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INFINITELY MANY SOLUTIONS FOR A CLASS OF THE ELLIPTIC SYSTEMS WITH EVEN FUNCTIONALS

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Abstract. We get a result that shows the existence of infinitely many solutions for a class of the elliptic systems involving subcritical Sobolev exponents nonlinear terms with even functionals on the bounded domain with smooth boundary. We get this result by variational method and critical point theory induced from invariant subspaces and invariant functional.

1. Introduction

In this paper we investigate existence of infinitely many solutions for a class of the elliptic systems on the bounded domain Ω of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n \geq 3$:

$$\begin{cases} Lu = \alpha u + \beta v + \frac{2p}{p+q}|u|^{p-1}|v|^q & \text{in } \Omega, \\ Lv = \beta u + \gamma v + \frac{2q}{p+q}|u|^p|v|^{q-1} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $L = -\Delta$ is the Laplace partial differential operator, α, β, γ are real constants and p, $q > 1$ are real constants with $2 < p + q < 2^*$, $2^* = \frac{2n}{n-2}$.

We know that a single elliptic boundary value problem

$$-\Delta u = u^p, \qquad 2 < p < \frac{2n}{n-2}, \qquad \text{in } \Omega, \tag{1.2}$$

$$u = 0 \qquad \text{on } \partial\Omega$$

has infinitely many solutions. In this paper we improve this single elliptic boundary value problem to the perturbation one for a class of elliptic systems. For the other system boundary problem we recommend the book [2].

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Let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ be eigenvalues of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, and ϕ_k be the eigenfunctions corresponding to the eigenvalues λ_k, $k \geq 1$. Let $W_0^{1,2}(\Omega)$ be a Sobolev space with norm

$$\|u\|^2_{W_0^{1,2}(\Omega)} = \int_\Omega |\nabla u|^2 dx.$$

Let $E = W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ be a Hilbert space endowed with the norm

$$\|(u,v)\|^2_E = \|u\|^2_{W_0^{1,2}(\Omega)} + \|v\|^2_{W_0^{1,2}(\Omega)}.$$

Let A be $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Let us set

$$W_{\lambda_i} = \text{span}\{\phi_i \mid -\Delta\phi_i = \lambda_i\phi_i\},$$
$$q_{\lambda_i}(\alpha,\beta,\gamma) = \det(\lambda_i I - A) = (\lambda_i - \alpha)(\lambda_i - \gamma) - \beta^2.$$

Let $\mu^1_{\lambda_i}$ and $\mu^2_{\lambda_i}$ be the eigenvalues of the matrix $\begin{pmatrix} \lambda_i-\alpha & -\beta \\ -\beta & \lambda_i-\gamma \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, i.e.,

$$\mu^1_{\lambda_i} = \frac{1}{2}\{-\gamma - \alpha - \sqrt{((-\gamma-\alpha))^2 - 4q_{\lambda_i}(\alpha,\beta,\gamma)}\},$$
$$\mu^2_{\lambda_i} = \frac{1}{2}\{-\gamma - \alpha + \sqrt{(-\gamma-\alpha)^2 - 4q_{\lambda_i}(\alpha,\beta,\gamma)}\}.$$

We are looking for weak solutions (u,v) of (1.1) in E. The weak solutions $(u,v) \in E$ satisfies

$$\int_\Omega [(-\Delta u, -\Delta v)\cdot(z,w) - (\alpha u + \beta v, \beta u + \gamma v)\cdot(z,w)$$
$$-(\frac{2p}{p+q}|u|^{p-1}|v|^q, \frac{2q}{p+q}|u|^p|v|^{q-1})\cdot(z,w)]dx = 0 \quad \forall (z,w) \in E.$$

We note that weak solutions of (1.1) correspond to critical points of the continuous and Frechét differentiable functional $I(u,v) \in C^1(E,\mathbb{R})$,

$$I(u,v) = \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx - \int_\Omega [\frac{2}{p+q}|u|^p|v|^q]dx$$
$$= Q_{\alpha,\beta,\gamma}(u,v) - \int_\Omega [\frac{2}{p+q}|u|^p|v|^q]dx,$$

where $Q_{\alpha,\beta,\gamma}(u,v) = \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx$. When $2 < p+q < \frac{2n}{n-2}$, the embedding $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \hookrightarrow L^{p+q}(\Omega)$ is compact, so we can assure that the functional $I(u,v)$ satisfies the $(P.S.)$ condition.

Our main result is as follows:

Theorem 1.1. *Assume that α, β, γ are real constants and p, $q > 1$ are real constants with $2 < p+q < 2^*$, $2^* = \frac{2n}{n-2}$, $n \geq 3$,*

(i) $\alpha > 0$, $\beta > 0$, $\gamma < 0$, $-\gamma > \alpha$.

(ii) $q_{\lambda_i}(\alpha,\beta,\gamma) = \det\begin{pmatrix} \lambda_i-\alpha & -\beta \\ -\beta & \lambda_i-\gamma \end{pmatrix} < 0$ *for* $1 \leq i \leq 2m$, $m \geq 1$,

and

(iii) $q_{\lambda_i}(\alpha, \beta, \gamma) > 0$, $\forall i \geq 2m+1$.
Then (1.1) *has infinitely many weak solutions.*

For the proof of Theorem 1.1 we approach variational method and use critical point theory on the invariant subspaces and the invariant functional. In Section 2, we introduce the eigenspaces spanned by the eigenfunctions corresponding to the eigenvalues of the matrix $\begin{pmatrix} \lambda_i-\alpha & -\beta \\ -\beta & \lambda_i-\gamma \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ and recall the critical point theory induced from the invariant subspaces and invariant functional. We also prove a multiplicity theorem for existence of infinitely many weak solutions which is a crucial role for the proof of main result. In Section 3, we prove that the corresponding functional of (1.1) satisfies $(P.S.)^*$ condition and prove Theorem 1.1.

