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THREE-DIMENSIONAL ALMOST KENMOTSU MANIFOLDS WITH η-PARALLEL RICCI TENSOR

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Abstract. In this paper, we prove that the Ricci tensor of a three-dimensional almost Kenmotsu manifold satisfying $\nabla_\xi h = 0$, $h \neq 0$, is η-parallel if and only if the manifold is locally isometric to either the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$ or a non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure.

1. Introduction

Let us first recall some basics regarding the geometry of almost contact metric manifolds. Let M^{2n+1} be a $(2n+1)$-dimensional smooth differentiable manifold on which there exists an *almost contact structure* (ϕ, ξ, η) defined by

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi,\ \eta(\xi) = 1,$$

where id denotes the identity map, ϕ a $(1,1)$-type tensor field, ξ a global vector field tangent to M^{2n+1} and η a global one form. Then M^{2n+1} is called an *almost contact manifold* and denoted by $(M^{2n+1}, \phi, \xi, \eta)$. If in addition there exists a Riemannian metric g on $(M^{2n+1}, \phi, \xi, \eta)$ satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

we say that g is compatible to the almost contact structure and M^{2n+1} is called an *almost contact metric manifold*, denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$. Generally, ξ is said to be the Reeb or characteristic vector field and η is a contact 1-form. The fundamental 2-form Φ of an almost contact metric manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y)$$

for any vector fields X, Y.

An almost contact metric manifold is called a *contact metric manifold* if $\mathrm{d}\eta = \Phi$ (see [1]) or an *almost Kenmotsu manifold* if $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\eta \wedge \Phi$ (see [12]) or an *almost cosymplectic* (*coKähler*) *manifold* if both η and Φ are

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closed (see [1, 3]). On the Riemannian product of an almost contact manifold M^{2n+1} and $\mathbb{R}$, there exists an almost complex structure defined by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a $\mathcal{C}^\infty$-function on $M^{2n+1} \times \mathbb{R}$. We denote by $[\phi, \phi]$ the Nijenhuis tensor of ϕ, if $[\phi, \phi] = -2d\eta \otimes \xi$ (or equivalently, the almost complex structure J is integrable), then the almost contact metric structure is said to be *normal*.

A normal contact metric (resp. almost Kenmotsu or almost coKähler) manifold is said to be a Sasakian (resp. Kenmotsu or coKähler) manifold. An almost Kenmotsu manifold but not Kenmotsu is called a strictly almost Kenmotsu manifold. Here we refer the reader to Blair [1] for more details on the geometry of almost contact manifolds.

It is known that a three-dimensional contact metric manifold is locally symmetric if and only if it is of constant sectional curvature 0 or 1 (see Blair and Sharma [2]). This implies that the local symmetry is a rather strong condition in contact metric manifolds. Thus, many new geometric conditions weaker than the local symmetry were considered. For example, Cho and Lee [6] proved that a three-dimensional contact metric manifold satisfying $\nabla_\xi h = 2ah\phi$, $a \in \mathbb{R}$, has an η-parallel Ricci tensor if and only if the manifold is locally isometric to a Sasakian ϕ-symmetric space or a unimodular Lie group with a left invariant non-Sasakian contact metric structure. Recently, Perrone in [16] proved that a three-dimensional almost coKähler manifold is locally symmetric if and only if it is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the Riemannian product $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$. Generalizing Perrone's results, the present author in [18] proved that a three-dimensional almost coKähler manifold satisfying $\nabla_\xi h = 2f\phi h$, $f \in \mathbb{R}$ and having an η-parallel Ricci tensor is locally isometric to either the product space $\mathbb{R} \times N^2(c)$ or a simply connected unimodular Lie group equipped with a left invariant almost coKähler structure.

In 1972, Kenmotsu [13] proved that a locally symmetric Kenmotsu manifold is of constant sectional curvature -1. Generalizing this result for dimension three, recently, the present author [17] and Cho [4] independently obtained that a three-dimensional almost Kenmotsu manifold is locally symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the product space $\mathbb{H}^2(-4) \times \mathbb{R}$. This implies that the local symmetry is also a rather strong condition for almost Kenmotsu manifolds. Therefore, in this paper we aim to generalize Wang and Cho's results to a special class of three-dimensional almost Kenmotsu manifolds under certain condition weaker than the local symmetry. Our main result is given as the following.

