GEOMETRIC RANK AND THE TUCKER PROPERTY

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Abstract. An open smooth manifold is said of *finite geometric rank* if it admits a handlebody decomposition with a finite number of 1-handles. We prove that, if there exists a proper submanifold W^{n+3} of finite geometric rank between an open 3-manifold V^3 and its stabilization $V^3 \times B^n$ (where B^n denotes the standard n-ball), then the manifold V^3 has the *Tucker property*. This means that for any compact submanifold $C \subset V^3$, the fundamental group $\pi_1(V^3 - C)$ is finitely generated. In the irreducible case this implies that V^3 has a well-behaved compactification.

1. Definitions and statement of the main result

A powerful tool in differential topology is the handle decomposition, which helps the study of a manifold by subdividing it into simpler objects (a sort of i-dimensional cells). More precisely, a handle decomposition of a smooth, compact n-manifold M is a union

$$\emptyset = M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{m-1} \subset M_m = M,$$

where each M_i is obtained from M_{i-1} by attaching *handles* H^j of index i (where $H^j = D^j \times D^{n-j}$). Handle decomposition is very useful for understanding compact manifolds, as showed by S. Smale in his well-known papers [24, 25]. Actually, any smooth, compact, simply-connected manifold of dimension at least 5 admits a handlebody decomposition without 1-handles. This was shown by Smale in his proof of the generalized Poincaré Conjecture. In dimension 3, the corresponding statement is just equivalent to the Poincaré Conjecture (and then true). Hence, finding handlebody decompositions without handles of index 1 may have very remarkable consequences.

For open manifolds the situation is much more delicate, and V. Poénaru intensively studied non-compact manifolds possessing a handlebody decomposition with no 1-handle (see [15, 16, 17, 18, 19, 20, 21]).

Definition 1. A smooth manifold M is said to be *geometrically simply connected* (or GSC) if it admits a proper handlebody decomposition without handles of index one.

Remark 1.1. Equivalently, M is GSC if it admits a proper Morse function f without critical points of index 1. For more details on this important and interesting notion see [2, 11, 15] and [3, 6, 10, 12, 13, 17, 18].

Since any smooth or piecewise-linear (PL-) manifold admits a handlebody decomposition (see [4] for PL-manifolds and handle-theory), for manifolds which are not necessarily simply connected, it is meaningful to propose the following definition:

Definition 2. Let V be an open smooth manifold. The *geometric rank* of V, denoted by $r_g(V)$, is the minimal number of 1-handles, for all possible handlebody decompositions of V.

Of course, $r_g(V) = 0$ if and only if the manifold V is *geometrically simply connected*, but in general, one may wonder whether there exists a handlebody decomposition with only a finite number of 1-handles. The so-defined geometric rank of a manifold V is obviously related to the (algebraic) rank of the fundamental group of V, defined as the minimal number of generators for $\pi_1(M)$; in fact, $\mathrm{rank}\big(\pi_1(M)\big) \leq r_g(V)$ (because any 1-handle that is not in canceling position with any other handle gives rise to (at most) a generator).

Definition 3. The smooth open n-manifold V^n is said of *finite geometric rank* if $r_g(V) < \infty$.

This means that there exists at least one handlebody decomposition with only a finite number of 1-handles. Let us explain with more details what we intend by this. An open n-manifold V admits a handlebody decomposition with finitely many 1-handles if there exists an increasing sequence of compact bounded PL-manifolds, X_i^n, exhausting V, and such that the following three conditions are fulfilled:

- the manifold X_i^n is obtained by X_{i-1}^n with the addition of a handle of index ≥ 2;
- any X_i^n is touched only by finitely many handles;
- $V^n = \bigcup_i X_i^n$.

Clearly, the first submanifold of the sequence, X_0^n, is just a compact, bounded PL-manifold for which the map $\pi_1(V^n) \to \pi_1(X_0^n)$ is surjective.

Remark 1.2.

- All compact manifolds have finite geometric rank.
- Manifolds which are geometrically simply connected (i.e., with $r_g = 0$) are simply connected.
- By Smale's results [24, 25], compact simply connected manifolds of dimension greater than four have $r_g = 0$. In dimension 4 the same result

is false (via Casson's handles), while in dimension 3 it is equivalent to the Poincaré Conjecture, and therefore true!

- Open simply connected n-manifolds of higher dimension ($n \geq 5$) are geometrically simply connected provided they are also simply connected at infinity (see [20]).

Our main theorem gives some information on this topic.

Theorem 1 (Main Theorem). *Let V^3 be an open 3-manifold, and B^n the ball of dimension n. If there exist a number n and a manifold W^{n+3} of finite geometric rank, such that $V^3 \subset W^{n+3} \subset V^3 \times B^n$, where the last inclusion is proper, then $\pi_1(V^3 - C)$ is finitely generated for any compact and connected submanifold $C \subset V$.*

Theorem 1 is a sort of generalization of the main result of [15], where V. Poénaru proves that, for an open simply connected 3-manifold V^3, the geometric simple connectivity of $V^3 \times D^n$ implies the simple connectivity at infinity of V^3. Note also that it is very likely that the proof of Theorem 1 can be generalized to any dimension ≥ 3.