2. Critical point theory on the invariant subspaces

Let $q_{\lambda_i}(\alpha, \beta, \gamma) = \det(\lambda_i I - A) = (\lambda_i - \alpha)(\lambda_i - \gamma) - \beta^2$ and $\mu^1_{\lambda_i}$ and $\mu^2_{\lambda_i}$ be the eigenvalues of the matrix $\begin{pmatrix} \lambda_i-\alpha & -\beta \\ -\beta & \lambda_i-\gamma \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. We note that

$$\begin{aligned} &\text{if} \ \ q_{\lambda_i}(\alpha,\beta,\gamma) < 0, \ \text{then} \ \mu^1_{\lambda_i} < 0 < \mu^2_{\lambda_i}, \\ &\text{if} \ \ -\gamma > \alpha \ \text{and} \ q_{\lambda_i}(\alpha,\beta,\gamma) > 0, \ \text{then} \ 0 < \mu^1_{\lambda_i} < \mu^2_{\lambda_i}, \\ &\text{if} \ \ -\gamma < \alpha \ \text{and} \ q_{\lambda_i}(\alpha,\beta,\gamma) > 0, \ \text{then} \ \mu^1_{\lambda_i} < \mu^2_{\lambda_i} < 0, \\ &\text{if} \ \ -\gamma = \alpha \ \text{and} \ q_{\lambda_i}(\alpha,\beta,\gamma) > 0, \ \text{then} \ \mu^1_{\lambda_i} = \mu^2_{\lambda_i} = 0. \end{aligned}$$

Let $(c^1_{\lambda_i}, d^1_{\lambda_i})$ and $(c^2_{\lambda_i}, d^2_{\lambda_i})$ be the eigenvectors of $\begin{pmatrix} \lambda_i-\alpha & -\beta \\ -\beta & \lambda_i-\gamma \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ corresponding to $\mu^1_{\lambda_i}$ and $\mu^2_{\lambda_i}$, respectively. Let us set

$$\begin{aligned} D_{\lambda_i} &= \{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \mid q_{\lambda_i}(\alpha,\beta,\gamma) < 0 \text{ for } \ 1 \leq i \leq 2m, \ m \geq 1, \\ &\qquad q_{\lambda_i}(\alpha,\beta,\gamma) > 0, \ \forall i \geq 2m+1\}, \\ D'_{\lambda_i} &= D_{\lambda_i} \cap \{-\gamma \leq \alpha\}, \\ D''_{\lambda_i} &= D_{\lambda_i} \cap \{-\gamma > \alpha\}, \\ E_{\lambda_i} &= \{(c\phi, d\phi) \in E \mid (c,d) \in \mathbb{R}^2, \phi \in H_{\lambda_i}\}, \\ E^1_{\lambda_i} &= \{(c^1_{\lambda_i}\phi, d^1_{\lambda_i}\phi) \in E \mid \phi \in H_{\lambda_i}\}, \\ E^2_{\lambda_i} &= \{(c^2_{\lambda_i}\phi, d^2_{\lambda_i}\phi) \in E \mid \phi \in H_{\lambda_i}\}, \\ H^+(\alpha,\beta,\gamma) &= (\oplus_{\mu^1_{\lambda_i} > 0} E^1_{\lambda_i}) \oplus (\oplus_{\mu^2_{\lambda_i} > 0} E^2_{\lambda_i}), \\ H^-(\alpha,\beta,\gamma) &= (\oplus_{\mu^1_{\lambda_i} < 0} E^1_{\lambda_i}) \oplus (\oplus_{\mu^2_{\lambda_i} < 0} E^2_{\lambda_i}), \\ H^0(\alpha,\beta,\gamma) &= (\oplus_{\mu^1_{\lambda_i} = 0} E^1_{\lambda_i}) \oplus (\oplus_{\mu^2_{\lambda_i} = 0} E^2_{\lambda_i}). \end{aligned}$$

Then $H^+(\alpha,\beta,\gamma)$, $H^-(\alpha,\beta,\gamma)$ and $H^0(\alpha,\beta,\gamma)$ are the positive, negative and null space relative to the quadratic form $Q_{\alpha,\beta,\gamma}(u,v)$ in E. Because $(\lambda_i - \alpha)(\lambda_i - \gamma) - b^2 \neq 0$,

$$H^0(\alpha,\beta,\gamma) = \{0\}.$$

Let us set

$$H_n = \oplus_{1 \le i \le n} E_{\lambda_i}, \qquad \dim\, H_n = 2n. \tag{2.1}$$

Then

$$H_1 \subset H_2 \subset \cdots \subset H_n \quad \text{and} \quad \overline{\cup_{n=1}^{\infty} H_n} = E.$$

Let us define an $\mathbb{Z}_2$-action T on E by

$$TU = -U, \qquad U = (u, v) \in E,$$

where $-U = (-u, -v)$. Let

$$\mathrm{Fix}_{\mathbb{Z}_2} = \{U \in E \mid TU = U \ \ \forall U \in E\}.$$

Then E has two orthogonal subspaces E_1 and $E_2 = E_1^{\perp}$ such that

$$E = E_1 \oplus E_2, \qquad E_1 = \mathrm{Fix}_{\mathbb{Z}_2}$$

and $\mathbb{Z}^2$-action has the representation

$$U \mapsto -U \qquad \forall U = (u, v) \in E_2.$$

Thus $\mathbb{Z}_2$ acts freely on the invariant subspace E_2. We say a subset B of E an $\mathbb{Z}_2$-invariant set if for all $U \in B$ and $TU \in B$. A function $f : E \to \mathbb{R}$ is called $\mathbb{Z}^2$-invariant if $f(TU) = f(U)$. Let $C(B, E)$ be the set of continuous functions from B into E. If B is an invariant set, we say $h \in C(B, E)$ is an equivariant map if $h(TU) = Th(U)$ for all $U \in B$.

