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ON RINGS WHOSE ANNIHILATING-IDEAL GRAPHS ARE BLOW-UPS OF A CLASS OF BOOLEAN GRAPHS

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Abstract. For a finite or an infinite set X, let 2^X be the power set of X. A class of simple graph, called strong Boolean graph, is defined on the vertex set $2^X \setminus \{X, \emptyset\}$, with M adjacent to N if $M \cap N = \emptyset$. In this paper, we characterize the annihilating-ideal graphs $\mathbb{AG}(R)$ that are blow-ups of strong Boolean graphs, complemented graphs and pre-atomic graphs respectively. In particular, for a commutative ring R such that $\mathbb{AG}(R)$ has a maximum clique S with $3 \leq |V(S)| \leq \infty$, we prove that $\mathbb{AG}(R)$ is a blow-up of a strong Boolean graph if and only if it is a complemented graph, if and only if R is a reduced ring. If assume further that R is decomposable, then we prove that $\mathbb{AG}(R)$ is a blow-up of a strong Boolean graph if and only if it is a blow-up of a pre-atomic graph. We also study the clique number and chromatic number of the graph $\mathbb{AG}(R)$.

1. Introduction and preliminary

Throughout this paper, all rings R are assumed to be commutative with identity 1_R. Following [10], for a ring R, let $\mathbb{I}(R)$ be the set of ideals of R, $\mathbb{A}(R)$ the set of annihilating-ideals of R, where a nonzero ideal I of R is called an *annihilating-ideal* if there exists a nonzero ideal J of R such that $IJ = \{0\}$. The *annihilating-ideal graph* $\mathbb{AG}(R)$ of R is a simple graph with vertex set $\mathbb{A}(R)$, such that distinct vertices I and J are adjacent, denoted as $I \sim J$, if and only if $IJ = \{0\}$. Clearly, the graph $\mathbb{AG}(R)$ is an empty graph if and only if R is an integral domain, and $\mathbb{A}(R) = \mathbb{I}(R) \setminus \{\{0\}, R\}$ if R is artinian. Note that $\mathbb{I}(R)$ is a commutative semigroup with zero element, under the binary operation of ideal multiplication, and $\mathbb{AG}(R) = \Gamma(\mathbb{I}(R))$, i.e., $\mathbb{AG}(R)$ is the zero-divisor graph of the semigroup $\mathbb{I}(R)$. For the definition and fundamental

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properties of zero-divisor graphs, refer to [8] and the listed references; for the recent development, one can refer to the recent comprehensive survey paper [4] for rings, and [5] for semigroups.

Annihilating-ideal graphs of rings, first introduced and studied in [10], provide an excellent setting for studying some aspects of algebraic property of a commutative ring, especially, the ideal structure of a ring. Some fundamental results on the concept have been established for both rings and semigroups. For example, $\mathbb{AG}(R)$ is always a simple, connected and undirected graph with diameter less than four; if $\mathbb{AG}(R)$ contains a cycle, then its girth is less than five; if R is a non-domain ring, then $\mathbb{AG}(R)$ is a finite graph if and only if R has finitely many ideals, if and only if every vertex of $\mathbb{AG}(R)$ has finite degree. Moreover, $\mathbb{AG}(R)$ has n vertices, $n \geq 1$, if and only if R has only n nonzero proper ideals. In [27, 28], the finite local rings R whose $\mathbb{AG}(R)$ are star graphs (consist only of triangles, respectively) are carefully characterized. For detailed further discussions, one can refer to, e.g., [3, 1, 9, 10, 11, 12, 27, 28].

For a simple graph G, the sets of vertices and edges of G are denoted by $V(G)$ and $E(G)$ respectively. For a vertex $v \in V(G)$, the neighbourhood of v, denoted by $N(v)$, consists of the vertices which are adjacent to v. We denote the set of the neighbourhoods by $N(G)$, and denote by $Max(N(G))$ all the maximal neighbourhoods in $N(G)$ (under inclusion). For a subset C of $V(G)$, if the subgraph induced on C is a complete graph, then C is called a clique of the graph G. The complete subgraph induced by a clique is also called a clique in the present paper. For a finite graph G, a maximum clique is a clique such that there is no clique with more vertices. The clique number of a graph G, denoted as $\omega(G)$, is the number of vertices in a maximum clique in G, while the vertex chromatic number $\chi(G)$ of graph G is the smallest number of colors needed to color the vertices of the graph G such that no two adjacent vertices have the same color.

Recall that a Boolean graph is defined to be the zero-divisor graph $\Gamma(R)$ of a Boolean ring R, see [6, 7, 15, 17, 22, 23] for details. It is well-known that for a zero-divisor graph $\Gamma(R)$ with no less than 3 vertices, $\Gamma(R)$ is a Boolean graph if and only if every vertex of $\Gamma(R)$ has a unique complement ([15, Theorem 2.5]). Theorem 3.8 in the present paper provides an analogue of this result for annihilating-ideal graphs.

In this paper, we study rings R whose annihilating-ideal graphs are strong Boolean graphs. Let X be a finite or an infinite set, and let $C(X) = \{\{v\} \mid v \in X\}$. A strong Boolean graph, denoted by $B_{C(X)}$, is a graph defined on the vertex set $2^X \setminus \{X, \emptyset\}$, with M adjacent to N if $M \cap N = \emptyset$. It is clear that a strong Boolean graph is a Boolean graph, and in the finite case, a Boolean graph is also a strong Boolean graph.

In order to consider the graph with infinite clique number, we use the definition about the maximum clique of a graph defined in [13].

Definition 1.1 ([13, Definition 2.1])**.** A clique S of a graph G is called a *maximum clique* of G if the following conditions are satisfied:

(1) $|V(S)|$ is the maximal in $\{|V(L)| \mid L$ is a clique of $G\}$.

(2) For any finite subset $A \subseteq V(S)$ and subset $B \subseteq V(G) \setminus V(S)$ with $|B| = |A| + 1$, the subgraph induced on $B \cup (V(S) \setminus A)$ is not a clique of the graph G.

By the above definition, it is easy to see that $C(X)$ is the unique maximum clique of the strong Boolean graph $B_{C(X)}$, no matter whether X is finite or infinite. From this point of view, a strong Boolean graph is uniquely determined by its maximum clique. So, if G is a strong Boolean graph with the unique maximum clique S, we also denote the strong Boolean graph G by B_S. Let $[n] = \{1, 2, \ldots, n\}$. A finite (strong) Boolean graph $B_{C([n])}$ is also denoted by B_n. Recall that in [13], an induced subgraph of the strong Boolean graph B_S is called a *pre-atomic graph*, denoted by A_S, if A_S contains the unique maximum clique S of B_S as a subgraph. Clearly, S is also the unique maximum clique of A_S. From [13], we know that the graph A_S shares some common properties with B_S. When $|S| = n$, A_S is also denoted by A_n.

Roughly speaking, to blow-up a graph G is to replace every vertex v of G by a set T_x to get a possibly new and larger graph G_T, where $|T_x| \geq 1$. The induced subgraph of G_T on T_x is a discrete graph, while for distinct vertices x, y of G, x is adjacent to y in G if and only if each vertex of T_x is adjacent to all vertices of T_y in G_T, see [14, 18, 20] for details. For a reduced ring, it is known that $\Gamma(R)$ is a blow-up of the compressed zero-divisor graph introduced in [19] and later studied in more depth in [21]. In this case, blow-up is a sort of inverse to being compressed. It is also well-known that the zero-divisor graph $\Gamma(R)$ is complemented if and only if $\Gamma(R)$ is a blow-up of the zero-divisor graph of a Boolean ring, if and only if the total quotient ring $T(R)$ of R is von Neumann regular ([7, Theorem 2.2, Theorem 3.5, and Proposition 4.5]). Theorem 3.5 in the present paper provides an analogue of these results for annihilating-ideal graphs. The previous work also shows that graph blow-up plays an essential role in the co-maximal ideal graphs of a ring, see [25, 26] for the concise definition, the history, the recent development, and a list of references.