Theorem 1.1. *Let M^3 be a three-dimensional almost Kenmotsu manifold satisfying $\nabla_\xi h = 0$ and $h \neq 0$. Then the Ricci tensor of M^3 is η-parallel if and* *only if the manifold is locally isometric to a non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure.*

The present paper is arranged as follows. In Section 2, after presenting some necessary preliminaries used in proofs of our main results, we construct some examples of three-dimensional almost Kenmotsu manifolds satisfying $\nabla_\xi h = 0$. In Section 3, we present the detailed proofs of our main results with some corollaries. Moreover, we give some examples of three-dimensional almost Kenmotsu manifolds on which $\nabla_\xi h \neq 0$ but the Ricci tensor is η-parallel.

2. Three-dimensional almost Kenmotsu manifolds

In this paper, we denote by $(M^3, \phi, \xi, \eta, g)$ a three-dimensional almost Kenmotsu manifold and we set $l = R(\cdot, \xi)\xi$, $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$, where $\mathcal{L}$ denotes the Lie differentiation and R is the Riemannian curvature tensor. From Dileo and Pastore [8, 9], we see that both h and h' are symmetric operators and we recall some properties of almost Kenmotsu manifolds as follows:

$$h\xi = l\xi = 0, \ \mathrm{tr}h = \mathrm{tr}(h') = 0, \ h\phi + \phi h = 0, \tag{2.1}$$

$$\nabla \xi = h' + \mathrm{id} - \eta \otimes \xi, \tag{2.2}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.3}$$

$$\nabla_\xi h = -\phi - 2h - \phi h^2 - \phi l. \tag{2.4}$$

Throughout this paper, we denote by $\mathcal{D}$ the distribution $\mathcal{D} = \ker \eta$ which is of dimension $2n$. Then it is easy to see that each integral manifold of $\mathcal{D}$, with the restriction of ϕ, admits an almost Kähler structure. If the associated almost Kähler structure is integrable, or equivalently (see [9]),

$$(\nabla_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX) \tag{2.5}$$

for any vector fields X, Y, then we say that M^{2n+1} is CR-integrable. Following [12, Theorem 2.1], we obtain from (2.5) that an almost Kenmotsu manifold is Kenmotsu if and only if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{2.6}$$

for any vector fields X, Y. Since a three-dimensional almost Kenmotsu manifold M^3 is CR-integrable, we obtain that M^3 is Kenmotsu if and only if h vanishes.

Let $\mathcal{U}_1$ be the open subset of a 3-dimensional almost Kenmotsu manifold M^3 such that $h \neq 0$ and $\mathcal{U}_2$ the open subset of M^3 which is defined by $\mathcal{U}_2 = \{p \in M^3 : h = 0 \text{ in a neighborhood of } p\}$. Therefore, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of M^3 and there exists a local orthonormal basis $\{\xi, e, \phi e\}$ of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$, we may set $he = \lambda e$ and hence $h\phi e = -\lambda \phi e$, where λ is a positive function on $\mathcal{U}_1$. Notice that the eigenvalue function λ is continuous on M^3 and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$.

Lemma 2.1 ([5, Lemma 6]). *On $\mathcal{U}_1$ we have*

$$\begin{aligned}
&\nabla_\xi\xi = 0,\ \nabla_\xi e = a\phi e,\ \nabla_\xi\phi e = -ae,\\
&\nabla_e\xi = e - \lambda\phi e,\ \nabla_e e = -\xi - b\phi e,\ \nabla_e\phi e = \lambda\xi + be,\\
&\nabla_{\phi e}\xi = -\lambda e + \phi e,\ \nabla_{\phi e}e = \lambda\xi + c\phi e,\ \nabla_{\phi e}\phi e = -\xi - ce,
\end{aligned} \tag{2.7}$$

where a, b, c are smooth functions.

Applying Lemma 2.1 in the following Jacobi identity

$$[[\xi, e], \phi e] + [[e, \phi e], \xi] + [[\phi e, \xi], e] = 0$$

yields

$$\begin{cases} e(\lambda) - \xi(b) - e(a) + c(\lambda - a) - b = 0,\\ \phi e(\lambda) - \xi(c) + \phi e(a) + b(\lambda + a) - c = 0. \end{cases} \tag{2.8}$$

Moreover, applying Lemma 2.1 we have (see also [5]) the following.

Lemma 2.2. *On $\mathcal{U}_1$, the Ricci operator can be written as*

$$Q\xi = -2(\lambda^2 + 1)\xi - (\phi e(\lambda) + 2\lambda b)e - (e(\lambda) + 2\lambda c)\phi e,$$

$$Qe = -(\phi e(\lambda) + 2\lambda b)\xi - (e(c) + \phi e(b) + b^2 + c^2 + 2\lambda a + 2)e + (\xi(\lambda) + 2\lambda)\phi e,$$

$$Q\phi e = -(e(\lambda) + 2\lambda c)\xi + (\xi(\lambda) + 2\lambda)e - (e(c) + \phi e(b) + b^2 + c^2 - 2\lambda a + 2)\phi e,$$

with respect to the local basis $\{\xi, e, \phi e\}$.

