 + \eta)(T_{jj} - D_{jj}) = 0$. Therefore, $T = A_{ii} + D_{jj}$.

Claim 2. *For every $A_{ii} \in \mathcal{A}_{ii}$, $B_{ij} \in \mathcal{A}_{ij}$, we have*
$$\Phi(A_{ii} + B_{ij}) = \Phi(A_{ii}) + \Phi(B_{ij}).$$

Let $T = T_{ii} + T_{ij} + T_{ji} + T_{jj} \in \mathcal{A}$ be such that
$$\Phi(T) = \Phi(A_{ii}) + \Phi(B_{ij}). \tag{2.3}$$
By applying Lemma 2.2 to (2.3) for $X = P_j$, we have
$$\Phi(P_jT + \eta TP_j) = \Phi(P_jA_{ii} + \eta A_{ii}P_j) + \Phi(P_jB_{ij} + \eta B_{ij}P_j),$$

or

$$\Phi(T_{ji}+T_{jj}+\eta T_{ij}+\eta T_{jj})=\Phi(0)+\Phi(\eta B_{jj}).$$

By injectivity, we can say

$$T_{ji}+T_{jj}+\eta T_{ij}+\eta T_{jj}=\eta B_{ij}.$$

It follows that $T_{ji}=T_{jj}=0$ and $T_{ij}=B_{ij}$.

Now, we apply Lemma 2.2 to (2.3) for X_{ij}, so we have

$$\Phi(X_{ij}T+\eta TX_{ij}=\Phi(X_{ij}A_{ii}+\eta A_{ii}X_{ij})+\Phi(X_{ij}B_{ij}+\eta B_{ij}X_{ij}),$$

equivalently

$$\Phi(X_{ij}T_{ji}+X_{ij}T_{jj}+\eta T_{ii}X_{ij}+\eta T_{ji}X_{ij})=\Phi(\eta A_{ii}X_{ij})+\Phi(0).$$

Since Φ is injective we have $\eta T_{ii}X_{ij}=\eta A_{ii}X_{ij}$. So, for each $X_{ij}\in\mathcal{A}_{ij}$ we have $T_{ii}X_{ij}=A_{ii}X_{ij}$. Since $\mathcal{A}_{ij}$ is prime, we can obtain $T_{ii}=A_{ii}$. Therefore, we showed $T=A_{ii}+B_{ij}$.

Claim 3. *For every $A_{ii}\in\mathcal{A}_{ii}$ and $C_{ji}\in\mathcal{A}_{ji}$, we have*

$$\Phi(A_{ii}+C_{ji})=\Phi(A_{ii})+\Phi(C_{ji}).$$

Since Φ is surjective, we can find an element $T=T_{ii}+T_{ij}+T_{ji}+T_{jj}\in\mathcal{A}$ such that

$$\Phi(T)=\Phi(A_{ii})+\Phi(C_{ji}), \tag{2.4}$$

we should show $T=A_{ii}+C_{ji}$. We apply Lemma 2.2 to (2.4) for P_j, then we have

$$\Phi(P_jT+\eta TP_j)=\Phi(P_jA_{ii}+\eta A_{ii}P_j)+\Phi(P_jC_{ji}+\eta C_{ji}P_j).$$

So, by the above equation and injectivity, we have

$$T_{ji}+T_{jj}+\eta T_{ij}+\eta T_{jj}=C_{ji},$$

hence, $T_{jj}=T_{ij}=0$ and $T_{ji}=C_{ji}$.

On the other hand, we apply Lemma 2.2 to (2.4) for X_{ji}, we have

$$\Phi(X_{ji}T+\eta TX_{ji})=\Phi(X_{ji}A_{ii}+\eta A_{ii}X_{ji})+\Phi(X_{ji}C_{ji}+\eta C_{ji}X_{ji}).$$

By similar calculations as above, We obtain $X_{ji}T_{ii}=X_{ji}A_{ii}$. We have $T_{ii}=A_{ii}$ by primeness of A_{ji}.