This paper is organized in the following way. In Section 1, some purely graphic characterizations for blow-ups of strong Boolean graphs are shown. In Section 2, the properties of a ring R whose annihilating-ideal graph $\mathbb{AG}(R)$ is a blow-up of a finite or an infinite strong Boolean graph are studied. The rings R whose annihilating-ideal graphs are complemented graphs are studied in Section 3, and the properties of a ring R whose annihilating-ideal graph is a blow-up of a pre-atomic graph are given in Section 4. In Section 5, we consider the clique number and the chromatic number of the annihilating-ideal graph of a ring with some special conditions.

The following purely graph-theoretic results were established in [13]:

Theorem 1.2 ([13, Theorem 2.2]). *Let G be a connected graph with a maximum clique S. Then G is isomorphic to the strong Boolean graph B_S if and only if the following properties are satisfied*:

(1) *For each nontrivial subset A of $V(S)$, there exists a vertex $v \in V(G)$ such that $A = N(v) \cap V(S)$;*

(2) *G is uniquely $S \cap N$-determined (or alternatively, G is uniquely N-determined), i.e., $V(S) \cap N(x) = V(S) \cap N(y)$ (respectively, $N(x) = N(y)$) implies $x = y$ for vertices $x, y \in V(G)$;*

(3) *For vertices $x, y \in V(G)$, $V(S) \subseteq N(x) \cup N(y)$ if and only if $x \in N(y)$.*

Note that under the assumption (3), the equality $V(S) \cap N(x) = V(S) \cap N(y)$ is equivalent to the equality $N(x) = N(y)$.

Theorem 1.3 ([13, Theorem 2.6]). *Let G be a connected graph with a maximum clique S. Then G is a blow-up of the strong Boolean graph B_S if and only if the following properties are satisfied*:

(1) *For each nontrivial subset $A \subsetneq V(S)$, there exists a vertex $v \in V(G)$ such that $N(v) \cap V(S) = A$;*

(2) *For vertices $x, y \in V(G)$, $V(S) \subseteq N(x) \cup N(y)$ if and only if $x \in N(y)$.*

Proposition 1.4 ([13, Proposition 2.8]). *For a connected graph G, G is isomorphic to a pre-atomic graph A_S if and only if in G there exists a maximum clique K such that $|V(K)| = |V(S)|$ and the following properties are satisfied*:

(1) *G is uniquely $K \cap N$-determined, i.e., $V(K) \cap N(x) = V(K) \cap N(y)$ implies $x = y$ for vertices $x, y \in V(G)$;*

(2) *For vertices $x, y \in V(G)$, $V(K) \subseteq N(x) \cup N(y)$ if and only if $x \in N(y)$.*

Proposition 1.5 ([13, Proposition 2.9]). *For a connected graph G, G is isomorphic to a blow-up of a pre-atomic graph A_S if and only if in G there exists a maximum clique K such that $|V(K)| = |V(S)|$ and for vertices $x, y \in V(G)$, $V(K) \subseteq N(x) \cup N(y)$ if and only if $x \in N(y)$.*

In this paper, we use the characterizations to study annihilating-ideal graph of a ring.

2. $\mathbb{AG}(R)$ that is a blow-up of a strong Boolean graph

Note that if S is a maximum clique of G, then there is no clique properly containing S. The following proposition follows from Definition 1.1.

Proposition 2.1. *Let S be a maximum clique of a graph G. If T is a clique of G and $|V(T) \setminus V(S)| = |V(S) \setminus V(T)| < \infty$ hold, then T is also a maximum clique of the graph G.*

Proof. It follows from $|V(T) \setminus V(S)| = |V(S) \setminus V(T)| < \infty$ that $|V(S)| = |V(T)|$. Since S is a maximum clique of G, clearly $|V(T)|$ is the maximal in $\{|V(L)| \mid L \text{ is a clique of } G\}$. Thus if T is not a maximum clique of G, then there exist a finite subset $A \subseteq V(T)$ and a subset $B \subseteq V(G) \setminus V(T)$ with

$|B| = |A| + 1$, such that the subgraph L of G induced on $B \cup (V(T) \setminus A)$ is a clique. Denote $C = V(S) \setminus V(T)$ and $D = V(T) \setminus V(S)$, then by assumption $|C| = |D| < \infty$. In the following Figure 1, let the three circles be $V(S)$, $V(T)$ and $V(L)$ respectively. Note that

$$\begin{aligned}|A \setminus D| + |C \setminus B| + |C \cap B| &= |A \setminus D| + |C| = |A \setminus D| + |D| \\ &= |A \setminus D| + |A \cap D| + |D \setminus A| \\ &= |A| + |D \setminus A| = |B| + |D \setminus A| - 1 \\ &= |C \cap B| + |B \setminus C| + |D \setminus A| - 1,\end{aligned}$$

so $|A \setminus D| + |C \setminus B| = |B \setminus C| + |D \setminus A| - 1$. It is easy to see that $|(A \setminus D) \cup (C \setminus B)| = |A \setminus D| + |C \setminus B| < \infty$ and $|(B \setminus C) \cup (D \setminus A)| = |B \setminus C| + |D \setminus A|$, and note that

$$((B \setminus C) \cup (D \setminus A)) \cup (V(S) \setminus ((A \setminus D) \cup (C \setminus B))) = V(L)$$

holds, a contradiction. This completes the proof. $\square$

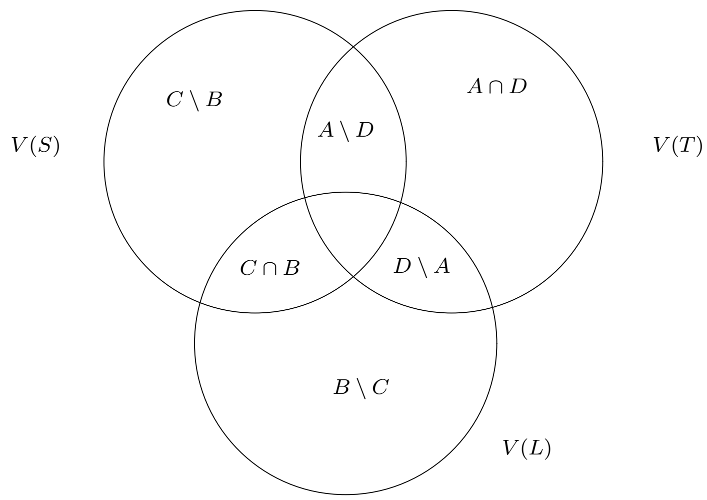

Figure 1

Lemma 2.2. *For a ring R, let $G = \mathbb{AG}(R)$ be a blow-up of a strong Boolean graph B_S, with $V(S) = \{I_i \,|\, i \in \Gamma\}$. If $3 \leq |V(S)| \leq \infty$, then the following statements hold*:

(1) *$N(I_i) \in Max(N(G))$ holds for each $i \in \Gamma$;*

(2) *For each pair of distinct $i, j \in \Gamma$, $N(I_i) \nsubseteq N(I_j)$ and $I_j \nsubseteq I_i$;*

(3) *For each pair of distinct i, j, $I_i \cap I_j = \{0\}$ holds*;

(4) *If another maximum clique C* (with $V(C) = \{J_i \,|\, i \in \Omega\}$) *exists, then there is a bijection σ from Γ to Ω, such that $N(I_i) = N(J_{\sigma(i)})$ for each $i \in \Gamma$.*

Proof. (1) and (4) follows directly from [13, Lemma 3.1].

(2) Let $A = \{I_i\}$. By Theorem 1.3, there exists $I \in V(G)$ such that $V(S) \cap N(I) = A = \{I_i\}$. Since $I_j \notin N(I) \cup N(I_j)$, so $I \in N(I_i) \setminus N(I_j)$. In the following, we will show that $I_j \not\subseteq I_i$. If on the contrary that $I_j \subseteq I_i$, then $N(I_i) \subseteq N(I_j) \cup \{I_j\}$. From the above discussion, $I \in N(I_i) \setminus N(I_j)$ implies $I = I_j$. Hence $\{I_i\} = V(S) \cap N(I) = V(S) \cap N(I_j) = V(S) \setminus \{I_j\}$, thus $V(S) = \{I_i, I_j\}$. It contradicts to $|V(S)| \geq 3$.