Following [9, Theorem 5.2] we now construct a left invariant almost Kenmotsu structure on a three-dimensional non-unimodular Lie group. A Lie group G is said to be unimodular if its left invariant Haar measure is also right invariant. It is known that a Lie group G is unimodular if and only if the endomorphism $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{ad}_X(Y) = [X, Y]$ has trace equal to zero for any $X \in \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra associated to G.

From Milnor [14], if G is a three-dimensional non-unimodular Lie group, then there exists a left invariant local orthonormal frame fields $\{e_1, e_2, e_3\}$ satisfying

$$[e_1, e_2] = \alpha e_2 + \beta e_3,\ [e_2, e_3] = 0,\ [e_1, e_3] = \gamma e_2 + \delta e_3 \tag{2.9}$$

and $\alpha+\delta = 2$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. We define a metric g on G by $g(e_i, e_j) = \delta_{ij}$ for $1 \leq i, j \leq 3$. We denote by $\xi = -e_1$ and by η the dual 1-form of ξ. We define a $(1, 1)$-type tensor field ϕ by $\phi(\xi) = 0$, $\phi(e_2) = e_3$ and $\phi(e_3) = -e_2$. Then, one can check that (G, ϕ, ξ, η, g) admits a left invariant almost Kenmotsu structure. Using the Koszul formula and (2.9) we obtain

$$\begin{aligned}
&\nabla_\xi\xi = 0,\ \nabla_{e_2}\xi = \alpha e_2 + \frac{1}{2}(\beta + \gamma)e_3,\ \nabla_{e_3}\xi = \frac{1}{2}(\beta + \gamma)e_2 + (2 - \alpha)e_3,\\
&\nabla_\xi e_2 = \frac{1}{2}(\gamma - \beta)e_3,\ \nabla_{e_2}e_2 = -\alpha\xi,\ \nabla_{e_3}e_2 = -\frac{1}{2}(\beta + \gamma)\xi,\\
&\nabla_\xi e_3 = \frac{1}{2}(\beta - \gamma)e_2,\ \nabla_{e_2}e_3 = -\frac{1}{2}(\beta + \gamma)\xi,\ \nabla_{e_3}e_3 = (\alpha - 2)\xi.
\end{aligned} \tag{2.10}$$

By using (2.2), it follows from the first term of (2.10) that

$$he_2 = (\alpha - 1)e_3 - \frac{1}{2}(\beta + \gamma)e_2 \text{ and } he_3 = \frac{1}{2}(\beta + \gamma)e_3 + (\alpha - 1)e_2. \tag{2.11}$$

Therefore, from (2.10) and (2.11) we have

$$\nabla_\xi h = (\beta - \gamma)h'. \tag{2.12}$$

Example 2.1. Let G be a three-dimensional non-unimodular Lie group with a left invariant local orthonormal frame fields $\{e_1, e_2, e_3\}$ satisfying

$$[e_1, e_2] = \alpha e_2 + \beta e_3,\ [e_2, e_3] = 0,\ [e_1, e_3] = \beta e_2 + (2 - \alpha)e_3$$

for $\alpha, \beta \in \mathbb{R}$. If either $\alpha \neq 1$ or $\beta \neq 0$, G admits a left invariant non-Kenmotsu almost Kenmotsu structure satisfying $\nabla_\xi h = 0$.

3. η-parallel Ricci tensors

On a three-dimensional Riemannian manifold (M, g), the curvature tensor R is given by using the Ricci operator Q, the Ricci tensor S and the scalar curvature r as follows

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ & - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned}$$

for any vector fields X, Y, Z. It follows directly that

$$\begin{aligned} (\nabla_V R)(X,Y)Z = {} & g(Y,Z)(\nabla_V Q)X - g(X,Z)(\nabla_V Q)Y + g((\nabla_V Q)Y,Z)X \\ & - g((\nabla_V Q)X,Z)Y - \frac{1}{2}V(r)(g(Y,Z)X - g(X,Z)Y) \end{aligned}$$

for any vector field V. From the above relation, it is easy to see that the local symmetry ($\nabla R = 0$) and the Ricci parallelism ($\nabla Q = 0$) are equivalent. Since the local classification problem of a three-dimensional almost Kenmotsu manifold M^3 under the local symmetry condition was completed by the present author [17] and Cho [4], thus in this paper we study M^3 under a weaker condition defined as follows.