Claim 4. *For every $B_{ij}\in\mathcal{A}_{ij}$ and $C_{ji}\in\mathcal{A}_{ji}$, we have*

$$\Phi(B_{ij}+C_{ji})=\Phi(B_{ij})+\Phi(C_{ji}).$$

Let $T=T_{ii}+T_{ij}+T_{ji}+T_{jj}\in\mathcal{A}$ be such that

$$\Phi(T)=\Phi(B_{ij})+\Phi(C_{ji}). \tag{2.5}$$

By applying Lemma 2.2 to (2.5) for X_{ij}, we get

$$\Phi(X_{ij}T+\eta TX_{ij})=\Phi(X_{ij}B_{ij}+\eta B_{ij}X_{ij})+\Phi(X_{ij}C_{ji}+\eta C_{ji}X_{ij}).$$

It implies that

$$\Phi(X_{ij}T_{ji}+X_{ij}T_{jj}+\eta T_{ii}X_{ij}+\eta T_{ji}X_{ij})=\Phi(0)+\Phi(X_{ij}C_{ji}+\eta C_{ji}X_{ij}).$$

It follows that

$$X_{ij}T_{ji} + X_{ij}T_{jj} + \eta T_{ii}X_{ij} + \eta T_{ji}X_{ij} = X_{ij}C_{ji} + \eta C_{ji}X_{ij}. \tag{2.6}$$

Multiply the above equation by P_j from the left, we obtain $\eta T_{ji}X_{ij} = \eta C_{ji}X_{ij}$, so by the primeness of $\mathcal{A}_{ij}$ we have $T_{ji} = C_{ji}$.

On the other hand, multiply equation (2.6) by P_i from the left and by P_j from the right, respectively. We have

$$X_{ij}T_{jj} + \eta T_{ii}X_{ij} = 0. \tag{2.7}$$

Similar to above calculations, applying Lemma 2.2 to (2.5) for X_{ji}, we can obtain $T_{ij} = B_{ij}$ and

$$X_{ji}T_{ii} + \eta T_{jj}X_{ji} = 0. \tag{2.8}$$

Multiply the above equation by X_{ij} from left, so $X_{ij}X_{ji}T_{ii} + \eta X_{ij}T_{jj}X_{ji} = 0$. Now, by (2.7) we obtain $X_{ij}X_{ji}T_{ii} - \eta^2 T_{ii}X_{ij}X_{ji} = 0$, or

$$X_{ii}T_{ii} - \eta^2 T_{ii}X_{ii} = 0.$$

Since the above equation is true for each X_{ii}, we can assume $X_{ii} = P_i$. Therefore, $(1 - \eta^2)T_{ii} = 0$, since $\eta \neq \pm 1$ then $T_{ii} = 0$. We put $T_{ii} = 0$ in equation (2.7) we obtain $T_{jj} = 0$.

Claim 5. *For every $A_{ii} \in \mathcal{A}_{ii}$, $B_{ij} \in \mathcal{A}_{ij}$, $C_{ji} \in \mathcal{A}_{ji}$, we have*

$$\Phi(A_{ii} + B_{ij} + C_{ji}) = \Phi(A_{ii}) + \Phi(B_{ij}) + \Phi(C_{ji}).$$

Let $T = T_{ii} + T_{ij} + T_{ji} + T_{jj} \in \mathcal{A}$ be such that

$$\Phi(T) = \Phi(A_{ii}) + \Phi(B_{ij}) + \Phi(C_{ji}). \tag{2.9}$$

By applying Lemma 2.2 to (2.9) for P_j, by Claim 4, we have

$$\begin{aligned}\Phi(P_jT + \eta TP_j) &= \Phi(P_jA_{ii}+\eta A_{ii}P_j)+\Phi(P_jB_{ij}+\eta B_{ij}P_j)+\Phi(P_jC_{ji}+\eta C_{ji}P_j)\\ &= \Phi(0) + \Phi(\eta B_{ij}) + \Phi(C_{ji})\\ &= \Phi(\eta B_{ij} + C_{ji}).\end{aligned}$$

Thus, by the injectivity of Φ we have $T_{ji} + T_{jj} + \eta T_{ij} + \eta T_{jj} = C_{ji} + \eta B_{ij}$. It follows that $T_{jj} = 0, T_{ij} = B_{ij}$ and $T_{ji} = C_{ji}$.