(3) If $I_i \cap I_j \neq \{0\}$, then by (2), since $I_i \cap I_j \subseteq I_i$, $I_i \cap I_j \notin V(S)$. Clearly, $I_i \cap I_j$ is adjacent to I_k for each $k \in \Gamma$ since $I_i \cap I_j \subseteq I_i$ and $I_i \cap I_j \subseteq I_j$. Hence $V(S) \cup \{I_i \cap I_j\}$ induces a clique properly containing S, a contradiction. □

Lemma 2.3. *Let $R = \prod_{i \in \Delta} R_i$ be a decomposition of a commutative ring R. If S is a maximum clique of $G = \mathbb{AG}(R)$ with $V(S) = \{I_i \mid i \in \Gamma\}$, and $I_i \not\subseteq I_j$ when $i \neq j$, then there exists a mutually disjoint decomposition of the set Γ, denoted by $\Gamma = \cup_{j \in \Delta} A_j$, such that $A_j = \{i \mid I_i \subseteq R_j\}$.*

Proof. Let $r = |\Delta|$. If $r = 1$, then the result is clear. For $r > 1$, it suffices to show that for every $i \in \Gamma$, there is only one $j \in \Delta$, such that $R_j \cap I_i \neq \{0\}$ holds. In fact, if there exist distinct j, k such that $R_j \cap I_i \neq \{0\}$ and $R_k \cap I_i \neq \{0\}$, then $R_j \cap I_i, R_k \cap I_i \in V(G)$ are adjacent since $(R_j \cap I_i)(R_k \cap I_i) \subseteq R_j R_k = \{0\}$. Note that $I_i \not\subseteq I_j$ for each pair $i, j \in \Gamma$ whenever $i \neq j$, hence neither $R_j \cap I_i$ nor $R_k \cap I_i$ is in $V(S)$. Hence $\{R_j \cap I_i, R_k \cap I_i\} \cup V(S) \setminus \{I_i\}$ induces a clique, contradicting Definition 1.1. □

The following corollary follows directly from Lemma 2.2(2) and Lemma 2.3.

Corollary 2.4. *For a ring $R = \prod_{i \in \Delta} R_i$, let $G = \mathbb{AG}(R)$ be a blow-up of a finite or an infinite strong Boolean graph B_S, with $V(S) = \{I_i \mid i \in \Gamma\}$. If $3 \leq |V(S)| \leq \infty$, then there exists a mutually disjoint decomposition $\Gamma = \cup_{j \in \Delta} A_j$, such that $A_j = \{i \mid I_i \subseteq R_j\}$.*

Lemma 2.5. *Let $G = \mathbb{AG}(R)$ be a blow-up of a finite strong Boolean graph $B_n\,(3 \leq n < \infty)$. Then for each $I \in V(G)$, there exists a maximum clique S with $V(S) = \{I_1, I_2, \ldots, I_n\}$, such that for each $1 \leq i \leq n$, either $I_i \subseteq I$ or $I_i \in N(I)$.*

Proof. Let $\{J_1, J_2, \ldots, J_n\}$ induces a maximum clique in G. For a vertex $I \in G$ and each $1 \leq i \leq n$, either $IJ_i = \{0\}$ or $I \cap J_i \supseteq IJ_i \neq \{0\}$. Let $I_i = J_i$ while $IJ_i = \{0\}$, and $I_i = I \cap J_i$ while $IJ_i \neq \{0\}$. In the following, we will show that S, induced by $\{I_1, I_2, \ldots, I_n\}$, is a maximum clique of G. Since it is clear that $I_i \in N(I_j)$ when $I_i \neq I_j$. It suffices to show that $I_i \neq I_j$ when $i \neq j$. Assume on the contrary that $I_i = I \cap J_i = I \cap J_j = I_j \neq \{0\}$ for some $i \neq j$. Note that $I_j \neq J_k$ for each $k \in [n] \setminus \{j\}$, so $J_i \in N(I_j) = N(I_i)$. Hence $\{I_i, J_1, J_2, \ldots, J_n\}$ induces a $(n+1)$-clique, a contradiction. Clearly, for the maximum clique induced by $\{I_1, I_2, \ldots, I_n\}$, either $I_i \subseteq I$ or $I_i \in N(I)$ for each $i \in [n]$. □

Proposition 2.6. *Let $G = \mathbb{AG}(R)$ be the annihilating-ideal graph of a commutative ring R. If G is a blow-up of a finite or an infinite strong Boolean graph B_S $(3 \leq |V(S)| \leq \infty)$, then R is reduced.*

Proof. Let S be a maximum clique of G with $V(S) = \{I_i \,|\, i \in \Gamma\}$. If R is not reduced, then there is an ideal $I \in V(G)$ such that $I^2 = \{0\}$. Since S is a maximum clique, there exists $i \in \Gamma$ such that $I \cap I_i \neq \{0\}$. It follows from Lemma 2.2(2) that $I \cap I_i \neq I_j$ and $(I \cap I_i)I_j = \{0\}$ for each $j \in \Gamma \setminus \{i\}$. By Proposition 2.1, $\{I \cap I_i\} \cup (V(S) \setminus \{I_i\})$ induces a maximum clique of G. For a fixed $j \in \Gamma \setminus \{i\}$. It is not hard to check that $\{I \cap I_i, I \cap I_i + I_j\} \cup (V(S) \setminus \{I_i, I_j\})$ induces a maximum clique. Again by Lemma 2.2(2), it is a contradiction. □

Now we prove the main result of this section.

Proposition 2.7. *Let R be a commutative ring such that the annihilating-ideal graph $G = \mathbb{AG}(R)$ has a maximum clique S with $3 \leq |V(S)| \leq \infty$. Then R is a reduced ring if and only if G is a blow-up of the strong Boolean graph B_S.*

Proof. $\Longleftarrow$: It follows from Proposition 2.6.

$\Longrightarrow$: Let S be a maximum clique of the graph $G = \mathbb{AG}(R)$ with $V(S) = \{I_i \,|\, i \in \Gamma\}$. We will prove the conclusion by taking advantage of Theorem 1.3.

First, for each nontrivial subset $A \subseteq V(S)$, let

$$I = \{x \in \sum_{I_i \in B} I_i \,|\, B \subseteq V(S) \setminus A,\ 1 \leq |B| < \infty\}.$$

Since R is a reduced ring, it is clear that $N(I) \cap V(S) = A$ holds.

Second, we will show that $V(S) \subseteq N(I) \cup N(J)$ if and only if $I \in N(J)$. Assume $V(S) \subseteq N(I) \cup N(J)$. Then we claim that $I \cap J = \{0\}$ holds. Otherwise, $\{0\} \neq I \cap J \notin N(I) \cup N(J)$ holds since R is reduced. Hence

$$V(S) \subseteq N(I) \cup N(J) \subseteq N(I \cap J),$$

and it follows that $\{I \cap J\} \cup V(S)$ induces a clique properly containing S, a contradiction. So, $I \cap J = \{0\}$, and thus $I \in N(J)$. On the other hand, if assume $V(S) \nsubseteq N(I) \cup N(J)$, then there exists an ideal I_i in $V(S)$ such that both $I_i I \neq \{0\}$ and $I_i J \neq \{0\}$ hold. We claim that $(I_i I)(I_i J) \neq \{0\}$, since otherwise, $\{I_i I, I_i J\} \cup (V(S) \setminus \{I_i\})$ induces a clique, a contradiction. Since $(I_i I)(I_i J) \neq \{0\}$ clearly implies $IJ \neq \{0\}$, it follows that $I \notin N(J)$. This proves that $I \in N(J)$ implies $V(S) \subseteq N(I) \cup N(J)$.

Note that since $\mathbb{AG}(R)$ is the zero-divisor graph of the semigroup $\mathbb{I}(R)$, it is connected. Thus by Theorem 1.3, G is a blow-up of the strong Boolean graph B_S. This completes the proof. □

For a simple graph G, the greatest distance between any two vertices is called the *diameter* of the graph G, denoted by $diam(G)$. The length of a shortest cycle contained in the graph G is called the *girth* of G, denoted by $girth(G)$.