Now we recall the multiplicity theorem.

Let X be a Hilbert space and S_r be the sphere centered at the origin of radius r. Let $I : X \to \mathbb{R}$ be a functional of the form

$$I(U) = \frac{1}{2}(LU)U - \psi(U),$$

where $L : X \to X$ is linear, continuous, symmetric and equivariant, $\psi : X \to \mathbb{R}$ is of class C^1 and invariant and $D\psi : X \to X$ is compact. The following result follows from [1].

Theorem 2.1. *Assume that*

(I1) $I \in C^1(X, \mathbb{R})$ *is* $\mathbb{Z}_2$*-invariant,*

(I2) *there exist two closed invariant linear subspaces* V, Y *of* X *and two regular values* $a < b$, $a > 0$ *with*

$$\frac{1}{2}\mathrm{codim} Y < \frac{1}{2}\dim V < \infty$$

such that

(1) $V + Y$ *is closed and of finite codimension in* X;

(2) $\mathrm{Fix}\{\mathbb{Z}_2\} \subseteq Y$ *and* $\mathrm{Fix}\{\mathbb{Z}_2\} \cap V = \{\theta\}$, *where* $\theta = (0, 0)$;

(3) $L(Y) \subseteq Y$;

(4) $\inf_{U \in S_r \cap Y} I(U) > a$ *for some* $r > 0$;

(5) $\sup_{U \in V} I(U) < b$, $b > a$;

(6) $u \notin \mathrm{Fix}_{\mathbb{Z}_2}$ *whenever* $DI(U) = \theta$ *and* $a \le I(U) \le b$.

(I3) $I(U)$ *satisfies* $(P.S.)_c$ *condition for all* $c \in [a, b]$.
Then I *has at least*

$$\frac{1}{2}\dim V - \frac{1}{2}\text{codim}Y$$

distinct critical points.

Let

$$M = \{B \mid B \subset E\backslash\{0\}, B \text{ is closed and invariant}\},$$

$$D_m = B_{R_m} \cap \big(((\oplus_{1\le i\le m} E^1_{\lambda_i}) \oplus (\oplus_{1\le i\le m} E^2_{\lambda_i})) \cap (\oplus_{i\ge 2m+1} E^1_{\lambda_i})\big).$$

Let G_m denote the class of mapping $h \in C(D_m, X)$ which satisfy the following properties:

i) h is equivariant.

ii) h is odd and $h(u) = u$ $\forall u \in (\partial B_{R_m} \cap \big(((\oplus_{1\le i\le 2m} E^1_{\lambda_i}) \oplus (\oplus_{1\le i\le m} E^2_{\lambda_i})) \cap (\oplus_{i\ge 2m+1} E^1_{\lambda_i})\big) \cup \{(0, 0)\}$.

iii) $Ph(u) = \alpha(u)Pu + \psi(u)$, where ψ is compact and $\alpha \in C(D_m, [1, \bar{\alpha}])$, $\bar{\alpha}$ depending on h.

By Theorem 2.1, we obtain the following result:

Theorem 2.2. *Suppose that* $I \in C^1(E, \mathbb{R})$ *is even with* $I(0, 0) = 0$*, and that*

(i) *there exist* ρ, $\tau > 0$*, and a finite dimensional linear subspace* F *such that* $I|_{F^\perp\cap S_\rho} \ge \tau$,

(ii) *there exists a sequence of linear subspaces* H_m, $\dim H_m = 2m$ *and* $R_m > 0$ *such that*

$$I(U) \le 0 \qquad \forall U \in H_m\backslash B_{R_m}, \qquad m = 1,\ 2, \ldots.$$

(iii) $I(U)$ *satisfies* $(P.S.)^*$ *condition with respect to* $\{H_n\}$*. Then* I *possesses infinitely many distinct critical points corresponding to positive critical values*

$$c_i = \inf_{h\in\Gamma} \sup_{U\in V_i} I(h(U))$$

for each i, $1 \le i \le m - j \le \dim(V\backslash F) - \text{codim}(V + F^\perp)$, $m \to \infty$*, where*

$$\Gamma = \{\overline{h(B_{R_m} \cap V\backslash Y)} \mid m \ge j,\ h \in G_m,\ \textit{odd and } Y \in M, \dim Y \le j\},$$

and where $V_i \subset B_{R_m} \cap (V\backslash Y)$ *a fixed subspace of dimension*

$$\dim(V_i\backslash F) = i.$$

Proof. By contradiction, we suppose that I has at least l critical points. Let $F = ((\oplus_{1\le i\le 2m} E^1_{\lambda_i}) \oplus (\oplus_{1\le i\le j} E^2_{\lambda_i})) \cap (\oplus_{i\ge 2m+1} E^1_{\lambda_i})$ and choose $m - j > l$. We note that $I \in C^1(E, \mathbb{R})$, $\text{Fix}_{\mathbb{Z}_2} = \{(0, 0)\}$ and by (iii), $I(U)$ satisfies $(P.S.)^*$ condition with respect to $\{H_m\}$. Let us set $V = ((\oplus_{1\le i\le 2m} E^1_{\lambda_i}) \oplus (\oplus_{1\le i\le m} E^2_{\lambda_i})) \cap (\oplus_{i\ge 2m+1} E^1_{\lambda_i})$, $Y = F^\perp$, $a = \tau$ and $b = \max_{U\in V} I(U) + 1$. Then by Theorem 2.1, I has at least $m - j$ distinct critical points, which is a contradiction. □

3. Proof of Theorem 1.1

We note that weak solutions of (1.1) coincide with critical points of the functional $I(u,v) \in C^{1,1}(E,\mathbb{R})$,

$$I(u,v) = \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx$$

$$= Q_{\alpha,\beta,\gamma} - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx, \tag{3.1}$$

where $Q_{\alpha,\beta,\gamma} = \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx$.