On the other hand, by Claim 2 we can write

$$\Phi(T) = \Phi(A_{ii} + B_{ij}) + \Phi(C_{ji}). \tag{2.10}$$

Now, we apply Lemma 2.2 to (2.10) for X_{ji}. So,

$$\Phi(X_{ji}T + \eta TX_{ji}) = \Phi(X_{ji}(A_{ii} + B_{ij}) + \eta(A_{ii} + B_{ij})X_{ji}) + \Phi(0).$$

Hence,

$$X_{ji}T + \eta TX_{ji} = X_{ji}A_{ii} + X_{ji}B_{ij} + \eta B_{ij}X_{ji}.$$

Multiply the above equation by P_j from the left and by P_i from the right, respectively. We obtain $X_{ji}T_{ii} = X_{ji}A_{ii}$. Since A_{ji} is prime we have $T_{ii} = A_{ii}$.

Claim 6. *For every $A_{ii} \in \mathcal{A}_{ii}$, $B_{ij} \in \mathcal{A}_{ij}$, $C_{ji} \in \mathcal{A}_{ji}$ and $D_{jj} \in \mathcal{A}_{jj}$ we have*

$$\Phi(A_{ii} + B_{ij} + C_{ji} + D_{jj}) = \Phi(A_{ii}) + \Phi(B_{ij}) + \Phi(C_{ji}) + \Phi(D_{jj}).$$

Assume $T = T_{ii} + T_{ij} + T_{ji} + T_{jj}$ which satisfies in

$$\Phi(T) = \Phi(A_{ii}) + \Phi(B_{ij}) + \Phi(C_{ji}) + \Phi(D_{jj}). \tag{2.11}$$

By using Lemma 2.2 to (2.11) for P_i, and Claim 5, we obtain

$$\begin{aligned}\Phi(P_iT + \eta TP_i) &= \Phi(A_{ii} + \eta A_{ii}) + \Phi(B_{ij}) + \Phi(\eta C_{ji})\\ &= \Phi(A_{ii} + \eta A_{ii} + B_{ij} + \eta C_{ji}).\end{aligned}$$

Since Φ is injective we have

$$T_{ii} + T_{ij} + \eta T_{ii} + \eta T_{ji} = A_{ii} + \eta A_{ii} + B_{ij} + \eta C_{ji}.$$

We obtain $T_{ii} = A_{ii}$, $T_{ij} = B_{ij}$ and $T_{ji} = C_{ji}$. Similarly, apply Lemma 2.2 to (2.11) for P_j and the same computation as above we can easily obtain $T_{jj} = D_{jj}$.

Claim 7. *For every $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ such that $1 \le i \ne j \le 2$, we have*

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

Let $S_{ij} = \frac{1}{\eta}A_{ij}$ and $T_{ij} = \frac{1}{\eta}B_{ij}$. So, by the assumptions of Theorem 2.3 and Claim 2, we have

$$\begin{aligned}&\Phi(A_{ij} + B_{ij})\\ &= \Phi(\eta S_{ij} + \eta T_{ij})\\ &= \Phi[(S_{ij}^* + P_j)^*(T_{ij} + P_i) + \eta(T_{ij} + P_i)(S_{ij}^* + P_j)^*]\\ &= \Phi(S_{ij}^* + P_j)^*\Phi(T_{ij} + P_i) + \eta\Phi(T_{ij} + P_i)\Phi(S_{ij}^* + P_j)^*\\ &= [\Phi(S_{ij}^*)^* + \Phi(P_j)^*][\Phi(T_{ij}) + \Phi(P_i)]\\ &\quad + \eta[\Phi(T_{ij}) + \Phi(P_i)][\Phi(S_{ij}^*)^* + \Phi(P_j)^*]\\ &= \Phi(S_{ij}^*)^*\Phi(T_{ij} + \Phi(S_{ij}^*)^*\Phi(P_i) + \Phi(P_j)^*\Phi(T_{ij}) + \Phi(P_j)^*\Phi(P_i)\\ &\quad + \eta[\Phi(T_{ij})\Phi(S_{ij}^*)^* + \Phi(P_i)\Phi(S_{ij}^*)^* + \Phi(T_{ij})\Phi(P_j)^* + \Phi(P_i)\Phi(P_j)^*]\\ &= \Phi(S_{ij}T_{ij} + \eta T_{ij}S_{ij}) + \Phi(S_{ij}P_i + \eta P_iS_{ij})\\ &\quad + \Phi(P_jT_{ij} + \eta T_{ij}P_j) + \Phi(P_jP_i + \eta P_iP_j)\\ &= \Phi(\eta S_{ij}) + \Phi(\eta T_{ij})\\ &= \Phi(A_{ij}) + \Phi(B_{ij}).\end{aligned}$$

Claim 8. *For every $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$, $1 \le i \le 2$ we have*

$$\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

Let $\Phi(T) = \Phi(A_{ii}) + \Phi(B_{ii})$. We apply Lemma 2.2 to the latter equation for P_j, we have

$$\Phi(P_jT + \eta TP_j) = 0.$$

It follows that $T_{ij} = T_{ji} = T_{jj} = 0$ and $T = T_{ii}$.