Definition 3.1. An almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to have an η-parallel Ricci tensor if there holds

$$g((\nabla_X Q)Y, Z) = 0 \tag{3.1}$$

for any X, Y, Z orthogonal to the Reeb vector field ξ.

If the Ricci tensor is parallel, then it must be η-parallel. However, the converse is not necessarily true.

Theorem 3.1. *Let M^3 be a three-dimensional strictly almost Kenmotsu manifold satisfying $\nabla_\xi h = 0$. Then the Ricci tensor of M^3 is η-parallel if and only if the manifold is locally isometric to either the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$ or a non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure.*

Proof. Since a three-dimensional almost Kenmotsu manifold M^3 is assumed to be strictly, i.e., non-Kenmotsu, or equivalently, $h \neq 0$, then $\mathcal{U}_1$ is a non-empty subset. By applying Lemma 2.1 and a direct calculation we obtain

$$(\nabla_\xi h)e = \xi(\lambda)e + 2a\lambda\phi e \text{ and } (\nabla_\xi h)\phi e = -\xi(\lambda)\phi e + 2a\lambda e. \tag{3.2}$$

By the assumption condition $\nabla_\xi h = 0$ and (3.2) we have

$$\xi(\lambda) = a = 0, \tag{3.3}$$

where we have used that λ is positive on $\mathcal{U}_1$. For simplicity, we denote f by

$$f = e(c) + \phi e(b) + b^2 + c^2 + 2. \tag{3.4}$$

Then, using (3.3) and (3.4) we obtain from Lemmas 2.1 and 2.2 that

$$(\nabla_\xi Q)\xi = -\xi(\phi e(\lambda) + 2\lambda b)e - \xi(e(\lambda) + 2\lambda c)\phi e, \tag{3.5}$$

$$(\nabla_\xi Q)e = -\xi(\phi e(\lambda) + 2\lambda b)\xi - \xi(f)e, \tag{3.6}$$

$$(\nabla_\xi Q)\phi e = -\xi(e(\lambda) + 2\lambda c)\xi - \xi(f)\phi e, \tag{3.7}$$

$$\begin{aligned}(\nabla_e Q)e = {}& (f - 2 - e(\phi e(\lambda) + 2\lambda b) - b(e(\lambda) + 2\lambda c))\xi \\ & - (e(f) + 2\phi e(\lambda))e + (e(\lambda) + \lambda\phi e(\lambda) + 2\lambda^2 b - 2\lambda c)\phi e,\end{aligned} \tag{3.8}$$

$$\begin{aligned}(\nabla_e Q)\phi e = {}& (2\lambda^3 - f\lambda + b(\phi e(\lambda) + 2\lambda b) - e(e(\lambda) + 2\lambda c))\xi \\ & + (e(\lambda) + \lambda\phi e(\lambda) - 2\lambda c + 2\lambda^2 b)e \\ & + (2\lambda(e(\lambda) + 2\lambda c) - e(f) - 4\lambda b)\phi e,\end{aligned} \tag{3.9}$$

$$\begin{aligned}(\nabla_{\phi e} Q)e = {}& (2\lambda^3 - f\lambda + c(e(\lambda) + 2\lambda c) - \phi e(\phi e(\lambda) + 2\lambda b))\xi \\ & - (\phi e(f) + 4\lambda c - 2\lambda(\phi e(\lambda) + 2\lambda b))e \\ & + (\phi e(\lambda) + \lambda e(\lambda) + 2\lambda^2 c - 2\lambda b)\phi e,\end{aligned} \tag{3.10}$$

$$\begin{aligned}(\nabla_{\phi e} Q)\phi e = {}& (f - 2 - \phi e(e(\lambda) + 2\lambda c) - c(\phi e(\lambda) + 2\lambda b))\xi \\ & + (\phi e(\lambda) + \lambda e(\lambda) + 2\lambda^2 c - 2\lambda b)e - (\phi e(f) + 2e(\lambda))\phi e,\end{aligned} \tag{3.11}$$

$$\begin{aligned}(\nabla_e Q)\xi = {}& 2(\phi e(\lambda) - 3\lambda e(\lambda) + 2\lambda b - 2\lambda^2 c)\xi \\ & + (f - 2 - e(\phi e(\lambda) + 2\lambda b) - b(e(\lambda) + 2\lambda c))e \\ & + (2\lambda^3 + b(\phi e(\lambda) + 2\lambda b) - e(e(\lambda) + 2\lambda c) - \lambda f)\phi e,\end{aligned} \tag{3.12}$$

$$\begin{aligned}(\nabla_{\phi e} Q)\xi = {}& 2(e(\lambda) - 3\lambda\phi e(\lambda) + 2\lambda c - 2\lambda^2 b)\xi \\ & + (2\lambda^3 + c(e(\lambda) + 2\lambda c) - \phi e(\phi e(\lambda) + 2\lambda b) - \lambda f)e \\ & + (f - 2 - \phi e(e(\lambda) + 2\lambda c) - c(\phi e(\lambda) + 2\lambda b))\phi e.\end{aligned} \tag{3.13}$$