Let us define

$$C_{p,q}(\Omega) = \inf_{(u,v)\in E\setminus(0,0)} \frac{\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx}{(\int_\Omega |u|^p|v|^q dx)^{\frac{2}{p+q}}} \quad \text{for } (u,v) \in E. \tag{3.2}$$

Lemma 3.1. *Assume that α, β, γ are real constants and p, $q > 1$ are real constants, $2 < p+q < 2^*$, $2^* = \frac{2n}{n-2}$,*

(i) $\alpha > 0$, $\beta > 0$, $\gamma < 0$, $-\gamma > \alpha$.

(ii) $q_{\lambda_i}(\alpha,\beta,\gamma) = \det\begin{pmatrix} \lambda_i-\alpha & -\beta \\ -\beta & \lambda_i-\gamma \end{pmatrix} < 0$ *for* $1 \le i \le 2m$, $m \ge 1$,

and

(iii) $q_{\lambda_i}(\alpha,\beta,\gamma) > 0$, $\forall i \ge 2m+1$.

Let $i \in N$ and $(\alpha_0,\beta_0,\gamma_0) \in \partial D'_{\lambda_i}$. Then there exist a neighborhood W of $(\alpha_0,\beta_0,\gamma_0)$ and two closed invariant subspaces V and Y of E such that for any $(\alpha,\beta,\gamma) \in W\setminus D'_{\lambda_i}$,

(1) *$V + Y$ is closed and of finite codimension in E;*

(2) *$\mathrm{Fix}\{\mathbb{Z}_2\} \subseteq V$ and $\mathrm{Fix}\{\mathbb{Z}_2\} \cap Y = \{\theta\}$, where $\theta = (0,0)$;*

(3) *$L(u,v) = -\Delta(u,v) - A\cdot(u,v)$, where $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Then $L(Y) \subseteq Y$;*

(4) *there exist a small number $r > 0$, $a > 0$ and $b > a$ such that*

$$\inf_{(u,v)\in S_r\cap Y} I(u,v) > a, \qquad \sup_{(u,v)\in V} I(u,v) < b \qquad \text{for some } b > a$$

and

$$\inf_{(u,v)\in B_r\cap Y} I(u,v) > -\infty.$$

Proof. (1) Let us set

$$V = ((\oplus_{1\le j\le 2m} E^1_{\lambda j}) \oplus (\oplus_{1\le j\le 2m} E^2_{\lambda j}) \cap (\oplus_{j\ge 2m+1} E^1_{\lambda j})),$$

$$Y = H^+(\alpha,\beta,\gamma) = ((\oplus_{1\le j\le 2m} E^2_{\lambda j}) \oplus (\oplus_{j\ge 2m+1} E^1_{\lambda j}) \oplus (\oplus_{j\ge 2m+1} E^2_{\lambda j})).$$

Then

$$E = V + Y, \qquad V \cap Y,$$

$$\dim(V\cap Y) \ge 2m, \quad \mathrm{codim}(V+Y) = 0, \quad 2m \le \mathrm{codim} Y < \dim V < \infty.$$

(2) Since $\mathrm{Fix}_{\mathbb{Z}_2} = \{\theta\}$, $\mathrm{Fix}\{\mathbb{Z}_2\} \subseteq Y$ and $\mathrm{Fix}\{\mathbb{Z}_2\} \cap V = \{\theta\}$.

(3) Let $(u,v) \in Y = H^+(\alpha,\beta,\gamma) = ((\oplus_{1\le j\le 2m} E^2_{\lambda j}) \oplus (\oplus_{j\ge 2m+1} E^1_{\lambda j}) \oplus (\oplus_{j\ge 2m+1} E^2_{\lambda j}))$. Then (u,v) can be expressed by

$$(u,v) \in \mathrm{span}\{(c^1\phi_j, d^1\phi_j) \mid j \ge 2m+1\} \oplus \mathrm{span}\{(c^2\phi_j, d^2\phi_j) \mid 1 \le j \le 2m\} \oplus \mathrm{span}\{(c^2\phi_j, d^2\phi_j) \mid j \ge 2m+1\}.$$

Then we have

$$\begin{aligned} L(u,v) &= (-\Delta - A)\cdot(u,v) \\ &\in (\mathrm{span}\{\mu^1_{\lambda_j}(c^1\phi_j, d^1\phi_j) \mid j \ge 2m+1\} \\ &\quad \oplus \mathrm{span}\{\mu^2_{\lambda_j}(c^2\phi_j, d^2\phi_j) \mid 1 \le j \le 2m\} \\ &\quad \oplus \mathrm{span}\{\mu^2_{\lambda_j}(c^2\phi_j, d^2\phi_j) \mid j \ge 2m+1\}) \cap X \subset H^+(\alpha,\beta,\gamma) = Y, \end{aligned}$$

so (3) is proved.