Corollary 2.8. *Let R be a commutative ring such that the annihilating-ideal graph $G = \mathbb{AG}(R)$ has a maximum clique S with $3 \leq |V(S)| \leq \infty$. If R is a reduced ring, then $diam(G) = 3$ and $girth(G) = 3$.*

Proposition 2.9. *Assume $3 \leq n < \infty$. Then for a ring R, $R \cong \prod_{i=1}^{n} F_i$ with every F_i being a field if and only if $\mathbb{AG}(R)$ is the strong Boolean graph B_n.*

Proof. $\Longrightarrow$: It is easy to check.

$\Longleftarrow$: Let S be the unique maximum clique of the graph B_n with $V(S) = \{I_1, I_2, \dots, I_n\}$. First, we will show that every vertex in S is a minimal ideal of R. Otherwise, assume without loss of generality that $\{0\} \neq J_1 \subset I_1$, then $N(I_1) \subseteq N(J_1) \cup \{J_1\}$. In view of R being a reduced ring, $J_1 \notin N(I_1)$, and hence $N(I_1) \subseteq N(J_1)$. So, $N(I_1) = N(J_1)$ follows from Lemma 2.2(1). Since B_n is neighbourhood determined, it is a contradiction.

Note that $I_i^2 \neq \{0\}$ for each minimal ideal $I_i \in V(S)$, so $I_i^2 = I_i$ for each $i \in [n]$. By Brauer's Lemma (see, e.g., [16, page 172]), $I_i = Re_i$ with e_i being an idempotent element in R for each $i \in [n]$. Clearly, $e_1, \dots, e_n$ is a set of orthogonal nonzero idempotent elements of R. We claim $e_1 + e_2 + \cdots + e_n = 1$. Otherwise, $e_1, \dots, e_n, e_{n+1}$ is also a set of orthogonal idempotent elements of R, where $e_{n+1} = 1 - \sum_{i=1}^{n} e_i$. Note that $Re_1, \dots, Re_n, Re_{n+1}$ induces an $(n+1)$-clique, a contradiction. So $R = Re_1 \times \cdots \times Re_n$. Finally, for each $i \in [n]$, Re_i is a field since $I_i = Re_i$ is a minimal ideal of R. □

The following is a known result, and it follows directly from [10, Theorem 2.6]. We include it here for completeness.

Proposition 2.10. *$G = \mathbb{AG}(R)$ is a strong Boolean graph B_2 if and only if R is one of the following two classes of rings*:

(1) *$R = F_1 \times F_2$, where both F_1 and F_2 are fields*;

(2) *$(R, \mathfrak{m})$ is a local principal ideal ring, with two nontrivial ideals $\mathfrak{m}, \mathfrak{m}^2$. In this case, $\mathfrak{m} = R\alpha$ for some $\alpha \in \mathfrak{m}$, where $\alpha^2 \neq 0, \alpha^3 = 0$.*

For $G = \mathbb{AG}(R)$, it is clear that G is a blow-up of the strong Boolean graph B_2 (i.e., K_2) if and only if G is a complete bipartite graph. In view of [11, Theorem 2.3] and [2, Corollary 23], we known that $\mathbb{AG}(R)$ is a complete bipartite graph if and only if either $\mathbb{AG}(R)$ is a star graph or R is a reduced ring with $|Min(R)| = 2$. By [10, Theorem 2.6] and a recent work of [28, Theorem A], for an artinian ring R, $\mathbb{AG}(R)$ is a star graph if and only if R satisfies one of the followings: (1) $R \cong F_1 \times F_2$; (2) $(R, \mathfrak{m})$ is a PIR, where $\mathfrak{m} \neq \{0\}$ and $\mathfrak{m}$ has nilpotency index less than or equal to 4; (3) $char(R) = 2$ or $char(R) = 4$, and $\mathfrak{m}$ has a minimal generating set $\{\beta_1, \beta_2\}$ with $\beta_1\beta_2 \neq 0, \beta_1^2 = \beta_2^2 = 0$; In the case (3), $\mathfrak{m}^2 \neq \{0\}$, $\mathfrak{m}^3 = \{0\}$. Furthermore, the structure of finite local rings satisfying (3) were carefully characterized in [28, Theorem B]. The structure of finite local rings satisfying (2) were carefully characterized in [24].

In the following, we will change to another idea and divide the class of rings R, whose annihilating-ideal graphs $\mathbb{AG}(R)$ are blow-ups of the strong Boolean graph B_2, into the following three types:

(1) R is a reduced ring with $|Min(R)| = 2$, which is called $B_2^{(1)}$*-type ring.*

(2) There is a unique square-zero ideal in R. We call this kind of rings $B_2^{(2)}$*-type rings.*

(3) There are at least two square-zero ideals in R. R is called a $B_2^{(3)}$*-type ring.*

If R is a $B_2^{(2)}$-type ring, it is not hard to see that $\mathbb{AG}(R)$ is a star graph, and the unique square-zero ideal is a minimal ideal of R. Furthermore, the unique square-zero ideal is adjacent to every other vertices in $\mathbb{AG}(R)$.

In the following, we provide two examples of non-artinian rings which are $B_2^{(1)}$ and $B_2^{(2)}$-type rings respectively.

Example 2.11. Let $R_1 = \mathbb{R}[[x,y]]/(xy)$. It is clear that R_1 is reduced, and $\mathbb{AG}(R_1)$ is a blow-up of B_2, as is shown in the following Figure 2. Let $R_2 = \mathbb{R}[[x,y]]/(xy, y^2)$. Clearly, (y) is the unique square-zero ideal of R_2, and $\mathbb{AG}(R_2)$ is a star graph, as is shown in the following Figure 3.

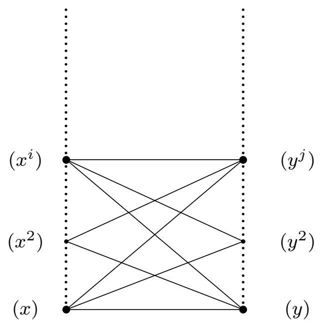

Figure 2

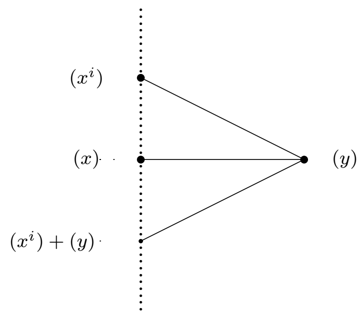

Figure 3

Proposition 2.12. *If R is a $B_2^{(3)}$-type ring, then $G = \mathbb{AG}(R)$ is a star graph. Further more, there is a smallest nonzero ideal in R, i.e., $\cap_{I\in V(G)} I \in V(G)$.*

Proof. It follows from $\omega(G) = 2$ that there exists a minimal ideal J such that $J^2 = \{0\}$. We claim that $J \subseteq K$ holds for any square-zero ideal $K \neq \{0\}$. In fact, assume that K is a square-zero ideal in $V(G)$ and $K \neq J$. Then $J \cap K$ is either J or $\{0\}$ since J is a minimal ideal. If assume further that $J \cap K = \{0\}$, then $J \not\subseteq K$ and $K \not\subseteq J$, and hence $\{J, K, J+K\}$ induces a 3-clique, a contradiction. So, $J \cap K = J$ and hence $J \subseteq K$ holds. Fix a square-zero ideal K $(K \neq J)$. Clearly, for each $L \in V(G) \setminus \{J, K\}$, $LK = \{0\}$ implies $LJ = \{0\}$. Note that G is a blow-up of B_2, so $JL = \{0\}$ holds for each $L \in V(G)$. Hence G is a star graph. Finally, we will show that J is the smallest ideal in $V(G)$. It is easy to see that for each $L \in V(G) \setminus \{J, K\}$, $L \cap K \neq \{0\}$

holds. Hence $J \subseteq L \cap K$ since $L \cap K$ is a square-zero ideal. This completes the proof. □

Example 2.13. Let $R_3 = \mathbb{R}[[x,y]]/(x^2,y^2)$. It is easy to see that (xy), (x), (y) are square-zero ideals of R_3, and $\mathbb{AG}(R_3)$ is also a star graph, see Figure 4.