2. Technical reminders

Since our proof is based on [14] and [15], let us introduce briefly Poénaru's main tools for relating the geometry of the fundamental group of a 3-manifold with the topology at infinity of its universal cover, via the *zipping-process* (or the Ψ/Φ-technology) developed in [14]. This theory is a practical strategy for getting rid of the singularities of simplicial maps, but still preserving some useful topological information (for more details see also [16, 17, 18, 19, 21]).

Poénaru considered the very general situation of a non-degenerate simplicial map $f : X \to M^n$, where M^n is a closed (triangulated) n-manifold and X a locally-finite simplicial complex. The map f being *non-degenerate* (i.e., the image of a simplex is a simplex of the same dimension), one always has $\dim X \leq \dim M^n$. Although the main applications of these techniques are suited for dimension 3, the whole theory works in any dimension (see [17]).

Given the map f, we denote by $\mathrm{Sing}(f) \subset X$ the set of *singularities* of f, where a point $x \notin \mathrm{Sing}(f)$ if, in a neighborhood of x, f is an embedding (i.e., in $X - \mathrm{Sing}(f)$ the map f is an immersion), and by $M^2(f)$ the set of *double points* of f, namely:

$$M^2(f) = \{(x, y) \in X \times X \mid x \neq y \text{ and } f(x) = f(y)\}.$$

We will also consider the set $M_2(f) = \{x \in X \mid \#f^{-1}f(\{x\}) \geq 2\} \subset X$, that is the set of double points of f viewed as a subset of X.

In [14] V. Poénaru analyzed in detail two equivalence relations on X related to f, denoted by $\Psi(f)$ and $\Phi(f)$. The first one is the equivalence relation $\Phi(f) \subset X \times X$ yielded by the map f, for which $x \sim y$ if and only if $f(x) = f(y)$ (equivalently, $\Phi(f) = M^2(f) \cup \mathrm{Diag}(X)$); while the second equivalence relation

attached to f, denoted by $\Psi(f)$, is "forced" by the singularities of f and can be defined as the "smallest" equivalence relation on X, compatible with f, killing all the singularities of f (i.e., such that the induced map $\bar{f} : X/\Psi(f) \to M^n$ is an immersion).

Remark 2.1.

- Note that, the quotient $X/\Phi(f)$ is just the image $f(X)$, and, in general, it has no memory of the topology of X.
- Usually, the passage from X to $X/\Psi(f)$ is called a "*zipping*" or a *zipping strategy*.

Before stating the main results concerning these two equivalence relations, let us explain what "compatible with f" means. Let σ_1 and σ_2 be two simplexes of X of the same dimension. An equivalence relation $\mathcal{R} \subset X \times X$ is *compatible* with f if the following holds. If $x \in$ int σ_1 and $y \in$ int σ_2, with $f(x) = f(y)$ and $f(\sigma_1) = f(\sigma_2)$, then $x \sim_\mathcal{R} y$ implies that $\mathcal{R}$ identifies σ_1 to σ_2. Compatible equivalence relations have the useful property to keep the simplicial structure of the quotient space $X/\mathcal{R}$.

The next theorem shows the existence of such a $\Psi(f)$, and defines it in a unique way:

Theorem 2 (Poénaru, [14, 17]). *Let f be a non-degenerate simplicial map $f : X \to M^n$, where M^n is a closed (triangulated) n-manifold and X a locally-finite simplicial complex.*

(1) *There exists a unique, well-defined equivalence relation $\Psi(f) \subset \Phi(f) \subset X \times X$ characterized by the following two properties:*

    - In the commutative diagram (1) below, the map $\bar{f}$ is an immersion (with $\mathrm{Sing}(\bar{f}) = \emptyset$) :

(1)

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;f\;\;} & M^n \\
& \searrow{\scriptstyle \pi} \quad \nearrow{\scriptstyle \bar{f} = \bar{f}_\Psi} & \\
& X/\Psi(f) &
\end{array}
$$

    - For any other equivalence relation $\mathcal{R} \subset \Phi(f)$ compatible with f, with $\mathrm{Sing}(\bar{f}_\mathcal{R}) = \emptyset$ and such that $\mathcal{R} \subset \Psi(f)$, one has $\mathcal{R} = \Psi(f)$.

(2) *The induced map on fundamental groups $\pi_* : \pi_1(X) \to \pi_1\big(X/\Psi(f)\big)$ is surjective.*

As stated, Theorem 2 is not constructive for Ψ, but one can construct it via successive *folding maps* (see [14]), by killing in a smart way all the singularities of f (in this sense Ψ is the equivalence relation forced by the singularities). Roughly speaking, $X/\Psi(f)$ is obtained by a sequence of successive quotients given by all the foldings commanded by the singular points of f. In this way all the singularities disappear and no new others are created. More precisely, this is how the procedure works:

Step 1. Pick a singularity $x_1 \in \mathrm{Sing}(f)$. There exists then a pair of distinct simplexes of the same dimension, $\sigma_1, \sigma_2 \subset X$ such that $x_1 \in \sigma_1 \cap \sigma_2$, with $f(x) = f(y)$ and $f(\sigma_1) = f(\sigma_2)$. Now, go to a first quotient of X, $X_1 = X/\mathcal{R}_1$, by identifying σ_1 with σ_2 with a *folding map*, which obviously kills the singularity x_1.