If the Ricci tensor is η-parallel, then from (3.1) and (3.8) we have

$$\begin{cases} e(f) + 2\phi e(\lambda) = 0, \\ e(\lambda) + \lambda\phi e(\lambda) + 2\lambda^2 b - 2\lambda c = 0. \end{cases} \tag{3.14}$$

Moreover, ir follows from (3.1) and (3.11) that

$$\begin{cases} \phi e(f) + 2e(\lambda) = 0, \\ \phi e(\lambda) + \lambda e(\lambda) + 2\lambda^2 c - 2\lambda b = 0. \end{cases} \tag{3.15}$$

We remark that if (3.14) and (3.15) are true, then there hold

$$g((\nabla_e Q)\phi e, X) = g((\nabla_{\phi e} Q)e, X) = 0$$

for any vector field X orthogonal to ξ. Notice that λ is assumed to be a positive engenvalue function of h. Thus, comparing the last term of (3.14) to that of (3.15) we see that either $\lambda = 1$ or $e(\lambda)$ and $\phi e(\lambda)$ are rewritten as follows.

$$e(\lambda) = \frac{4\lambda^2 b - 2\lambda^3 c - 2\lambda c}{\lambda^2 - 1}, \quad \phi e(\lambda) = \frac{4\lambda^2 c - 2\lambda^3 b - 2\lambda b}{\lambda^2 - 1}. \tag{3.16}$$

The following proof is divided into two cases.

Case i: $\lambda = 1$. Using this in the last term of (3.14) or (3.15) gives $b = c$. Then, it follows from Lemma 2.2 that

$$\begin{cases} Q\xi = -4\xi - 2be - 2b\phi e, \\ Qe = -2b\xi - fe + 2\phi e, \\ Q\phi e = -2b\xi + 2e - f\phi e. \end{cases} \tag{3.17}$$

By using the above relations we directly obtain

$$Qh\phi - h\phi Q + g(Q\xi, \cdot)\xi - \eta \otimes Q\xi = 0. \tag{3.18}$$

Applying $\lambda = 1$, $b = c$ and (3.3) in relation (2.8) implies $\xi(b) = 0$. Next, let us recall the well known formula $\mathrm{div}Q = \frac{1}{2}\mathrm{grad}r$. By using (3.17) we obtain the scalar curvature $r = -4 - 2f$. Therefore, using $b = c$, $\lambda = 1$, equations (3.4), (3.5), (3.8) and (3.11) in this formula we obtain $\xi(r) = 2g((\nabla_\xi Q)\xi + (\nabla_e Q)e + (\nabla_{\phi e} Q)\phi e, \xi) = 0$ and hence we obtain $\xi(f) = 0$. Using this, $\xi(b) = 0$ and $\lambda = 1$ in (3.5)-(3.7) we have

$$\nabla_\xi Q = 0. \tag{3.19}$$

Combining (3.18) with (3.19) and applying (2.2) we see that the Ricci tensor is invariant along the Reeb flow, i.e., $\mathcal{L}_\xi Q = 0$.

Cho in [5] proved that a three-dimensional non-Kenmotsu almost Kenmotsu manifold M^3 satisfies $\mathcal{L}_\xi Q = 0$ if and only if the manifold is locally isometric to a non-unimodular Lie group with a left invariant almost Kenmotsu structure. Moreover, he obtained $b = c = 0$ on M^3 under the condition $\mathcal{L}_\xi Q = 0$ (see [5, p. 272]). Thus, using $\lambda = 1$ and $a = b = c = 0$ in relations (3.5)-(3.13) we see that the Ricci tensor is parallel and hence the manifold is locally symmetric. Since the present author [17] and Cho [4] proved that a three-dimensional locally symmetric strictly almost Kenmotsu manifold is locally isometric to the product space $\mathbb{H}^2(-4) \times \mathbb{R}$, then the proof for the first case follows.