(4) Let

$$(u,v) \in Y = H^+(\alpha,\beta,\gamma) = ((\oplus_{1\le j\le 2m} E^2_{\lambda j}) \oplus (\oplus_{j\ge 2m+1} E^1_{\lambda j}) \oplus (\oplus_{j\ge 2m+1} E^2_{\lambda j})).$$

Then we have

$$\begin{aligned} I(u,v) &= \frac{1}{2}\int_\Omega (-\Delta - A)(u,v)\cdot(u,v)dx - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx \\ &\ge \frac{1}{2}\min\{\frac{\mu^2_{\lambda_1}}{\lambda_1}, \frac{\mu^1_{\lambda_{2m+1}}}{\lambda_{2m+1}}\}\|(u,v)\|^2_E - \frac{2}{p+q} C_{p,q}^{-\frac{2}{p+q}} \|(u,v)\|_E^{p+q}. \end{aligned}$$

Since $\min\{\frac{\mu^2_{\lambda_1}}{\lambda_1}, \frac{\mu^1_{\lambda_{2m+1}}}{\lambda_{2m+1}}\} > 0$ and $p+q > 2$, there exists a small number $r > 0$ such that if $(u,v) \in Y$, then $\inf_{(u,v)\in S_r\cap Y} I(u,v) > a$ for $a > 0$. Moreover if $(u,v) \in B_r \cap Y$, then $I(u,v) \ge -\frac{2}{p+q} C_{p,q}^{-\frac{2}{p+q}} \|(u,v)\|_E^{p+q} > -\infty$. Let $(u,v) \in V = ((\oplus_{1\le j\le 2m} E^1_{\lambda_j}) \oplus (\oplus_{1\le j\le 2m} E^2_{\lambda_j}) \oplus (\oplus_{j\ge 2m+1} E^1_{\lambda_j}))$. Then we have

$$\begin{aligned} I(u,v) &= \frac{1}{2}\int_\Omega (-\Delta - A)(u,v)\cdot(u,v)dx - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx \\ &\le \frac{1}{2}\max\{\frac{\mu^2_{\lambda_{2m}}}{\lambda_{2m}}, \frac{\mu^1_{\lambda_{2m+1}}}{\lambda_{2m+1}}\}\|(u,v)\|^2_E - \frac{2}{p+q}\int_\Omega |u|^p|v|^q \\ &\le \frac{1}{2}\max\{\frac{\mu^2_{\lambda_{2m}}}{\lambda_{2m}}, \frac{\mu^1_{\lambda_{2m+1}}}{\lambda_{2m+1}}\}\|(u,v)\|^2_E = b < \infty \end{aligned}$$

for $b > a > 0$. Thus the lemma is proved. □

Lemma 3.2. *Assume that α, β, γ are real constants, p, $q > 1$ are real constants with $2 < p+q < 2^*$ and the conditions* (i), (ii), (iii) *of Theorem 1.1 hold. Let $i \in \mathbb{N}$ and $(\alpha_0,\beta_0,\gamma_0) \in \partial D'_{\lambda_i}$. Then there exists a neighborhood W of $(\alpha_0,\beta_0,\gamma_0)$ such that for any $(\alpha,\beta,\gamma) \in W\backslash D'_{\lambda_i}$. If (u,v) is a critical point of $I(u,v)$, i.e., $DI(u,v) = \theta$ and $(u,v) \in \mathrm{Fix}\{\mathbb{Z}_2\}$, then $I(u,v) = 0$.*

Proof. We note that $\mathrm{Fix}\{\mathbb{Z}_2\} = \{\theta\}$. Thus we have that if $(u,v) \in \mathrm{Fix}\{\mathbb{Z}_2\} = \{\theta\}$, then $I(u,v) = 0$. □

Lemma 3.3. *Assume that* α, β, γ *are real constants,* p, $q > 1$ *are real constants with* $2 < p+q < 2^*$ *and the conditions* (i), (ii), (iii) *of Theorem 1.1 hold. Then if* $\|(u_n, v_n)\|_E \to \infty$ *and* $(u_n, v_n)_n$ *is a sequence such that*

$$\frac{\int_\Omega [(\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q, \frac{2q}{p+q}|u_n|^p|v_n|^{q-1}) \cdot (u_n, v_n) - \frac{4}{p+q}|u_n|^p|v_n|^q]dx}{\|(u_n, v_n)\|_E} \longrightarrow 0,$$

then there exist $(u_{h_n}, v_{h_n})_n$ *and* $(z, w) \in E$ *such that*

$$(\frac{2p}{p+q}|u_{h_n}|^{p-1}|v_{h_n}|^q, \frac{2q}{p+q}|u_{h_n}|^p|v_{h_n}|^{q-1}) \to (z, w) \in E,$$

$$\frac{(u_{h_n}, v_{h_n})}{\|(u_{h_n}, v_{h_n})\|_E} \to (0, 0).$$

Proof. We note that

$$\begin{aligned}
&\int_\Omega [\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n + \frac{2q}{p+q}|u_n|^p|v_n|^{q-1})v_n]dx - \frac{4}{p+q}\int_\Omega |u_n|^p|v_n|^q dx \\
\le\ &\int_\Omega [\frac{2p}{p+q}|u_n|^p|v_n|^q + \frac{2q}{p+q}|u_n|^p|v_n|^q)]dx - \frac{4}{p+q}\int_\Omega |u_n|^p|v_n|^q dx \\
\le\ &(\frac{2p}{p+q} + \frac{2q}{p+q} - \frac{4}{p+q})\int_\Omega |u_n|^p|v_n|^q dx \\
\le\ &C_{p,q}^{-\frac{2}{p+q}}(\Omega)(\frac{2p}{p+q} + \frac{2q}{p+q} - \frac{4}{p+q})\|(u_n, v_n)\|_E^{p+q}, \quad 2 < p+q < 2^*.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|\frac{\int_\Omega [\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n + \frac{2q}{p+q}|u_n|^p|v_n|^{q-1})v_n]dx}{\|(u_n, v_n)\|_E}\|_{L^r} \\
\le\ &C_{p,q}^{-\frac{2}{p+q}}(\Omega)(\frac{2p}{p+q} + \frac{2q}{p+q})\|\|\|(u_n, v_n)\|_E^{p+q-1}\|_{L^r} \\
\le\ &C(\frac{\|(u_n, v_n)\|_E^{p+q}}{\|(u_n, v_n)\|_E})^{\frac{p+q-1}{p+q}}\|(u_n, v_n)\|^l,
\end{aligned}$$