Step 2. If it exists, take then another singularity x_2 of the natural map $f_1 : X_1 \to M^n$, and proceed as in the previous step. In this way one gets a second quotient $X_2 = X/\mathcal{R}_2$ together with a map $f_2 : X_2 \to M^n$, for which neither x_1 nor x_2 belongs to $\mathrm{Sing}(f_2)$.

Step 3. Repeat the same operations again and again. If the starting complex X was finite, then the process will stop after finitely many steps and the last quotient is $X/\Psi(f)$. On another hand, for an infinite X, this process provides an infinite sequence of quotients, corresponding to the sequence of equivalence relations $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \cdots \subset \Phi(f)$. Taking $\mathcal{R}_\omega = \bigcup_i \mathcal{R}_i$ the union of all $\mathcal{R}_i$, one creates a new quotient $X_\omega = X/\mathcal{R}_\omega$ with its associated map $f_\omega : X_\omega \to M$.

Step 4. Executing afresh Steps 1, 2 and 3 for $f_\omega : X_\omega \to M$, one can go on with a transfinite induction which provides a transfinite sequence of equivalence relations, $\mathcal{R}_\omega \subset \mathcal{R}_{\omega+1} \subset \cdots \subset \Phi(f)$. Now, since X is assumed to have (at most) a countable number of simplexes, this process eventually comes to an end at some countable ordinal ω_1 with $\mathrm{Sing}(f_{\omega_1}) = \emptyset$.

Since the equivalence relation $\Psi(f)$ is unique by Theorem 2, then $\Psi(f) = \mathcal{R}_{\omega_1}$. This also implies that $\Psi(f)$ is independent of the various choices made during the process just described. Of course, there are several ways leading to it, but in [14] it is proved that one may opportunely choose already the first sequence $(f_i)_{\mathbb{N}}$ of folding maps, in order to have $\Psi(f) = \mathcal{R}_\omega = \bigcup_i \mathcal{R}_i$ (i.e., without going to the transfinite number ω_1).

Remark 2.2.

- Let us stress that the equivalence relation $\Phi(f)$ kills all the double points, while $\Psi(f)$ kills all the singularities. The special case when the equality $\Psi(f) = \Phi(f)$ holds, means in particular that when one kills all the singularities, there are no more double points left. Or, in other words, $\Psi(f) = \Phi(f)$ implies that the smallest equivalence relation $\mathcal{R}$ compatible with f which is such that $X/\mathcal{R} \to Y$ is an immersion (i.e., without singularities), is simply the trivial equivalence relation Φ defined by f.
- Notice also that whenever the condition $\Psi(f) = \Phi(f)$ is satisfied, then one may obtain interesting results relating the geometry of the fundamental group of a closed manifold with the topology at infinity of its universal cover (see [1, 15, 16, 19] and also [5, 7, 8, 9, 10, 17, 18, 21]).

For the proofs and all the details on these equivalence relations Ψ and Φ see [14] and [17].

2.1. Cellulations and bisections

We end our technical reminders section by recalling a class of "good subdivisions" of topological spaces (from [22]) which we will need afterwards for proving some of our lemmas. Whenever nice subdivisions are considered, one normally thinks of linear subdivisions of simplicial complexes, that include e.g. both barycentric and stellar subdivisions. But, from our point of view, only these last subdivisions are pretty good because they clearly preserve both geometric simple connectivity and the finite geometric rank condition. On the other hand, generally speaking, not all linear subdivisions preserve them. Nevertheless, there exists a very useful and clear tool invented by L. Siebenmann [22], which perfectly fits into this issue.

Actually, in [22], it is introduced the notion of *cellulation*, which is a sort of generalization of (triangulations of) simplicial complexes, where simplexes are replaced by compact cells with a linear-convex structure. More precisely:

Definition 4. A *cellulation* of a metric space X is given by a locally finite covering by compact cells such that: (a) each cell is convex with respect to the linear structure induced by an homeomorphism with the convex hull of a finite set of points in an Euclidean space; (b) the formal interiors of the convex cells form a partition of X; (c) the formal boundary ∂C of any convex cell C, is the union of a finite numbers of cells, which are linearly included in C.

Remark 2.3. Recall that, the *formal interior* of a convex compact subset C of a vector space is the set of all points x such that, for any line l passing through x, the segment $l \cap C$ contains x in its interior. The complementary, in C, of the formal interior is called the *formal boundary* of C, denoted by ∂C.

In this new setting, the notions of linear subdivision and sub-complex extend trivially for cellulations. The real nice novelty is that any subdivision $\hat{C}$ of a sub-cellulation C of a cellulation X extends, in a canonical way, to a subdivision of X, not influencing the open cells of $X - C$. The main class of subdivisions for cellulations is constituted by the so-called *bisections*. Bisections are, for cellulations, the natural transformations corresponding to subdivisions of triangulations.