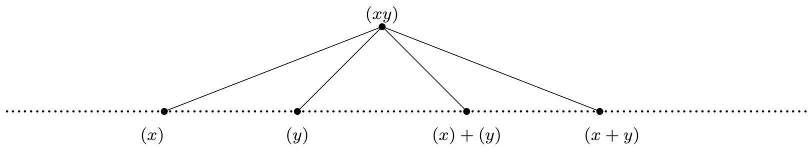

Figure 4

3. $\mathbb{AG}(R)$ that is a complemented graph

Recall from [7] that in a graph G, a vertex $w \in V(G)$ is called a complement of v, denoted by $w \perp v$, if v is adjacent to w, and no vertex is adjacent to both v and w. A graph G is called *complemented* if every vertex of G has a complement. Recall from [7] that a complemented graph G is called *uniquely complemented*, if further $a \perp b$ and $a \perp c$ implies $N(b) = N(c)$. In the rest part of this paper, we call a complemented graph G to be *strongly complemented*, if every vertex of G has a unique complement. It is clear that for a strongly complemented graph G, $N(a) \neq N(b)$ holds for each pair of distinct vertices $a, b \in V(G)$.

In the following of this section, we will study about the ring whose annihilating-ideal graph is a complemented graph.

Lemma 3.1. *Let G be a complemented graph. For each pair of distinct vertices $a, b \in V(G)$, if $\{a\} \cup N(a) \subseteq \{b\} \cup N(b)$, then $N(a) = \{b\}$.*

Proof. Since G is a complemented graph, there exists a vertex $c \in V(G)$ such that $c \perp a$, i.e., $c \in N(a)$ and there is no vertex adjacent to both c and a. Hence $N(c) \cap N(a) = \emptyset$. We claim that $c = b$. Otherwise, $c \neq b$ and it implies $c \in N(a) \subseteq \{b\} \cup N(b)$, whence $c \in N(b)$. Since $\{a\} \cup N(a) \subseteq \{b\} \cup N(b)$, it follows that $a \in N(b)$. So, $b \in N(c) \cap N(a) \neq \emptyset$, a contradiction. The contradiction implies that $N(a) = \{b\}$. □

Corollary 3.2. *Let G be a complemented graph. If S is a clique of G with $3 \leq |V(S)| \leq \infty$, then for each pair of distinct $a, b \in V(S)$, $\{a\} \cup N(a) \nsubseteq \{b\} \cup N(b)$.*

Corollary 3.3. *Let R be a commutative ring. If $G = \mathbb{AG}(R)$ is a complemented graph and S is a clique of G with $3 \leq |V(S)| \leq \infty$, then for each pair of distinct $I, J \in V(S)$, $I \nsubseteq J$.*

Proposition 3.4. *Let R be a commutative ring such that $G = \mathbb{AG}(R)$ is a complemented graph. If further G has a maximum clique S with $3 \leq |V(S)| \leq \infty$, then there is no square-zero ideal in R. In this case, R is a reduced ring.*

Proof. Let $V(S) = \{I_i \mid i \in \Gamma\}$. Assume to the contrary that there is a nonzero ideal I such that $I^2 = \{0\}$.

If $I \in V(S)$, assume $I = I_1$. In this case, fix an $I_2 \in V(S)$, and it follows from Corollary 3.3 that $\{I_1 + I_2\} \not\in V(S) \setminus \{I_2\}$. By Proposition 2.1, $\{I_1 + I_2\} \cup (V(S) \setminus \{I_2\})$ induces a maximum clique of G, contradicting Corollary 3.3.

If $I \notin V(S)$, then we claim that there exists an $I_i \in V(S)$ such that $I \cap I_i \neq \{0\}$. Otherwise, if $I \cap I_i = \{0\}$ for each $i \in \Gamma$, then $\{I\} \cup V(S)$ induces a clique of G properly containing S, a contradiction. Without loss of generality, assume $I \cap I_1 \neq \{0\}$. Then $(I \cap I_1)^2 = \{0\}$ and $\{I \cap I_1\} \cup (V(S) \setminus \{I_1\})$ induces a maximum clique of G. By a similar discussion as above, there is a contradiction to Corollary 3.3.

In conclusion, there is no square-zero ideal in R, and hence R is a reduced ring. □

Here is the first main result of this section, which provides an analogue to Theorem 3.5 of [7].

Theorem 3.5. *For a commutative ring R, let $G = \mathbb{AG}(R)$ be its annihilating-ideal graph. If G has a maximum clique S with $3 \leq |V(S)| \leq \infty$, then the following statements are equivalent*:

(1) *R is a reduced ring.*

(2) *G is a blow-up of a strong Boolean graph.*

(3) *G is a complemented graph.*

Proof. (1) $\Longleftrightarrow$ (2). By Proposition 2.7.

(2) $\Longrightarrow$ (3). It follows from Theorem 1.3 that for each $I \in V(G)$, $\emptyset \neq V(S) \cap N(I) \subset V(S)$ holds. Again by Theorem 1.3, there exists a vertex $J \in V(G)$ such that $V(S) \cap N(J) = V(S) \setminus N(I)$. It is not hard to check that $J \perp I$.

(3) $\Longrightarrow$ (1). By Proposition 3.4. □

It is shown in Theorem 3.5 of [7] that, for a reduced ring R, the zero-divisor graph $\Gamma(R)$ is complemented if and only if the total quotient ring $T(R)$ of R is von Neumann regular. However, the following example shows that it is not true for an annihilating-ideal graph.

Example 3.6. Let x be an indeterminate, set

$$R_1 = \{r \in \prod_{i \in \mathbb{Z}} \mathbb{Z}_2 \mid r(i) = r(j) \text{ for all even integers } i \text{ and } j\},$$

and define $R = R_1 + \oplus_{i \in \mathbb{Z}} x\mathbb{Z}_2[x]$. Then R is a reduced commutative ring with identity. For every $i \in \mathbb{Z}$, let $I_i = \{r \in R \mid r(j) = 0 \text{ for all } j \neq i\}$. Then $S = \{I_i \mid i \in \mathbb{Z}\}$ induces a maximum clique, so $\mathbb{AG}(R)$ is a blow-up of a strong

Boolean graph B_S by Theorem 3.5. However, $T(R)$ is not von Neumann regular by Theorem 3.5 of [7] since, for example, the element $r \in R$ given by $r(0) = x$ and $r(i) = 0$ for $i \neq 0$ has no complement in the zero-divisor graph $\Gamma(R)$, since $s \in R$ with $sr = 0$ implies $s(i) = 0$ for all but finitely many even i.

Lemma 3.7. *For a commutative ring R, let $G = \mathbb{AG}(R)$ be its annihilating-ideal graph. If G is strongly complemented and has a maximum clique S with $3 \leq |V(S)| \leq \infty$, then each ideal in a maximum clique of G is a minimal ideal.*

Proof. First, it follows from Proposition 3.4 that R is reduced. Assume that $V(S) = \{I_i \,|\, i \in \Gamma\}$. Without loss of generality, it suffices to show that I_1 is a minimal ideal. Assume to the contrary that I_1 is not a minimal ideal. Then there exists a nonzero ideal J_1 such that $J_1 \subseteq I_1$ and $I_1 \neq J_1$. Then $J_1 \notin N(I_1)$, hence $N(I_1) \subseteq N(J_1)$. If there exists an ideal $K \in N(J_1) \setminus N(I_1)$, then $K \cap I_1 \neq \{0\}$ and $J_1(K \cap I_1) = \{0\}$. Since R is reduced, $K \cap I_1 \notin \{J_1\} \cup (V(S) \setminus \{I_1\})$. Hence $\{J_1, K \cap I_1\} \cup (V(S) \setminus \{I_1\})$ induces a clique, a contradiction. Thus $N(I_1) = N(J_1)$. Note that G is a strongly complemented graph, so $I_1 = J_1$, another contradiction. This completes the proof. □

The following is the second main result of this section, which provides an analogue to Theorem 2.5 of [15].