Case ii: $\lambda \neq 1$. Now we suppose that $\lambda \neq 1$ holds on an open subset of $\mathcal{U}_1$ and hence (3.16) is true. From Lemma 2.2 we get

$$r = -2\lambda^2 - 2 - 2f. \tag{3.20}$$

Using (3.20) in the first terms of (3.14) and (3.15) we have

$$\begin{cases} e(r) = 4(\phi e(\lambda) - \lambda e(\lambda)), \\ \phi e(r) = 4(e(\lambda) - \lambda \phi e(\lambda)). \end{cases} \tag{3.21}$$

On the other hand, using again the formula $\mathrm{div}Q = \frac{1}{2}\mathrm{grad}r$ and applying equations (3.5), (3.8) and (3.11) we obtain $e(r) = -2\xi(\phi e(\lambda) + 2\lambda b)$. Putting the second term of relation (3.16) and the first term of (3.21) into this relation gives

$$\lambda\xi(c) - \xi(b) = 3\lambda^2 b + b - 3\lambda c - \lambda^3 c, \tag{3.22}$$

where we have used the first term of relation (3.3) and $\lambda \neq 1$.

Similarly, using the formula $\mathrm{div}Q = \frac{1}{2}\mathrm{grad}r$ and applying equations (3.5), (3.8) and (3.11) we obtain $\phi e(r) = -2\xi(e(\lambda) + 2\lambda c)$. Using (3.3) and putting the first term of (3.16) and the second term of (3.21) into this relation gives

$$\lambda\xi(b) - \xi(c) = 3\lambda^2 c + c - 3\lambda b - \lambda^3 b. \tag{3.23}$$

By a direct calculation, it follows from (3.22) and (3.23) that

$$\begin{cases} \xi(b) = 2\lambda c - (\lambda^2 + 1)b, \\ \xi(c) = 2\lambda b - (\lambda^2 + 1)c, \end{cases} \tag{3.24}$$

where we have used the second terms of (3.14) and (3.15) and that λ is a positive function $\neq 1$.

Moreover, by using (3.3) in (2.8) we have

$$\begin{cases} e(\lambda) = \xi(b) + b - \lambda c, \\ \phi e(\lambda) = \xi(c) + c - \lambda b. \end{cases}$$

Putting (3.24) into the above relation we get

$$\begin{cases} e(\lambda) = -\lambda^2 b + \lambda c, \\ \phi e(\lambda) = -\lambda^2 c + \lambda b. \end{cases} \tag{3.25}$$

For simplicity, we set

$$\theta = -\lambda(3\lambda^2 + 1), \quad \rho = \lambda^2(\lambda^2 + 3).$$

Thus, comparing (3.25) with (3.16) we get

$$\begin{pmatrix} \theta & \rho \\ \rho & \theta \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since we have assumed that $\lambda \neq 1$ holds on an open subset of $\mathcal{U}_1$ and that λ is positive, then it is easy to see that $\theta^2 - \rho^2 \neq 0$ and hence there exists a unique solution for the above equation, i.e., $b = c = 0$. Using this in (3.25) implies that $e(\lambda) = \phi e(\lambda) = 0$. Since λ is continuous on $\mathcal{U}_1$, then by (3.3) we conclude that λ is a global positive constant on M^3. In this context, from (2.7) we have

$$[\xi, e] = \lambda\phi e - e,\ [e, \phi e] = 0,\ [\phi e, \xi] = -\lambda e + \phi e. \tag{3.26}$$

Following Milnor [14], here we state that M^3 is locally isometric to a three-dimensional non-unimodular Lie group. Actually, from (3.26) we observe that its unimodular kernel $\{X \in \mathfrak{g} : \mathrm{trace}(\mathrm{ad}_X) = 0\}$ is commutative and of 2-dimension and $\mathrm{trace}(\mathrm{ad}_\xi) = -2$.

Conversely, since the product space $\mathbb{H}^2(-4) \times \mathbb{R}$ is locally symmetric then it has a η-parallel Ricci tensor. Finally, we need only to show that the Ricci tensor of a non-Kenmotsu almost Kenmotsu structure defined on any three-dimensional non-unimodular Lie group satisfying $\nabla_\xi h = 0$ is η-parallel. In this context, from (2.12) we have $\beta = \gamma$. By a direct calculation, we obtain from (2.10) that

$$\begin{aligned}
R(e_2, \xi)\xi &= -(\alpha^2 + \beta^2)e_2 - 2\beta e_3,\\
R(e_3, \xi)\xi &= -2\beta e_2 - ((\alpha - 2)^2 + \beta^2)e_3,\\
R(\xi, e_2)e_2 &= -(\alpha^2 + \beta^2)\xi,\\
R(\xi, e_3)e_3 &= -((\alpha - 2)^2 + \beta^2)\xi,\\
R(e_2, e_3)e_3 &= (\alpha^2 - 2\alpha + \beta^2)e_2,\\
R(e_3, e_2)e_2 &= (\alpha^2 - 2\alpha + \beta^2)e_3.
\end{aligned}$$