where $l = -1 + \frac{p+q-1}{p+q} < 0$. When $2 < p+q < \frac{2n}{n-2}$, the embedding $W_0^{1,2}(\Omega, R^2) \hookrightarrow L^{p+q}(\Omega)$ is compact. Thus there exist $(u_{h_n}, v_{h_n})_n$ in E such that

$$\begin{aligned}
(3.3)\quad &\frac{\int_\Omega [\frac{2p}{p+q}|u_{h_n}|^{p-1}|v_{h_n}|^q u_{h_n} + \frac{2q}{p+q}|u_{h_n}|^p|v_{h_n}|^{q-1})v_{h_n}]dx}{\|(u_{h_n}, v_{h_n})\|_E} \\
&= \int_\Omega (\frac{2p}{p+q}|u_{h_n}|^{p-1}|v_{h_n}|^q, \frac{2q}{p+q}|u_{h_n}|^p|v_{h_n}|^{q-1}) \cdot \frac{(u_{h_n}, v_{h_n})}{\|(u_{h_n}, v_{h_n})\|_E} dx \longrightarrow 0.
\end{aligned}$$

It follows that there exists $(z, w) \in E$ such that

$$(\frac{2p}{p+q}|u_{h_n}|^{p-1}|v_{h_n}|^q, \frac{2q}{p+q}|u_{h_n}|^p|v_{h_n}|^{q-1}) \to (z, w) \in E,$$

$$\frac{(u_{h_n}, v_{h_n})}{\|(u_{h_n}, v_{h_n})\|_E} \to (0,0).$$

□

Lemma 3.4 ((P.S.) condition)**.** *Assume that α, β, γ are real constants, p, $q > 1$ are real constants with $2 < p+q < 2^*$ and the conditions* (i), (ii), (iii) *of Theorem 1.1 hold. Let $i \in \mathbb{N}$ and $(\alpha_0, \beta_0, \gamma_0) \in \partial D'_{\lambda_i}$. Then there exists a neighborhood W of $(\alpha_0, \beta_0, \gamma_0)$ such that for any $(\alpha, \beta, \gamma) \in W \setminus \cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in \partial D'_{\lambda_i}} D'_{\lambda_i}$, the functional $I(u,v)$ satisfies $(P.S.)_c$ condition for any $c \in [a,b]$.*

Proof. Let $i \in N$, $(\alpha_0, \beta_0, \gamma_0) \in \partial D'_{\lambda_i}$ and W be a neighborhood of $(\alpha_0, \beta_0, \gamma_0)$. Let $(\alpha, \beta, \gamma) \in W \setminus \cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in \partial D'_{\lambda_i}} D'_{\lambda_i}$. Let $c \in \mathbb{R}$ and $(u_n, v_n)_n \subset E$ be a sequence such that $I(u_n, v_n) \to c$ and $DI(u_n, v_n) \to \theta$, $\theta = (0,0)$. We claim that $(u_n, v_n)_n$ is bounded in E. By contradiction we suppose that $\|(u_n, v_n)\|_E \to \infty$ and set $(\hat{u_n}, \hat{v_n}) = \frac{(u_n,v_n)}{\|(u_n,v_n)\|_E}$. Since $(\hat{u_n}, \hat{v_n})_n$ is bounded, up to a subsequence, $(\hat{u_n}, \hat{v_n})_n$ converges weakly to some $(\hat{u}, \hat{v})$ in E. Let $(\alpha, \beta, \gamma) \in W \setminus \cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in \partial D'_{\lambda_i}} D'_{\lambda_i}$. Since $DI(u_n, v_n) \to 0$, we have

$$\begin{aligned} &\langle (-\Delta - A)\cdot(\hat{u_n}, \hat{v_n}), (\hat{u_n}, \hat{v_n})\rangle \\ &\quad - \langle \frac{\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n + \frac{2q}{p+q}|u_n|^p|v_n|^{q-1})v_n}{\|(u_n,v_n)\|_E}, (\hat{u_n}, \hat{v_n})\rangle \longrightarrow 0. \end{aligned} \tag{3.4}$$

Since $DI(u_n, v_n) \to 0$ and $I(u_n, v_n) \to c$, we also have

$$\begin{aligned} &\frac{DI(u_n,v_n)\cdot(u_n,v_n)}{\|(u_n,v_n)\|} \\ &= \frac{2I(u_n,v_n)}{\|(u_n,v_n)\|_E} - \frac{\int_\Omega(\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n + \frac{2q}{p+q}|u_n|^p|v_n|^{q-1}v_n - \frac{4}{p+q}|u_n|^p|v_n|^q)dx}{\|(u_n,v_n)\|_E} \longrightarrow 0. \end{aligned}$$

Thus we have

$$\frac{\int_\Omega(\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n + \frac{2q}{p+q}|u_n|^p|v_n|^{q-1}v_n - \frac{4}{p+q}|u_n|^p|v_n|^q)dx}{\|(u_n,v_n)\|_E} \longrightarrow 0. \tag{3.5}$$