Definition 5. A *bisection* cuts a cell C with a linear hyperplane H_C of codimension 1, and splits C into two non-empty pieces C_+ and C_-. The inverse operation is called a *coupling*. The support of a bisection is the closure of the hyperplane section in C (i.e., $H_C \cap C$), and an infinite family of bisections whose union of supports does not have accumulation points is said *proper*.

The main reasons for using bisections and cellulations instead of simplicial complexes and subdivisions are given by the following two facts (see [22]):

- Let X be a cellulation and Y a sub-complex. Then, any subdivision of Y by bisections induces canonically a subdivision (by bisections too) of X not affecting those cells of X which are not cells of Y too.
- Let X be a cellulation and consider two subdivisions X' and X'' of X. There exists then a third cellulation X''', which is a common refinement of them, namely such that one can go from both X' and X'' to X''' via a proper family of bisections.

Finally, if X is a cellulation, one can subdivide it to a simplicial complex X_s in a simple and canonical way. This simply works as follows: for any 2-dimensional cell, one takes a point in its interior and then subdivides the cell in the obvious way. After, one performs the same procedure for 3-cells, then for 4-cells and so forth. Eventually, one gets a simplicial complex X_s, and the operation going from X to X_s is called a *stellation*.

Remark 2.4. The main interest for us is that bisections and stellations preserve both the GSC property and the FGR property.

From now on, we will consider only subdivisions such as barycentric, stellar or Siebenmann bisections. We call them *nice*. These kinds of subdivisions have the property that, given a triangulation whose 3-skeleton is FGR, then it stays FGR after nice subdivisions.

3. Technical lemmas

In this section we will start the proof of our main theorem, by giving several preliminary lemmas. Let us begin with some definitions:

Definition 6. A finite simplicial complex Y is obtained from the subcomplex X by a *Whitehead dilatation* [27] (also called an *internal expansion*) if $\text{int}(Y - X)$ is one simplex whose closure intersects X along a disk which can be either a simple face or a connected union of several faces. The inverse operation is called an *internal collapse*.

Definition 7. Let X^3 be a 3-dimensional complex. We say that X^3 has the FGR *property* if it admits an exhaustion

$$K_1 \overset{j_1}{\subset} K_2 \overset{j_2}{\subset} \cdots \overset{j_\infty}{\subset} X^3 = \cup_i K_i$$

by finite simplicial complexes for which any inclusion j_n is either an addition of a cell of dimension ≥ 2, or a Whitehead dilatation.

Remark 3.1.

- The fundamental group of K_1 does not increase when we go towards K_n.
- A simplicial complex is FGR if and only if its 3-dimensional skeleton is FGR.

- Consider a simplicial complex X with the property FGR and a linear subdivision X_1 of it. By means of Siebenmann's bisections [22], it is easy to prove that there exists a subdivision of X_1 with the property FGR.

Proposition 1. *Let W^m be a smooth open manifold of finite geometric rank. Then there exists a smooth triangulation Θ of W^m whose 3-skeleton Θ^3 is FGR.*

Proof. Since the manifold W^m has finite geometric rank, we can write it as $W^m = B^m + \{\text{finite number of 1-handles}\} + \{h\text{-handles, with } h \geq 2\}$. Consider now a triangulation Θ of W^m compatible with this decomposition, in the sense that $B^m + \{\text{all the 1-handles}\}$ is a subcomplex, $B^m + \{\text{all the 1-handles}\} + \{\text{one 2-handle}\}$ is a subcomplex too, and so on.

Denote now by K_1 the 3-skeleton of $B^m + \{\text{all the 1-handles}\}$, K_2 the 3-skeleton of $B^m + \{\text{all the 1-handles}\} + \{\text{one 2-handle}\}$, and so on. We have then a sequence of finite complexes as in Definition 7. For instance this can be done in such a way that any addition of a handle of index ≥ 2 can be realized as a Whitehead dilatation or an addition of a cell of dimension ≥ 2. This means that the attaching zone $S^{\lambda-1} \times B^{m-\lambda} \subset B^\lambda \times B^{m-\lambda}$, with $\lambda > 1$, can be triangulated in such a way that the passage from the 3-skeleton of the attaching zone to the 3-skeleton of the handle can be realized by Whitehead dilatations or additions of cells of dimension $\lambda \geq 2$ (this may be easily done by means of Siebenmann's bisections). Hence, the 3-skeleton $\Theta^3 = \cup_i K_i$ of the triangulation Θ, is FGR. $\square$

For the next lemmas we will need also a geometric triangulation for the manifold $V^3 \times B^n$. Consider then some smooth triangulations for V^3 and B^n, and take the product cellulation for $V^3 \times B^n$. By subdividing it sufficiently many times, we can obtain

($*$) a triangulation τ, for which $V^3 \times \{q\} \subset V^3 \times B^n$ is a subcomplex, for some point $q \in B^n$.

Moreover, if there exists a manifold W^{n+3} such that $V^3 \subset W^{n+3} \subset V^3 \times B^n$, where the last inclusion is proper, then the triangulation τ can be chosen so that W^{n+3} is a subcomplex too.