It follows directly that

$$\begin{aligned}
Q\xi &= -2(\alpha^2 - 2\alpha + 2 + \beta^2)\xi,\\
Qe_2 &= -2\alpha e_2 - 2\beta e_3,\\
Qe_3 &= -2\beta e_2 + 2(\alpha - 2)e_3.
\end{aligned}$$

Therefore, using relation (2.10), by a direct calculation we obtain that $(\nabla_{e_i} Q)e_j$ is colinear with the Reeb vector field ξ for any $i, j \in \{2, 3\}$ and hence the Ricci tensor is η-parallel. This completes the proof. □

Remark 3.1. An almost Kenmotsu structure defined on a three-dimensional non-unimodular Lie group whose Lie algebra admits a local orthonormal basis satisfying

$$[e_1, e_2] = 2e_2,\ [e_2, e_3] = 0,\ [e_3, e_1] = 0$$

is in fact locally a Riemannian product space $\mathbb{H}^2(-4) \times \mathbb{R}$ (see also [9, pp. 59–60]). Then Theorem 1.1 follows.

In what follows, we say that the Ricci tensor of an almost contact metric manifold is strong η-parallel if it satisfies

$$g((\nabla_X Q)Y, Z) = 0$$

for any vector field X and any vector fields Y, Z orthogonal to ξ. In view of the almost Kenmotsu structures considered in Theorem 3.1 being non-Kenmotsu, then the theorem can be rewritten as the following.

Corollary 3.1. *Let M^3 be a three-dimensional almost Kenmotsu manifold satisfying $\nabla_\xi h = 0$. Then the Ricci tensor of M^3 is strong η-parallel if and only if the manifold is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$, the product space $\mathbb{H}^2(-4) \times \mathbb{R}$ or a non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure.*

Proof. Notice that a three-dimensional almost Kenmotsu manifold is Kenmotsu if and only if h vanishes. It is proved by De and Pathak [7, Proposition 5.1] that a three-dimensional Kenmotsu manifold having a strong η-parallel Ricci tensor is of constant scalar curvature. Inoguchi [11] proved that a three-dimensional Kenmotsu manifold with constant scalar curvature is of constant sectional curvature -1. Thus, the proof for Kenmotsu case follows.

Finally, we consider the non-Kenmotsu cases. Obviously, a strong η-parallel Ricci tensor must be η-parallel. Thus, the proof follows immediately from Theorem 3.1. This completes the proof. □

Since on a locally symmetric almost Kenmotsu manifold there holds $\nabla_\xi h = 0$ (see [8, Proposition 6]), therefore, the following corollary follows from Theorem 3.1 and [13, Corollary 6].

Corollary 3.2 ([4, 17])**.** *A three-dimensional almost Kenmotsu manifold is locally symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the product space $\mathbb{H}^2(-4) \times \mathbb{R}$.*

Proof. In [17, p. 83], the present author proved that on any three-dimensional locally symmetric non-Kenmotsu almost Kenmotsu manifold there holds $h^2 = -\phi^2$. Therefore, in this context we observe that **Case ii** in proof of Theorem 3.1 can not occur. For the Kenmotsu case, the proof follows from [13, Corollary 6], i.e., a locally symmetric Kenmotsu manifold is of constant sectional curvature -1. This completes the proof. □

An almost Kenmotsu manifold is called a (k, μ, ν)-almost Kenmotsu manifold if the Reeb vector field ξ satisfies the (k, μ, ν)-nullity condition, that is,

$$\begin{aligned} R(X,Y)\xi = {} & k(\eta(Y)X - \eta(X)Y) \\ & + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)h'X - \eta(X)h'Y) \end{aligned} \tag{3.27}$$

for any vector fields X, Y, Z and some smooth functions k, μ, ν. Putting $Y = \xi$ into (3.27) gives

$$l = -k\phi^2 + \mu h + \nu h'.$$

Using the above relation in equation (2.3) we obtain $h^2 = (k+1)\phi^2$.

Remark 3.2. By relation (2.4) we obtain that on a three-dimensional (k, μ, ν)-almost Kenmotsu manifold there holds $\nabla_\xi h = \mu h' - (\nu+2)h$. Thus, one can find many examples of three-dimensional non-Kenmotsu (k,μ,ν)-almost Kenmotsu manifolds with either $\mu\neq 0$ or $\nu\neq -2$ on which $\nabla_\xi h\neq 0$. For these examples, we refer the reader to [15, Examples 6.3 and 6.4].