Now, we use Lemma 2.2 to $\Phi(T_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii})$ for X_{ji}. By Claim 7 we obtain

$$\begin{aligned}\Phi(X_{ji}T_{ii}) &= \Phi(X_{ji}A_{ii}) + \Phi(X_{ji}B_{ii})\\ &= \Phi(X_{ji}A_{ii} + X_{ji}B_{ii}).\end{aligned}$$

By the injectivity of Φ, we have $X_{ji}T_{ii} = X_{ji}(A_{ii} + B_{ii})$. Since A_{ji} is prime, then $T_{ii} = A_{ii} + B_{ii}$.

Hence, additivity of Φ comes from above Claims. $\square$

In the rest of this paper we show that Φ is a $*$-isomorphism by assuming that $\Phi(I)$ is a projection.

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras with identity such that $\mathcal{A}$ is prime. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective map which satisfies $\Phi(A^*B + \eta BA^*) = \Phi(A)^*\Phi(B) + \eta\Phi(B)\Phi(A)^*$ for all $A, B \in \mathcal{A}$ such that $\Phi(I)$ is a projection and a non-zero scalar η such that $\eta \neq \pm 1$. Then, Φ is a $*$-isomorphism.*

Proof. We prove our theorem in several steps.

Step 1. $\Phi(I) = I$.

Let $A = B = I$ in

$$\Phi(A^*B + \eta BA^*) = \Phi(A)^*\Phi(B) + \eta\Phi(B)\Phi(A)^*, \tag{2.12}$$

so we have

$$\Phi(I + \eta I) = \Phi(I^*)\Phi(I) + \eta\Phi(I)\Phi(I)^*,$$

since Φ is additive and $\Phi(I)$ is a projection, we have from the above equation

$$\Phi(\eta I) = \eta\Phi(I). \tag{2.13}$$

On the other hand, let $\Phi(A) = I$ for some $A \in \mathcal{A}$ and assume that $B = I$ in (2.12) we have

$$\Phi(A^*I + \eta IA^*) = \Phi(A)^*\Phi(I) + \eta\Phi(I)\Phi(A)^*,$$

therefore, by the additivity of Φ and (2.13), we obtain

$$\Phi(A^* + \eta A^*) = \Phi(I + \eta I).$$

Since Φ is in injective we have $A = I$.

Step 2. $\Phi(\eta B) = \eta\Phi(B)$ *for all* $B \in \mathcal{A}$.

By Theorem 2.3, Φ is additive. Let $A = I$ in the following equation

$$\Phi(A^*B + \eta BA^*) = \Phi(A)^*\Phi(B) + \eta\Phi(B)\Phi(A)^*,$$

so we have $\Phi(B + \eta B) = \Phi(I)^*\Phi(B) + \eta\Phi(B)\Phi(I)^*$. Since Φ is additive and unital (by Step 1), we obtain $\Phi(\eta B) = \eta\Phi(B)$ for all $B \in \mathcal{B}$.

Step 3. Φ *is $*$-preserving.*

Let $B = I$ in the following equation

$$\Phi(A^*B + \eta BA^*) = \Phi(A)^*\Phi(B) + \eta\Phi(B)\Phi(A)^*,$$

so we have $\Phi(A^* + \eta A^*) = \Phi(A)^*\Phi(I) + \eta\Phi(I)\Phi(A)^*$. Since Φ is additive and unital, from Step 2, we obtain $\Phi(A^*) = \Phi(A)^*$.

Step 4. Φ *is multiplicative.*

Let change A with A^* in equation

$$\Phi(A^*B + \eta BA^*) = \Phi(A)^*\Phi(B) + \eta\Phi(B)\Phi(A)^*,$$

so we have

$$\Phi(AB + \eta BA) = \Phi(A^*)^*\Phi(B) + \eta\Phi(B)\Phi(A^*)^*$$

for all $A, B \in \mathcal{A}$. From the above equation, Steps 2, 3 and the additivity of Φ, we have

$$\Phi(AB) + \eta\Phi(BA) = \Phi(A)\Phi(B) + \eta\Phi(B)\Phi(A).$$

Similarly, we can write

$$\Phi(BA) + \eta\Phi(AB) = \Phi(B)\Phi(A) + \eta\Phi(A)\Phi(B).$$

From these two equations, we can obtain

$$(\eta^2 - 1)\Phi(AB) = (\eta^2 - 1)\Phi(A)\Phi(B).$$

Hence, $\Phi(AB) = \Phi(A)\Phi(B)$, as $\eta \neq \pm 1$.