By Lemma 3.1, (3.3) and (3.5), there exists a sequence $(u_{h_n}, v_{h_n})_n$ such that

$$\frac{\int_\Omega[\frac{2p}{p+q}|u_{h_n}|^{p-1}|v_{h_n}|^q u_{h_n} + \frac{2q}{p+q}|u_{h_n}|^p|v_{h_n}|^{q-1})v_{h_n}]dx}{\|(u_{h_n}, v_{h_n})\|_E} \longrightarrow 0$$

and

$$\frac{(u_{h_n}, v_{h_n})}{\|(u_{h_n}, v_{h_n})\|_E} \to (0,0).$$

Thus we have $(\hat{u}, \hat{v}) = (0,0)$, which is absurd because $\|(\hat{u}, \hat{v})\|_E = 1$. Thus $(u_n, v_n)_n$ is bounded. Thus $(u_n, v_n)_n$ has a subsequence converging weakly to some (u,v) in E. Let $P_- : E \to H^-(\alpha,\beta,\gamma) = \oplus_{\mu^1_{\lambda_i}<0,\ 1\le i\le 2m} E^1_{\lambda_i}$ and $P_+ : E \to H^+(\alpha,\beta,\gamma) = ((\oplus_{\mu^2_{\lambda_i}>0, 1\ \le i\le 2m} E^2_{\lambda_i}) \oplus (\oplus_{\mu^1_{\lambda_i}>0,\ i\ge 2m+1} E^1_{\lambda_i}))$ denote the orthogonal projections. We claim that the subsequence of (u_n, v_n) converges to $(u, v) \in E$ strongly. Since $DI(u_n, v_n) \to (0, 0)$, we have

$$\begin{aligned}
&\langle DI(u_n, v_n), (u_n, v_n)\rangle \\
= &\int_\Omega [(-\Delta u_n)u_n + (-\Delta v_n)v_n - \alpha u_n^2 - \beta v_n v_n - \beta u_n v_n - \gamma v_n^2]dx \\
&- \int_\Omega (\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n + \frac{2q}{p+q}|u_n|^p|v_n|^{q-1}v_n)dx \longrightarrow 0.
\end{aligned}$$

Since (u_n, v_n) has a subsequence converging to (u, v) weakly and the embedding $W_0^{1,2}(\Omega, \mathbb{R}^2) \hookrightarrow L^{p+q}(\Omega)$ for $2 < p+q < \frac{2n}{n-2}$ is compact, the sequence $(\int_\Omega (\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n + \frac{2q}{p+q}|u_n|^p|v_n|^{q-1}v_n)dx)_n$ has a subsequence converging to $\int_\Omega (\frac{2p}{p+q}|u|^{p-1}|v|^q u + \frac{2q}{p+q}|u|^p|v|^{q-1}v)dx$. Since

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega (-\Delta - A)(u_n, v_n) \cdot (u_n, v_n)dx \\
= &\lim_{n\to\infty} (\|P_+(u_n, v_n)\|_E^2 - \|P_-(u_n, v_n)\|_E^2) \\
= &\lim_{n\to\infty} \int_\Omega (\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n + \frac{2q}{p+q}|u_n|^p|v_n|^{q-1}v_n)dx,
\end{aligned}$$

$(\int_\Omega (-\Delta - A)(u_n, v_n) \cdot (u_n, v_n)dx)_n$ has a subsequence converging to $\int_\Omega (-\Delta - A)(u, v) \cdot (u, v)dx$. Since $(u_n, v_n)_n$ is bounded, $(-\Delta - A)(u_n, v_n)$ has a subsequence converging weakly to $(-\Delta - A)(u, v)$. Since $(-\Delta - A)^{-1}$ is compact, (u_n, v_n) has a subsequence converging strongly to (u, v). Thus the lemma is proved. □

Let us set

$$H_m = \oplus_{1\le i\le n} E_{\lambda_i}, \qquad \dim\ H_m = 2m.$$

Then

$$H_1 \subset H_2 \subset \cdots \subset H_m \quad \text{and} \quad \overline{\cup_{m=1}^\infty H_m} = E.$$

Lemma 3.5 ($(P.S.)^*$ condition)**.** *Assume that α, β, γ are real constants, p, $q > 1$ are real constants with $2 < p+q < 2^*$ and the conditions* (i), (ii), (iii) *of Theorem 1.1 hold. Let $i \in N$ and $(\alpha_0, \beta_0, \gamma_0) \in \partial D'_{\lambda_i}$. Then there exists a neighborhood W of $(\alpha_0, \beta_0, \gamma_0)$ such that for any $(\alpha, \beta, \gamma) \in W \setminus \cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}} D'_{\lambda_i}$, the functional $I(u, v)$ satisfies $(P.S.)^*$ condition with respect to $\{H_m\}$.*

Proof. Let us set

$$\begin{aligned}
I(u, v) &= \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx \\
&= Q_{\alpha,\beta,\gamma} - \Psi(u, v),
\end{aligned}$$

where $Q_{\alpha,\beta,\gamma} = \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx$ and $\Psi(u, v) = \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx$. For fixed m, choose any sequence $\{(U_m)_i\} \subset H_m$, $m =$