Lemma 1. *Consider the manifold $V^3 \times B^n$ with the triangulation τ as in ($*$), and suppose that there exists a manifold W^{n+3} of finite geometric rank, such that $V^3 \subset W^{n+3} \subset V^3 \times B^n$, where the last inclusion is proper. Then, we can find a triangulation $\mathcal{T}$ of W^{n+3} whose 3-dimensional skeleton $\mathcal{T}^3$ has the property FGR, and such that, for some point $q \in B^n$, $V^3 \times \{q\} \subset W^{n+3}$ is a subcomplex of $\mathcal{T}^3$.*

Proof. Since both triangulations Θ (from Proposition 1) and τ (from above) are compatible with the same differential structure on W^{n+3}, the (classical) smooth Hauptvermutung theorem of Whitehead (that proves the unicity of a PL structure compatible with a given differential structure, see [4]) implies that

there exists a subdivision τ_1 of τ isomorphic to some subdivision of Θ. Since the 3-skeleton of Θ is FGR, by the second item of Remark 3.1, there will exist a subdivision $\mathcal{T}$ of τ_1 whose 3-skeleton $\mathcal{T}^3$ is FGR. This ends the proof. $\qquad \square$

Before going on, we need to refine again our triangulation τ of $V^3 \times B^n$ and to provide an affine structure on it (as done in [1, 15]). We will consider totally geodesic simplexes which become smaller and smaller as they approach to infinity. Since 3-manifolds are parallelizables, by the Smale-Hirsh immersion theory [23] there exists a smooth immersion of V^3 in $\mathbb{R}^3$ inducing a flat (non-complete) Riemannian metric on V^3. Hence, whenever exist geodesics connecting two points, they are unique.

Consider now an exhaustion of V^3 by compact submanifolds with boundary M_j such that $M_j \subset \operatorname{int}(M_{j+1})$, and a function $f : \mathbb{N} \to \mathbb{R}$ which decreases sufficiently fast so that the following requirements are fulfilled: for each point $x \in M_n - M_{n+1}$ there is a closed subset $C(x) \subset V^3$ such that, first, any point $y \in V^3$ which can be joined to x by a geodesic arc of length less than $f(n)$ belongs to $C(x)$, and second, there is an isometry I_x between $C(x)$ and the Euclidean ball of radius $f(n)$ centered at 0, which maps x into 0. Another technical condition we need is that for any vertex v of our triangulation, the iterated star should be contained in $\operatorname{int}(C(v))$.

Now, the factor B^n of $V^3 \times B^n$ will be considered simply as a simplex linearly embedded in $\mathbb{R}^n$, endowed with a very fine linear subdivision. In this way our $V^3 \times B^n$ is endowed with a flat product metric and all the pieces $C(v) \times B^n$ have a canonical affine structure.

Lemma 2. *Let V^3 be an open smooth manifold and suppose the existence of an open smooth manifold of finite geometric rank W^{n+3} containing V^3 and properly included in $V^3 \times B^n$. Then there exists a smooth triangulation $\mathcal{D}$ of $V^3 \times B^n$ such that*:

    (1) *W^{n+3} is a subcomplex of $\mathcal{D}$;*
    (2) *the 3-skeleton of W^{n+3}, denoted by Δ^3, is FGR;*
    (3) *V^3 is a subcomplex of Δ^3;*
    (4) *the projection map $p : V^3 \times B^n \to V^3 \times \{0\}$ restricted to Δ^3, namely $p|_{\Delta^3} : \Delta^3 \to V^3$, is simplicial and non-degenerate (possibly after nice subdivisions, or generic perturbations).*

Proof. For the proof of this lemma we will make use of the affine structure defined above (we will follow [15], but the reader may find further details in [1]). We first need to slightly modify all the simplexes σ of the triangulation τ. Since the projection $p(\sigma)$ of any simplex σ is contained in the interior of a $C(v)$, for some vertex v, we can suppose that our simplexes σ verify the following condition:

There exists a locally finite cover $\bigcup_{x_i} \operatorname{int} C(x_i)$ of V^3 such that, if $p(\sigma_{i_1}) \cap p(\sigma_{i_2}) \cap \cdots \cap p(\sigma_{i_l}) \neq \emptyset$, then we can find a $C(x_j)$ containing in its interior $p(\sigma_{i_1}) \cup p(\sigma_{i_2}) \cup \cdots \cup p(\sigma_{i_l})$.

We can now perform a small perturbation inside $V^3 \times B^n$ changing the position of the vertices v of the simplexes $\sigma \in \tau$ in such a way that the perturbed vertices w satisfy all the conditions below:

(1) If $v_j \in$ int $C(v_j)$, then $w_j \in$ int $C(w_j)$.

(2) If $v_{l_1}, v_{l_2}, \ldots, v_{l_q}$ span a $(q-1)$-simplex σ of τ, then the corresponding perturbed vertices are in general position and span an affine $(q-1)$-simplex σ'. In this way, the set of all the perturbed simplexes σ' form a new triangulation $\mathcal{D}$ of $V^3 \times B^n$ which is isomorphic to τ.

(3) If the dimension of σ' is ≤ 3, then $p|_{\sigma'}$ is an affine isomorphism, and all the $p(\sigma')$ are in general position (for dim $\sigma' \leq 3$).