Step 5. Φ *preserves positivity.*

Since each operator is positive if and only if it can be represented as $A = B^*B$ for some operator B. So, let A be a positive then by Step 3 and Step 4, we have $\Phi(A) = \Phi(B^*B) = \Phi(B)^*\Phi(B)$. Therefore, $\Phi(B)$ is positive.

Step 6. $\Phi(|A|) = |\Phi(A)|$ (*i.e.,* Φ *preserves absolute values*).

Since Φ is multiplicative, we have $\Phi(A^*A) = \Phi(A^*)\Phi(A)$ for all $A \in \mathcal{A}$. Moreover, by Step 3, we can obtain $\Phi(A^*A) = \Phi(A)^*\Phi(A)$. It follows that

$$\Phi(|A|)^2 = \Phi(|A|^2) = |\Phi(A)|^2.$$

The result follows by Step 5.

Step 7. Φ *is an* $\mathbb{R}$*-linear continuous map sending self-adjoint* (*rep. positive*) *elements of* $\mathcal{A}$ *into self-adjoint* (*resp. positive*) *elements of* $\mathcal{B}$.

The proof of this step is exactly as one of the first step of the proof of [18, Theorem 1]. The details are omitted.

Step 8. Φ *is a* $*$*-isomorphism.*

For showing that Φ is a $*$-isomorphism, by Steps 3 and 4, it is enough to prove that Φ is $\mathbb{C}$-linear. Since Φ is $\mathbb{R}$-linear, by Steps 1 and 2, we have

$$\begin{aligned}\eta_1 I + \eta_2 \Phi(iI) &= \Phi(\eta_1 I) + \Phi(\eta_2 iI)\\ &= \Phi(\eta_1 I + i\eta_2 I)\\ &= (\eta_1 + i\eta_2)\Phi(I),\end{aligned}$$

where η_1 and η_2 are real and imaginary parts of scalar η, respectively. It follows that $\Phi(iI) = iI$. Since Φ is multiplicative, we have the result. $\square$

Corollary 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras with $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ the identities of them, respectively. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a bijective map which satisfies $\Phi(A^*B + \eta BA^*) = \Phi(A)^*\Phi(B) + \eta\Phi(B)\Phi(A)^*$ for all $A, B \in \mathcal{A}$ and η is a non-zero rational number such that $\eta \neq \pm 1$, then Φ is a $*$-isomorphism.*

Proof. By Theorem 2.4, it is enough to show that Φ is unital. Since Φ is additive by Theorem 2.3, we can easily obtain $\Phi(\eta A) = \eta\Phi(A)$, for all $A \in \mathcal{A}$ and rational number η. So, we have the following

$$\Phi(A^*B) + \eta\Phi(BA^*) = \Phi(A)^*\Phi(B) + \eta\Phi(B)\Phi(A)^*$$

for all $A, B \in \mathcal{A}$. Let $A = I$ in the above equation, we have

$$(1 + \eta)\Phi(B) = \Phi(I)^*\Phi(B) + \eta\Phi(B)\Phi(A)^* \tag{2.14}$$

for all $B \in \mathcal{B}$. Since Φ is surjective we can find an element B such that $\Phi(B) = I$. Hence, by equation (2.14), we have $\Phi(I) = I$. $\square$

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Vahid Darvish
Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran
P. O. Box 47416-1468
Babolsar, Iran
E-mail address: vahid.darvish@mail.com

Haji Mohammad Nazari
Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran
P. O. Box 47416-1468
Babolsar, Iran
E-mail address: m.nazari@stu.umz.ac.ir

Hamid Rohi
Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran
P. O. Box 47416-1468
Babolsar, Iran
E-mail address: h.rohi@stu.umz.ac.ir

Ali Taghavi
Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran
P. O. Box 47416-1468
Babolsar, Iran
E-mail address: taghavi@umz.ac.ir