$1, 2, \ldots$, $U = (u, v)$. Let $L^m = L|_{H_m}$, $I^m = I|_{H_m}$, $Q^m_{\alpha,\beta,\gamma} = Q_{\alpha,\beta,\gamma}|_{H_m}$ and $\Psi^m(u,v) = \Psi(u,v)|_{H_m}$. For $\{(U_m)_1^\infty\} \subset H_m$, $U = (u, v)$, $m = 1, 2, \ldots$ such that $DI^m(U_m) \to (0, 0)$ and $I^m(U_m)$ is bounded, we shall find a convergent subsequence. Let $P_+ : E \to H^+(\alpha, \beta, \gamma)$ be an orthogonal projection from E onto $H^+(\alpha, \beta, \gamma)$ and $P_- : E \to H^-(\alpha, \beta, \gamma)$ be one from E onto $H^-(\alpha, \beta, \gamma)$ respectively. From $DI^m(U_m) \to (0, 0)$, it follows that $\forall\epsilon > 0$, there exists $N = N(\epsilon)$ such that for $m > N$

$$\begin{aligned}
&\langle DI^m(U_m, P_\pm U_m)\rangle \\
&= \langle (L^m - A)U_m, P_\pm U_m\rangle - \langle (\frac{2p}{p+q}|u_m|^{p-1}|v_m|^q, \frac{2q}{p+q}|u_m|^p|v_m|^{q-1}), P_\pm U_m\rangle \\
&\le \epsilon \|P_\pm U_m\|_E.
\end{aligned}$$

By the same process of the proof of Lemma 3.4, $\|P_\pm U_m\|_E$ is bounded, and then $\langle (L^m - A)U_m, P_\pm U_m\rangle$ are bounded. If $I^m(U_m)$ is bounded, then by the same process of the proof of Lemma 3.4, the sequence

$$\langle (\frac{2p}{p+q}|u_m|^{p-1}|v_m|^q, \frac{2q}{p+q}|u_m|^p|v_m|^{q-1}), P_\pm U_m\rangle$$

has a convergent subsequence. Thus there exists a subsequence (U_{m_i}) such that $\langle (\frac{2p}{p+q}|u_m|^{p-1}|v_m|^q, \frac{2q}{p+q}|u_m|^p|v_m|^{q-1}), P_\pm U_m\rangle$ is convergent. By

$$\begin{aligned}
DI^m(U_{m_i}) &= P_+(L^m - A)P_+U_{m_i} + P_-(L^m - A)P_-U_{m_i} \\
&\quad - (\frac{2p}{p+q}|u_{m_i}|^{p-1}|v_{m_i}|^q, \frac{2q}{p+q}|u_{m_i}|^p|v_{m_i}|^{q-1}) \longrightarrow (0, 0)
\end{aligned}$$

and by the compactness of $(P_\pm(L^m - A))^{-1}$, $P_\pm U_{m_i}$ is convergent. Thus $(P.S.)^*$ condition holds. □

Proof of Theorem 1.1. We note that I is $C^1(E, R)$ and even functional with $I(0, 0) = 0$, so I is $\mathbb{Z}_2$-invariant functional. In fact,

$$\begin{aligned}
I(-u, -v) &= \frac{1}{2}\int_\Omega [|\nabla(-u)|^2 + |\nabla(-v)|^2 - \alpha(-u)^2 - 2\beta(-u)(-v) - \gamma(-v)^2]dx \\
&\quad - \frac{2}{p+q}\int_\Omega |-u|^p|-v|^q dx \\
&= \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx \\
&= I(u, v),
\end{aligned}$$

so I is even functional. Let us set

$$F = \oplus_{1\le i\le 2m} E^1_{\lambda_i}.$$

Then $F^\perp = H^+(\alpha, \beta, \gamma) = Y$. By Lemma 3.1, there exist $r > 0$ and $a > 0$ such that

$$\inf_{(u,v)\in F^\perp\cap S_r} I(u, v) = \inf_{(u,v)\in Y\cap S_r} I(u, v) > a,$$

so the condition (i) of Theorem 2.2 is satisfied. Let us set

$$H_m = \oplus_{1\le i\le n} E_{\lambda_i}, \qquad \dim\ H_m = 2m.$$

Then

$$H_1 \subset H_2 \subset \cdots \subset H_m \quad \text{and} \quad \overline{\cup_{m=1}^{\infty} H_m} = E.$$

Let $(u,v) \in H_m$. Then we have

$$\begin{aligned} I(u,v) &= \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx \\ &= \frac{1}{2}(\|P_+(u,v)\|_E^2 - \|P_-(u,v)\|_E^2) - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx \\ &\le \frac{1}{2}\|(u,v)\|_E^2 - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx. \end{aligned}$$

Since $p+q>2$, there exists $R_m>0$ such that $\|(u,v)\|_E > R_m$, $|u| > R_m$, $|v| > R_m$ and if $(u,v) \in H_m \backslash B_{R_m}$, then

$$I(u,v) \quad \le \quad \frac{1}{2}R_m^2 - \frac{1}{2}R_m^{p+q}|\Omega| < 0,$$

so the condition (ii) of Theorem 2.2 is satisfied. By Lemma 3.5, $I(u,v)$ satisfies $(P.S.)^*$ condition with respect to $\{H_m\}$, so (iii) of Theorem 2.2 is satisfied. Thus by Theorem 2.2, I has infinitely many distinct critical points corresponding to positive critical values

$$c_i = \inf_{h\in\Gamma}\sup_{U\in V_i} I(h(U))$$

for each i, $1 \le i \le m - j \le V\backslash F - \operatorname{codim}(V + F^\perp)$, $m \to \infty$, where

$$\Gamma = \{\overline{h(B_{R_m}\cap V\backslash Y)} \mid m \ge j,\ h \in G_m,\ \text{odd and } Y \in M,\ \dim Y \le j\},$$

and where $V_i \subset B_{R_m} \cap (V\backslash Y)$ a fixed subspace of dimension

$$\dim(V_i\backslash F) = i. \qquad \square$$

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References

[1] K. C. Chang, *Infinite dimensional Morse theory and multiple solution problems*, Birkhäuser, 1993.

[2] M. Ghergu and V. D. Rădulescu, *Singular Elliptic Problems Bifurcation and Asymptotic Analysis*, Clarendon Press Oxford, 2008.

[3] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, 65. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

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