(4) If $v \in V^3 \times \{0\}$, then $v' = v$.

Recall by the previous Lemma 1 that W^{n+3} admits a triangulation $\mathcal{T}$ whose 3-skeleton $\mathcal{T}^3$ is FGR. But since W^{n+3} is also a subcomplex of the triangulation τ, it will be a subcomplex of $\mathcal{D}$ too. Appealing again to Whitehead's Hauptvermutung and Remark 3.1 (second item), we can find a subdivision of the triangulation of W^{n+3} whose 3-dimensional skeleton is FGR. With this, we can produce the required Δ^3 by subdividing opportunely. $\qquad \square$

Lemma 2 tells us that there is a triangulation of W^{n+3} whose 3-skeleton Δ^3 is exhausted by an increasing union of finite simplicial complexes

$$K_1 \overset{i_1}{\subset} K_2 \overset{i_2}{\subset} \cdots \overset{i_\infty}{\subset} \Delta^3 = \cup_i K_i,$$

where each inclusion i_n is either an addition of a cell of dimension ≥ 2, or a Whitehead dilatation. From now on, for the projection map $p : V^3 \times B^n \to V^3 \times \{0\}$, we will use the following notation:

$$\Psi_\infty = \Psi(p|_{\Delta^3}), \ \Phi_\infty = \Phi(p|_{\Delta^3}), \ \Psi_n = \Psi(p|_{K_n}) \text{ and } \Phi_n = \Phi(p|_{K_n}).$$

Lemma 3. *The equality $\Psi_\infty = \Phi_\infty$ holds.*

Proof. In the diagram below

(2)
$$\begin{array}{ccc} \Delta^3 & \xrightarrow{\ p\ } & V^3 \\ & \searrow \quad \nearrow_{p_1} & \\ & \Delta^3/\Psi_\infty & \end{array}$$

the map p_1 is a surjection (because $V^3 \subset \Delta^3$) without singularities (by definition of Ψ_∞). Let us show it is also injective. For this, consider the following commutative diagram:

(3)
$$\begin{array}{ccc} \Delta^3/\Psi_\infty & \xrightarrow{\ p_1\ } & V^3 \\ & \searrow_i \quad \nearrow_{id} & \\ & V^3/\Psi_\infty & \end{array}$$

If we show that i is surjective, we will automatically have the injectivity of p_1, and hence it will be a bijection. Suppose i not surjective. Then there exist two simplexes σ_1 and σ_2 such that $\sigma_1 \in \text{Im}(i)$ and $\sigma_2 \in \Delta^3/\Psi_\infty - \text{Im}(i)$. But since we are in a connected space, we can find a continuous sequence of simplexes between σ_1 and σ_2. Hence, there will exist a point x belonging to two simplexes α and β of the sequence, which sits at the boundary of $\text{Im}(i)$, i.e., for which $\alpha \in \text{Im}(i)$ and $\beta \in \Delta^3/\Psi_\infty - \text{Im}(i)$. This point x is then a singularity for the map $p_1 : \Delta^3/\Psi_\infty \to V^3$, because V^3 is contained in Δ^3/Ψ_∞. But this is not possible because the map p_1 has no singularities (by definition of Ψ_∞).

Hence, in the diagram (2), the map p_1 is bijective. Now, we want to prove that $\Psi_\infty = \Phi_\infty$. Putting together diagram (2) with the same diagram where instead of Δ^3/Ψ_∞ one puts Δ^3/Φ_∞, we obtain two bijections

$$\Delta^3/\Psi_\infty \xleftrightarrow{p_1} V^3 \text{ and } \Delta^3/\Phi_\infty \xleftrightarrow{id} V^3.$$

Knowing that by definition $\Psi_\infty \subset \Phi_\infty$, the desired equality $\Psi_\infty = \Phi_\infty$ obviously follows. $\square$

Lemma 4. *For any n there exists an $N(n)$ such that $\Psi_N|_{K_n} = \Phi_n$.*

Proof. We know that $\Psi_\infty|K_n = \Phi_n$, and so the result is true for $N = \infty$. However, since we need a finite N, we will use a compactness argument. First of all, we know that there exists a sequence of folding maps ρ_n such that $\cup_n \rho_n = \Psi_\infty$. Now, our K_n is a finite complex so it involves only a finite number of simplexes. Hence, for some finite $N(n)$, the folding $\rho_{N(n)}$ exhausts all the foldings touching K_n. The result follows. $\square$

Recall that our objective is to show that for any compact subset C of V^3, $\pi_1(V^3 - C)$ is finitely generated. In order to prove this, we will work with the tubular neighborhood K of C (for which holds the inclusion $C \subset \text{int}(K)$).