Theorem 3.8. *Let R be a commutative ring, and let $G = \mathbb{AG}(R)$ be its annihilating-ideal graph. If G has a maximum clique S with $3 \leq |V(S)| < \infty$, then the following statements are equivalent*:

(1) *R is a finite direct product of fields.*
(2) *G is a strong Boolean graph.*
(3) *G is a strongly complemented graph.*

Proof. (1) $\Longleftrightarrow$ (2). By Proposition 2.9.

(2) $\Longrightarrow$ (3). If $G = B_n$, then for each nontrivial $A \subsetneq [n]$, it is easy to see that $[n] \setminus A$ is the unique complement of A.

(3) $\Longrightarrow$ (1). Assume that $V(S) = \{I_1, I_2, \ldots, I_n\}$. By Proposition 3.4 and Lemma 3.7, $I_i^2 = I_i$ for each $i \in [n]$. Hence, as in the proof of Proposition 2.9, $I_i = Re_i$ for each $i \in [n]$ with $e_1, e_2, \ldots, e_n$ being a collection of orthogonal idempotent elements of R with $e_1 + \cdots + e_n = 1$. So,

$$R = Re_1 \times Re_2 \times \cdots \times Re_n.$$

By Lemma 3.7, each Re_i is a minimal ideal in R, and hence it is a field. □

4. $\mathbb{AG}(R)$ that is a pre-atomic graph

Recall from [13] that a graph G is said to satisfy the *N-condition*, if for each pair of nonadjacent vertices $u, v \in V(G)$, there exists a vertex w such that $N(u) \cup N(v) \subseteq N(w)$. By [13], For a graph G satisfying the N-condition, S is a maximum clique of G if and only if $N(S) = \{N(I) \,|\, I \in V(S)\} = Max(N(G))$. Recall from [13] that each connected graph satisfying the N-condition is a blow-up of a pre-atomic graph. So, we have the following property.

Lemma 4.1. *If G is a blow-up of a pre-atomic graph, then a subgraph S with $V(S) = \{I_i \mid i \in \Gamma\}$ is a maximum clique of G if and only if*

$$Max(N(G)) = N(S) = \{N(I) \mid I \in V(S)\}$$

and $N(I_i) \neq N(I_j)$ while $I_i, I_j \in V(S)$ and $I_i \neq I_j$.

Lemma 4.2. *Let $G = \mathbb{AG}(R)$ be a pre-atomic graph, and let S be a maximum clique of G with $V(S) = \{I_i \mid i \in \Gamma\}$. If $\{0\} \neq J \subset I_i$ for some $i \in \Gamma$, then $J \in V(S)$ and $J^2 = \{0\}$.*

Proof. It follows from G being a pre-atomic graph that S is the unique maximum clique of G. Assume that $\{0\} \neq J \subset I_i$ for some $i \in \Gamma$. If further that $J \notin V(S)$, then replace I_i by J get another maximum clique induced on $\{J\} \cup (V(S) \setminus \{I_i\})$, a contradiction. Since $J, I_i \in V(S)$, $J^2 \subseteq JI_i = \{0\}$ holds and it completes the proof. □

Proposition 4.3. *Let $G = \mathbb{AG}(R)$ be a pre-atomic graph, and let S be the maximum clique of G with $V(S) = \{I_i \mid i \in \Gamma\}$. Then for each $i \in \Gamma$, either $(I_i)^3 = \{0\}$ or I_i is a minimal ideal of R.*

Proof. Assume that $I_i \in V(S)$ and is not a minimal ideal.

If I_i is not a principal ideal, then for each nonzero $x \in I_i$, we have $Rx \subseteq I_i$ and $Rx \neq I_i$. By Lemma 4.2, we have $Rx \in V(S)$ and thus $I_i \cdot Rx = \{0\}$. Hence $(I_i)^2 = \{0\}$ and thus $I_i^3 = \{0\}$. In the following, we assume that I_i is a principal ideal of R and let $I_i = Rx$.

If further $(I_i)^2 \neq \{0\}$, then we claim that $(I_i)^2 \neq I_i$. In fact, if $Rx = Rx^2$, then there exists a nonzero $r \in R$ such that $x = rx^2$. Let $e = rx$. Clearly, $e^2 = e$ and $I_i = Re$. Since I_i is not a minimal ideal, there exists a nonzero proper $J \subset I_i$. By Lemma 4.2, $J \in V(S)$, and hence $J = JRe = JI_i = \{0\}$, a contradiction. The contradiction shows $\{0\} \neq (I_i)^2 \subset I_i$ and hence, $(I_i)^2 \in V(S)$ holds. Finally, $I_i(I_i)^2 = (I_i)^3 = \{0\}$ holds. This completes the proof. □

Proposition 4.4. *For a ring R, let $G = \mathbb{AG}(R)$ be its annihilating-ideal graph with a finite or an infinite maximum clique S. If G is a pre-atomic graph and there exists an idempotent ideal $I \in V(S)$, then R is a reduced ring.*

Proof. Let $V(S) = \{I_i \mid i \in \Gamma\}$ with $I_1^2 = I_1 = I$. Then by Proposition 4.3, I_1 is a minimal ideal of R. By Brauer's Lemma, $I = Re$, where e is an idempotent element of R. Clearly, $I = Re$ is a field. In the following, we will show that R is reduced. Otherwise, if there exists a nonzero ideal J of R such that $J^2 = \{0\}$, then consider the following two possible cases:

Case 1: $J \in V(S)$. Clearly, $J \neq I$. Assume without loss of generality that $J = I_2$. Then $V(S) \cap N(I) = V(S) \setminus \{I\} = V(S) \cap N(I + I_2)$, so $I = I + I_2$ by Proposition 1.4, contradicting the assumption on I.

Case 2: $J \notin V(S)$. It follows from I being a field that $J \cap I$ is I or $\{0\}$. Note that $J^2 = \{0\}$, so $J \cap I = \{0\}$. Since S is a maximum clique of G, J is not adjacent to every vertices of $V(S)$. Assume without loss of generality that

$J \cap I_2 \neq \{0\}$. Then $J \cap I_2 \in V(S)$ by Lemma 4.2. Hence S is a maximum clique with a square-zero ideal $J \cap I_2$ and it reduces to the case 1.

In conclusion, there exists no square-zero ideal of R, and thus R is reduced. □

Corollary 4.5. *For a ring R, let $G = \mathbb{AG}(R)$ be its annihilating-ideal graph with a maximum clique S. If $2 \leq |V(S)| \leq \infty$, and there is an idempotent ideal $I \in V(S)$, then the following statements are equivalent*:

(1) *G is a strong Boolean graph*;

(2) *G is a pre-atomic graph.*

Proof. Note that $B_2 = A_2$, so it suffices to consider the case when $3 \leq |V(S)| \leq \infty$. In the following, we only prove (2) $\Longrightarrow$ (1) since (1) $\Longrightarrow$ (2) is clear.

Assume that G is a pre-atomic graph. By Proposition 4.4, R is a reduced ring. Then G is a blow-up of a strong Boolean graph by Theorem 2.7. Note that the three conditions of Theorem 1.2 are actually Theorem 1.3(1) adding Proposition 1.4(1)(2), so G is a strong Boolean graph. □

Proposition 4.6. *For a decomposable ring R, let $G = \mathbb{AG}(R)$ be its annihilating-ideal graph with a finite or an infinite maximum clique. If G is a blow-up of a pre-atomic graph, then R is a reduced ring.*

Proof. Let $R = \prod_{i=1}^{r} R_i \, (1 < r < \infty)$, and let S be a maximum clique of G, with $V(S) = \{I_i \,|\, i \in \Gamma\}$. If R is not reduced, then there exists an $I \in V(G)$ such that $I^2 = \{0\}$. Since S is a maximum clique, I is not adjacent to every $I_i \in V(S)$. Assume without loss of generality that $I \cap I_1 \neq \{0\}$. In the following, consider the following two possible cases.