Finally, we point out that on [15, Example 6.3] the Ricci tensor of the manifold is η-parallel in spite of $\nabla_\xi h\neq 0$.

Example 3.1. Let (x,y,z) be the canonical coordinates on $\mathbb{R}^3$. We now consider a three-dimensional manifold $M^3\subset\mathbb{R}^3$ defined by

$$M^3:=\{(x,y,z)\in\mathbb{R}^3\,|\,z>0\}.$$

In what follows, on M^3 we set

$$\begin{aligned}
&\xi:=\frac{\partial}{\partial z},\ \eta:=dz,\\
&\phi\left(\frac{\partial}{\partial x}\right)=z\frac{\partial}{\partial y},\ \phi\left(\frac{\partial}{\partial y}\right)=-\frac{1}{z}\frac{\partial}{\partial x},\\
&g=ze^{2z}dx^2+\frac{e^{2z}}{z}dy^2+dz^2.
\end{aligned}$$

Saltarelli [15] proved that (M^3,ϕ,ξ,η,g) is an almost Kenmotsu manifold for which ξ belongs to the $(-1-\frac{1}{4z^2},-2+\frac{1}{z},0)$-nullity distribution. Then it follows from (3.27) that

$$Q\xi=-\left(2+\frac{1}{2z^2}\right)\xi.$$

Moreover, using the Koszul formula we have

$$\begin{aligned}
&\nabla_\xi\xi=0,\ \nabla_\xi\frac{\partial}{\partial x}=\left(1+\frac{1}{2z}\right)\frac{\partial}{\partial x},\\
&\nabla_\xi\frac{\partial}{\partial y}=\left(1-\frac{1}{2z}\right)\frac{\partial}{\partial y},\ \nabla_{\frac{\partial}{\partial x}}\xi=\left(1+\frac{1}{2z}\right)\frac{\partial}{\partial x},\\
&\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}=-\frac{1+z^2}{z}e^{2z}\xi,\ \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}=\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}=0,\\
&\nabla_{\frac{\partial}{\partial y}}\xi=\left(1-\frac{1}{2z}\right)\frac{\partial}{\partial y},\ \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}=\frac{1-2z}{2z^2}e^{2z}\xi,
\end{aligned}$$

where ∇ denotes the Levi-Civita connection of the Riemannian metric g. Using (2.2) we also have from the above relation that

$$h\frac{\partial}{\partial x}=\frac{1}{2}\frac{\partial}{\partial y},\ h\frac{\partial}{\partial y}=\frac{1}{2z^2}\frac{\partial}{\partial x}.$$

By a straightforward calculation we have

$$R(\frac{\partial}{\partial x},\xi)\xi=\left(\frac{1}{4z^2}-\frac{1}{z}-1\right)\frac{\partial}{\partial x},$$
$$R(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\frac{\partial}{\partial y}=\frac{1-4z^2}{4z^3}e^{2z}\frac{\partial}{\partial x},$$
$$R(\frac{\partial}{\partial y},\xi)\xi=\left(-\frac{3}{4z^2}+\frac{1}{z}-1\right)\frac{\partial}{\partial y},$$
$$R(\frac{\partial}{\partial y},\frac{\partial}{\partial x})\frac{\partial}{\partial x}=\frac{(1+z^2)(1-2z)}{2z^2}e^{2z}\frac{\partial}{\partial y},$$

and hence we get

$$Q\frac{\partial}{\partial x}=\left(\frac{1}{2z^2}-\frac{1}{z}-2\right)\frac{\partial}{\partial x}$$

and

$$Q\frac{\partial}{\partial y}=\left(\frac{1}{2z^3}-\frac{7}{4z^2}+\frac{3}{2z}-2\right)\frac{\partial}{\partial y},$$

where Q is the Ricci operator. Finally, by a direct calculation we obtain that $(\nabla_{\frac{\partial}{\partial x}}Q)\frac{\partial}{\partial x}$, $(\nabla_{\frac{\partial}{\partial x}}Q)\frac{\partial}{\partial y}$, $(\nabla_{\frac{\partial}{\partial y}}Q)\frac{\partial}{\partial x}$ and $(\nabla_{\frac{\partial}{\partial y}}Q)\frac{\partial}{\partial y}$ are all colinear with the Reeb vector field ξ. This means that the Ricci tensor is η-parallel, but not parallel.

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