Lemma 5. *For any such a K (tubular neighborhood of some compact subset), there exist a natural number $N \in \mathbb{N}$ and an inclusion i which make commutative the following diagram:*

(4)

$$K \overset{i}{\hookrightarrow} i(K) \subset K_N/\Psi_N \subset \Delta^3/\Psi_N$$
$$\searrow \qquad \swarrow_{p_N}$$
$$V^3$$

and such that, for the map $g : \Delta^3/\Psi_N \to V^3$, the Dehn-condition holds:

$$i(K) \cap \{M_2(g), \text{Sing}(g)\} = \emptyset.$$

Proof. For the compact K we have the following inclusions $K \subset V^3 = V^3 \times \{0\} \subset X^{n+3} \subset V^3 \times B^n$ and $V^3 \times \{0\} \hookrightarrow \Delta^3 = \cup_i K_i \hookrightarrow X^{n+3}$. Now, being K compact, there exists an m_0 such that $K_{m_0} \cap V^3 \times \{0\} \supset K$, where K_{m_0} is a compact of the exhaustion of Δ^3. Consider now the subspace $p^{-1}|_{\Delta^3}[(p|_{\Delta^3})(K_{m_0})]$: it is compact because B^n is compact and the inclusion of X^{n+3} is proper.

Hence $p^{-1}|_{\Delta^3}[(p|_{\Delta^3})(K_{m_0})]$ is contained in some other compact subset K_m. With these two compacts we have that if a double point x belongs to $M_2(p)$ and $y \in K_{m_0}$ is such that $p(y) = p(x)$, then $x \in K_m$; in other words all the singularities and the double points which touch K_{m_0} cannot come from $\Delta^3 - K_m$. The previous Lemma 4 tells us that there exists a $N = N(m)$ such that $\Psi_N/K_m = \Phi_m$. Thus we have that:

$$K_N/\Psi_N \supset K_m/\Psi_N = K_m/\Phi_m = p(K_m) \supset K.$$

Consider now the inclusions $K \hookrightarrow K_N/\Psi_N \subset \Delta^3/\Psi_N$, and the following maps:

$$i : K \hookrightarrow V^3, \ p_N : K_N/\Psi_N \to V^3 \text{ and } g : \Delta^3/\Psi_N \to V^3.$$

If $(x, y) \in M^2(g)$ with $x \in K$, then $y \in K_m$, and so $(x, y) \in M^2(g|_{K_m})$; but the map $g|_{K_m}$ is just p_N and our equivalence relation Ψ_N does not have neither singularities nor double points in K_m. This completes the proof. $\qquad \square$

Lemma 6. *If $X \xhookrightarrow{i} Y$ is a good inclusion, meaning that Y is obtained by X via either an addition of a cell of dimension ≥ 2, or a Whitehead dilatation, and if $\mathcal{R}$ is an equivalent relation on X, then $X/\mathcal{R} \hookrightarrow Y/\mathcal{R}$ is a good inclusion too.*

Proof. Obvious. $\qquad \square$

4. Proof of Theorem 1

We are ready to prove our Main Theorem.

Proof. By Lemma 2, we have a first exhaustion

$$K_1 \overset{i_1}{\subset} K_2 \overset{i_2}{\subset} \cdots \overset{i_\infty}{\subset} \Delta^3$$

and its corresponding sequence of inclusions

$$K_N/\Psi_N \overset{j_N}{\hookrightarrow} K_{N+1}/\Psi_N \overset{j_{N+1}}{\hookrightarrow} K_{N+2}/\Psi_N \overset{j_{N+2}}{\hookrightarrow} \cdots \overset{j_\infty}{\hookrightarrow} \Delta^3/\Psi_N$$

which are good inclusions by Lemma 6.

Now, take a compact subset C in V^3 and denote by K its tubular neighborhood. Of course, there exists an N such that $K \subset K_N/\Psi_N$. Now, denoting by H_j the quotient K_{N+j}/Ψ_N, consider the following sequence of inclusions:

$$K \subset K_N/\Psi_N = H_0 \overset{j_0}{\hookrightarrow} H_1 \overset{j_1}{\hookrightarrow} H_2 \overset{j_2}{\hookrightarrow} \cdots \overset{j_\infty}{\hookrightarrow} H_\infty = \Delta^3/\Psi_N.$$

What we know for this sequence is that all the inclusions j_m are good, and that the map $g : H_\infty \to V^3$ is simplicial and non degenerate, for which $\Psi(g) = \Phi(g)$ (by Lemma 3). Furthermore, by Lemma 5, we know that $K \subset H_\infty$ satisfies the Dehn-condition: $K \cap \{M_2(g), \text{Sing}(g)\} = \emptyset$.

We can then consider the corresponding sequence of inclusions of complements of K, namely:

$$H_0 - K \overset{j_0}{\hookrightarrow} H_1 - K \overset{j_1}{\hookrightarrow} H_2 - K \overset{j_2}{\hookrightarrow} \cdots \overset{j_\infty}{\hookrightarrow} H_\infty - K \overset{g'}{\to} V^3 - K.$$

These are also good inclusions with the property that $\Psi(g') = \Phi(g')$. Now, from this last sequence of subspaces, we go to the associated sequence of fundamental groups obtaining:

$$\pi_1(H_0 - K) \to \pi_1(H_1 - K) \to \pi_1(H_2 - K) \to \cdots \to \pi_1(H_\infty - K) \to \pi_1(V^3 - K).$$

Since $\pi_1(H_0 - K)$ is finitely generated (obviously) and all the inclusions j_i are good, in the sequence above our π_1 does not grow! Hence $\pi_1(H_\infty - K)$ is finitely generated too.