Case 1: $I \cap I_1 \in V(S)$. Assume without loss of generality that $R_1 \cap I \cap I_1 \neq \{0\}$. It follows from $(I \cap I_1)^2 = \{0\}$ and S being a maximum clique that $R_1 \cap I \cap I_1 \in V(S)$. Otherwise, $\{R_1 \cap I \cap I_1\} \cup V(S)$ induces a clique properly containing S, a contradiction. Furthermore, it is not hard to check

$$V(S) \cap N(R_2) = \{J \,|\, J \in V(S),\, J \cap R_2 = \{0\}\} = V(S) \cap N(R_1 \cap I \cap I_1 + R_2).$$

By Proposition 1.5 and the description after Theorem 1.2, $N(R_2) = N(R_1 \cap I \cap I_1 + R_2)$ holds. On the other hand, $R_1 \in N(R_2) \setminus N(R_1 \cap I \cap I_1 + R_2)$ holds, a contradiction.

Case 2: $I \cap I_1 \notin V(S)$. Note that S is a maximum clique of G, thus $(I \cap I_1)I_1 \neq \{0\}$ holds, since otherwise $\{I \cap I_1\} \cup V(S)$ induces a clique properly containing S, a contradiction. So, $\{I \cap I_1\} \cup (V(S) \setminus \{I_1\})$ induces a maximum clique with a square-zero ideal $I \cap I_1$. By a similar discussion as case 1, one can deduce a contradiction. □

The following theorem follows directly from Proposition 3.5 and Proposition 4.6, So that the proof is omitted.

Theorem 4.7. *For a decomposable ring R, let $G = \mathbb{AG}(R)$ be its annihilating-ideal graph. If G has a maximum clique S and $3 \leq |V(S)| \leq \infty$, then the following statements are equivalent*:
(1) *R is a reduced ring*;
(2) *G is a blow-up of a strong Boolean graph*;
(3) *G is a blow-up of a pre-atomic graph*;
(4) *G is a complemented graph.*

It is worth mentioning that, for a decomposable ring R, its zero-divisor graph $\Gamma(R)$ may not be a blow-up of a strong Boolean graph even though it is a blow-up of a pre-atomic graph, as the following example shows:

Example 4.8. Let R be the Boolean ring of finite and cofinite subsets of a infinite set X. It is easy to see that its zero-divisor graph $\Gamma(R)$ is a pre-atomic graph with the unique maximum clique $C(X)$. However, $\Gamma(R)$ is not a blow-up of any strong Boolean graph.

The following theorem follows from Theorem 4.7 and Proposition 2.9.

Theorem 4.9. *For a decomposable ring R, let $G = \mathbb{AG}(R)$ be its annihilating-ideal graph. If G has a maximum clique S and $3 \leq |V(S)| < \infty$, then the following statements are equivalent*:
(1) *R is a finite product of fields*;
(2) *G is a finite strong Boolean graph*;
(3) *G is a finite pre-atomic graph*;
(4) *G is a strongly complemented finite graph.*

Note that for a finite maximum clique S of $G = \mathbb{AG}(R)$, if there is an idempotent ideal $I \in V(S)$, then by Proposition 4.3, I is a minimal ideal of R. By Brauer's Lemma, R is decomposable. So, Corollary 4.5 follows also from Theorem 4.9.

5. Clique number and chromatic number of $\mathbb{AG}(R)$

In [11], the authors conjecture that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R))$ holds for every commutative ring. In this section, we will partially consider about this problem.

Since a strong Boolean graph has an identical clique number and chromatic number, and blow-up preserves the clique number and chromatic number respectively, so a blow-up of a strong Boolean graph is a graph with the clique number and chromatic number identical.

Proposition 5.1. *For a ring R, if $G = \mathbb{AG}(R)$ is a pre-atomic graph or a blow-up of a pre-atomic graph, then $\omega(G) = \chi(G)$.*

Proof. As an induced subgraph of B_n, $\chi(A_n) \leq \chi(B_n)$ clearly holds. Note that A_n contains the unique maximum clique of B_n, so $\omega(B_n) \leq \omega(A_n)$. Hence $\omega(A_n) \leq \chi(A_n) \leq \chi(B_n) = \omega(B_n) \leq \omega(A_n)$. Since a blow-up of a graph does not change the clique number and chromatic number of the graph, the proof is completed. □

It follows from Proposition 2.7 and its proof that if R is a reduced ring, even if $\omega(\mathbb{AG}(R)) = 2$, $\mathbb{AG}(R)$ is a blow-up of a strong Boolean graph. So, the following proposition is clear.

Proposition 5.2. *If R is a reduced commutative ring, then $\mathbb{AG}(R)$ is a graph with an identical clique number and chromatic number.*

Theorem 5.3. *Let $R = \prod_{i=1}^{r} R_i$ be a decomposable ring with $\omega(\mathbb{AG}(R_i)) = \chi(\mathbb{AG}(R_i)) = n_i < \infty$ for each $1 \leq i \leq r$. If for each R_i, there exists a maximum clique containing all the square-zero ideals of R_i, then $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R))$.*

Proof. Let S_i be a maximum clique of $\mathbb{AG}(R_i)$ with $V(S_i) = \{I_{i1}, \ldots, I_{in_i}\}$ containing all the square-zero ideals of R_i. Since $\omega(\mathbb{AG}(R_i)) = \chi(\mathbb{AG}(R_i))$, each $\mathbb{AG}(R_i)$ can be divide into a mutually disjoint union of subsets $C(I_{i1})$, $C(I_{i2}), \ldots, C(I_{in_i})$ ($I_{ij} \in C(I_{ij})$, $1 \leq j \leq n_i$, $1 \leq i \leq r$), with each pair of elements (ideals) in the same part being nonadjacent.

Let $A = \{I \mid I \in \cup_{i=1}^{r} V(S_i)$ and $I^2 = \{0\}\}$. For a subset $B \subseteq A$, let $I_B = \Sigma_{I\in B} I$. It is easy to see that $(\cup_{i=1}^{r} V(S_i)) \cup (\cup_{B\subseteq A}\{I_B\})$ induces a clique of $\mathbb{AG}(R)$, and then assume it is an n-clique. For any $J = J_1 \times \cdots \times J_r \in V(G) \setminus ((\cup_{i=1}^{r} V(S_i)) \cup (\cup_{B\subseteq A}\{I_B\}))$ with $J_i \subseteq R_i$, denote $in(J) = J_i$ if $(J_i)^2 \neq \{0\}$ and $(J_j)^2 = \{0\}$ for each $j \in [i-1]$. Note that $V(S_i)$ contains all the square-zero ideals of R_i, so such a J_i does exist.

In the following, we will define a collection of mutually disjoint subsets $D(I_{ij})$ of $V(\mathbb{AG}(R))$ such that $C(I_{ij}) \subseteq D(I_{ij})$ for each $I_{ij} \in \cup_{i=1}^{r} V(S_i)$. In fact, let $C(I_{ij}) \subseteq D(I_{ij})$ hold first. Then for vertices not in $\cup_{i=1}^{r} \cup_{j=1}^{n_i} C(I_{ij})$, we choose vertices of $D(I_{ij})$ in the following way:

For each $J \in V(G) \setminus ((\cup_{i=1}^{r} V(S_i)) \cup (\cup_{B\subseteq A}\{I_B\}))$, if $in(J) \subseteq R_i$ holds, there are two possible cases:

Case 1: $in(J) \in C(I_{ij})$ holds for some $j \in [n_i]$. In this case, let $J \in D(I_{ij})$.

Case 2: $in(J) \notin C(I_{ij})$ holds for each $j \in [n_i]$. In this case, let $J \in D(I_{i1})$.

It is easy to check that $\mathbb{AG}(R) \setminus \cup_{i=1}^{r} \cup_{j=1}^{n_i} D(I_{ij}) \subseteq \cup_{B\subseteq A}\{I_B\}$. Thus $V(\mathbb{AG}(R))$ is a mutually disjoint union of n subsets $D(I_{ij})$ ($I_{ij} \in \cup_{i=1}^{r} V(S_i)$) together with all singletons $\{I_B\}$ ($B \subseteq A$).