To end up we need to observe that $V^3 - K$ is just $g'(H_\infty - K)$ which is, in turn, equal to $(H_\infty - K)/\Phi(g') = (H_\infty - K)/\Psi(g')$. Knowing that the equivalence relation Ψ does not increase the π_1 (by point 2 of Theorem 2), we conclude that $\pi_1(V^3 - K)$ is also finitely generated, as we wanted to show. $\quad\square$

Corollary 1. *If, in addition to the hypotheses of Theorem 1, the 3-manifold V^3 is also irreducible, then V^3 is a missing boundary manifold (also called tame manifold), meaning that it is obtained from a compact manifold with boundary by removing a closed subset of its boundary.*

Proof. By a result of Tucker [26], an irreducible and open 3-manifold V^3 is a missing boundary manifold if and only if for any compact submanifold C of V^3, $\pi_1(V^3 - C)$ is finitely generated. $\quad\square$

References

[1] L. Funar, *On proper homotopy type and the simple connectivity at infinity of open 3-manifolds*, Atti Sem. Mat. Fis. Univ. Modena **49** (2001), 15–29.

[2] L. Funar and S. Gadgil, *On the geometric simple connectivity of open manifolds*, Int. Math. Res. Not. **2004** (2004), no. 24, 1193–1248.

[3] L. Funar and D. E. Otera, *On the WGSC and QSF tameness conditions for finitely presented groups*, Groups Geom. Dyn. **4** (2010), no. 3, 549–596.

[4] R. Kirby and L. C. Siebenmann, *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, Princeton University Press, 1977.

[5] D. E. Otera, *On the proper homotopy invariance of the Tucker property*, Acta Math. Sin. (Engl. Ser.) **23** (2007), no. 3, 571–576.

[6] ———, *Topological tameness conditions of spaces and groups: results and developments*, Lith. Math. J. **56** (2016), no. 3, 357–376.

[7] ———, *An application of Poénaru's zipping theory*, Indag. Math. (N.S.) **27** (2016), no. 4, 1003–1012.

[8] D. E. Otera and V. Poénaru, *"Easy" representations and the QSF property for groups*, Bull. Belg. Math. Soc. Simon Stevin **19** (2012), no. 3, 385–398.

[9] _____ , *Tame combings and easy groups*, Forum Math., (to appear).

[10] _____ , *Finitely presented groups and the Whitehead nightmare*, Groups, Geom., Dyn. (to appear).

[11] D. E. Otera, V. Poénaru, and C. Tanasi, *On geometric simple connectivity*, Bull. Math. Soc. Sci. Math. Roumanie **53(101)** (2010), no. 2, 157–176.

[12] D. E. Otera and F. G. Russo, *On the* WGSC *property in some classes of groups*, Mediterr. J. Math. **6** (2009), no. 4, 501–508.

[13] _____ , *On topological filtrations of groups*, Period. Math. Hungar. **72** (2016), no. 2 (2016), 218–223.

[14] V. Poénaru, *On the equivalence relation forced by the singularities of a non-degenerate simplicial map*, Duke Math. J. **63** (1991), no. 2, 421–429.

[15] _____ , *Killing handles of index one stably and* π_1^∞, Duke Math. J. **63** (1991), no. 2, 431–447.

[16] _____ , *Almost convex groups, Lipschitz combing, and* π_1^∞ *for universal covering spaces of closed* 3-*manifolds*, J. Differential Geom. **35** (1992), no. 1, 103–130.

[17] _____ , *Equivariant, locally finite inverse representations with uniformly bounded zipping length, for arbitrary finitely presented groups*, Geom. Dedicata **167** (2013), 91–121.

[18] _____ , *Geometric simple connectivity and finitely presented groups*, Preprint (2014), arXiv:1404.4283 [math.GT].

[19] V. Poénaru and C. Tanasi, *Hausdorff combing of groups and* π_1^∞ *for universal covering spaces of closed* 3-*manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **20** (1993), no. 3, 387–414.

[20] _____ , *Some remarks on geometric simple connectivity*, Acta Math. Hung. **81** (1998), no. 1-2, 1–12.

[21] _____ , *Equivariant, almost-arborescent representations of open simply-connected 3-manifolds; A finiteness result*, Mem. Amer. Math. Soc. **169** (2004), no. 800, 88 pp.

[22] L. C. Siebenmann, *Les bisections expliquent le théorème de Reidemeister-Singer, un retour aux sources*, Prepublications mathématiques d'Orsay 80T16, 1980.

[23] S. Smale, *The classification of immersions of spheres in Euclidean spaces*, Ann. of Math. **69** (1959), 327–344.

[24] _____ , *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. (2) **74** (1961), 391–406.

[25] _____ , *On the structure of manifolds*, Amer. J. Math. **84** (1962), 387–399.

[26] T. W. Tucker, *Non-compact 3-manifolds and the missing boundary problem*, Topology **13** (1974), 267–273.

[27] J. H. C. Whitehead, *Simplicial spaces, nuclei and m-groups*, Proc. Lond. Math. Soc. II. Ser. **45** (1939), 243–327.

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