In order to complete the proof, it suffices to show that each pair of ideals in $D(I_{ij})$ is nonadjacent, i.e., for each pair $J, K \in D(I_{ij})$, $JK \neq \{0\}$. The result is clear for either $J = I_{ij}$ or $K = I_{ij}$. If $J \neq I_{ij}$ and $K \neq I_{ij}$, then no matter $in(J) = in(K)$ or $in(J) \neq in(K)$, $in(J)in(K) \neq \{0\}$ holds by the definition of $C(I_{ij})$ and the construction of $D(I_{ij})$. So, $JK \neq \{0\}$. This completes the proof. □

In the following, we are interested in the product of rings whose annihilating-ideal graphs are blow-ups of strong Boolean graphs. Let $R = \prod_{i=1}^{r} R_i$. If for each $i \in [r]$, $\mathbb{AG}(R_i)$ is a blow-up of B_{n_i} with $n_i \geq 3$, then R_i is a reduced ring for each $i \in [r]$. Hence R is a reduced ring, and thus $\mathbb{AG}(R)$ is a blow-up of a strong Boolean graph B_n by Proposition 2.7. It is not hard to check that

$n = \Sigma_{i=1}^{r} n_i$. In fact, it follows directly from the following theorem, in which, maybe $\omega(\mathbb{AG}(R_i)) = 2$ hold for some $i \in [r]$.

Theorem 5.4. *Let $R = \prod_{i=1}^{r} R_i$ with each $\mathbb{AG}(R_i)$ being a blow-up of B_{n_i} ($2 \le n_i < \infty$). If denote $l = |\{i \,|\, R_i$ is a B_2^2-type ring$\}|$ and $m = |\{i \,|\, R_i$ is a B_2^3-type ring $\}|$, then $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = \Sigma_{i=1}^{r} n_i + 2^l 3^m - l - 2m - 1$.*

Proof. Let $R = R_1 \times \cdots \times R_l \times R_{l+1} \times \cdots \times R_{l+m} \times R_{l+m+1} \times \cdots \times R_r$, with $\mathbb{AG}(R_i)$ containing a maximum clique S_i with $V(S_i) = \{I_{i1}, \ldots, I_{in_i}\}$. Without loss of generality, assume that R_i is a B_2^2-type ring with the unique square-zero ideal I_{i1} $(1 \le i \le l)$, and assume that R_{l+i} is a B_2^3-type ring with a pair of square-zero ideals $I_{l+i,1} \subseteq I_{l+i,2}$ $(1 \le i \le m)$. By Proposition 2.12, for each $i \in [m]$, $\mathbb{AG}(R_{l+i})$ is a star graph, and $I_{l+i,1}$ is the smallest nonzero ideal of R_{l+i}. Let $A = \{I \,|\, I \in \cup_{i=1}^{l+m} V(S_i)$ and $I^2 = \{0\}\}$. For a subset $B \subseteq A$, let $I_B = \Sigma_{I \in B} I$. It is easy to see that $(\cup_{i=1}^{r} V(S_i)) \cup (\cup_{B \subseteq A}\{I_B\})$ induces a clique of $\mathbb{AG}(R)$. In fact, it is a $(\Sigma_{i=1}^{r} n_i + 2^l 3^m - l - 2m - 1)$-clique, since $|\cup_{i=1}^{r} V(S_i)| = \Sigma_{i=1}^{r} n_i$, $|\cup_{B \subseteq A}\{I_B\}| = 2^l 3^m - 1$ and $|(\cup_{i=1}^{r} V(S_i)) \cap (\cup_{B \subseteq A}\{I_B\})| = l + 2m$. This shows that $\omega(\mathbb{AG}(R)) \ge \Sigma_{i=1}^{r} n_i + 2^l 3^m - l - 2m - 1$.

For each $I \in (\cup_{i=1}^{r} V(S_i)) \cup (\cup_{B \subseteq A}\{I_B\})$, we are going to define a subset $C(I)$ containing I. For this purpose, for any $J = J_1 \times \cdots \times J_r \in V(G) \setminus ((\cup_{i=1}^{r} V(S_i)) \cup (\cup_{B \subseteq A}\{I_B\}))$ with $J_i \subseteq R_i$, in a way similar to the above Theorem 5.3, let $in(J) = J_i$ whenever $(J_i)^2 \ne \{0\}$ and $(J_j)^2 = \{0\}$ holds for each $j \in [i-1]$; and let $in(J) = \{0\}$ whenever $(J_i)^2 = \{0\}$ holds for each $i \in [r]$.

In the following, we will show that $V(G)$ can be divided into mutually disjoint union of $\Sigma_{i=1}^{r} n_i + 2^l 3^m - l - 2m - 1$ subsets in the following way:

If $in(J) = J_i \ne \{0\}$, then let $J \in C(I_{i1})$ whenever $J_i \notin N(I_{ik})$ holds for each $k \in [n_i]$; and let $J \in C(I_{ij})$ whenever $J_i \notin N(I_{ij})$ and $J_i \in N(I_{ik})$ holds for each $k \in [j-1]$.

If $in(J) = \{0\}$, then clearly $J \subseteq R_1 \times \cdots \times R_{l+m}$. Let

$$B_J = \{I \in A \,|\, I \in \{J_1, \ldots J_{l+m}\} \text{ or } I \notin N(J_i) \text{ for some } i \in [l+m]\}.$$

Clearly, $B_J \subseteq A$. In this case, let $J \in C(I_{B_J})$.

Next, we will show that each pair of ideals J, K in the same set $C(I)$ is nonadjacent. If $J = I$ or $K = I$, then the result is clear. If $J \ne I$ and $K \ne I$, then there are two possible cases.

Case 1: $in(J) = J_i \ne \{0\}$ or $in(K) = K_i \ne \{0\}$. Assume without loss of generality that $in(J) = J_i \ne \{0\}$. In this case, $I \in V(S_i)$, and clearly $K_i \ne \{0\}$ holds. Then consider further the following two subcases.

Subcase 1: R_i is reduced. Since $J_i, K_i \notin N(I)$, $J \cap I \ne \{0\}$ and $K \cap I \ne \{0\}$. If $JK = \{0\}$, then $(J \cap I)(K \cap I) = \{0\}$. So, either $J \cap I = K \cap I$, or $\{J \cap I, K \cap I\} \cup (V(S_i) \setminus \{I\})$ induces a $(n_i + 1)$-clique. Since R_i is reduced and $\omega(\mathbb{AG}(R_i)) = n_i$, each of them deduces a contradiction.

Subcase 2: R_i is not reduced. Then $\mathbb{AG}(R_i)$ is a star graph. Note that $(J_i)^2 \ne \{0\}$, no matter $J_i = K_i$ or $J_i \ne K_i$, $J_i K_i \ne \{0\}$ always holds. Hence $JK \ne \{0\}$.

Case 2: $in(J) = in(K) = \{0\}$. In this case, clearly $J^2 = K^2 = I^2 = \{0\}$ hold. Hence $J, K \in R_1 \times \cdots \times R_{l+m}$. Note that $J \neq K$, by the construction of $C(I)$, there exists an I_{i2} for some $l + 1 \leq i \leq l + m$ such that $J_i \neq K_i$ and $J_i, K_i \notin N(I_{i2})$. Note that $\mathbb{AG}(R_i)$ is a star graph, so $J_i K_i \neq \{0\}$ and hence, $JK \neq \{0\}$ holds.

This shows $\chi(\mathbb{AG}(R)) \leq \Sigma_{i=1}^{r} n_i + 2^l 3^m - l - 2m - 1$. Since $\omega(\mathbb{AG}(R)) \leq \chi(\mathbb{AG}(R))$ is a known result, the proof is completed. □

Note that even though $\mathbb{AG}(R_i)$ is a blow-up of a strong Boolean graph for each $i \in [r]$, $\mathbb{AG}(R)$ may not be a blow-up of a strong Boolean graph. In fact, if some R_i is not reduced, then by Proposition 2.7, $\mathbb{AG}(R)$ is not a blow-up of a strong Boolean graph since R is not reduced.

In fact, Theorem 5.3 and Theorem 5.4 provide two distinct ideas to consider about the clique number and chromatic number. One is established on a perfect "Basis", the other one is by providing a way to well distribute all the vertices of $\mathbb{AG}(R)$ to a collection of subsets. By these ideas, maybe there is a way to deal with the conjecture given by Behboodi in [11].

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