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ATTRACTORS OF LOCAL SEMIFLOWS ON TOPOLOGICAL SPACES

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Abstract. In this paper we introduce a notion of an attractor for local semiflows on topological spaces, which in some cases seems to be more suitable than the existing ones in the literature. Based on this notion we develop a basic attractor theory on topological spaces under appropriate separation axioms. First, we discuss fundamental properties of attractors such as maximality and stability and establish some existence results. Then, we give a converse Lyapunov theorem. Finally, the Morse decomposition of attractors is also addressed.

1. Introduction

Invariant sets are of crucial importance in the theory of dynamical systems. This is because that for a given dynamical system, they are the carriers of much information on the longtime behavior of the system. Of special interest are attractors. An attractor, if exists, is the depository of "all" the dynamics of a system near the attractor.

The attractor theories in metric spaces (especially nonlocally compact metric spaces) were fully developed in the past decades for both autonomous and nonautonomous systems [1, 3, 4, 8, 10, 13, 16, 19–21]. Here we are interested in the case where the phase space is a topological one that may not be metrizable. There are many motivations for this consideration. For example, for an infinite dimensional system on a Banach space X, in some cases one has to study the dynamics of the system under weak topologies of X; see e.g. [5]. However, when X is endowed with its weak topology, it may fail to be metrizable. This makes the usual theory of dynamical systems in metric spaces inapplicable. Another example is closely related to some recent work on the study of invariant sets. Let N be an isolating neighborhood of a semiflow Φ on a complete metric space X. Denote N^- the *exit set* of N. Then one can define a quotient flow $\widetilde{\Phi}$ on the quotient space N/N^-. It can be shown under reasonable assumptions that $\widetilde{\Phi}$ is

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well-defined and continuous [11,22]. Thus many problems concerning invariant sets (including the existence of invariant sets) of Φ can be transformed into that of attractors of the quotient flow $\widetilde{\Phi}$. But now a major obstacle preventing us from taking a further step is that, in general N/N^- may not be metrizable, which makes the attractor theories on metric spaces fail to work. To overcome this difficulty, a suitable attractor theory on topological spaces needs to be developed.

In [15] Marzocchi and Necca introduced a notion of attractors and established some existence results for global semiflows on topological spaces. Informally speaking, the authors defined an attractor $\mathscr{A}$ of a semiflow on a topological space X to be a compact invariant set that attracts each element $B \in \mathcal{B}$, where $\mathcal{B}$ is a given family of subsets of X. Based on this notion Giraldo et al. further introduced the notion of global attractors for global semiflows on topological spaces, using which they successfully extended the shape theory of attractors from metric spaces to topological spaces [6].

In practice, for an infinite dimensional system Φ, instead of imposing compactness conditions on the phase space X one usually requires Φ to have some *asymptotic compactness* properties. In the case where X is non-metrizable, such a property seems to be more consistent with *sequential compactness* of subsets of X. This simple observation stimulates us to introduce another notion of attractors for local semiflows on topological spaces. Specifically, we define an attractor to be a sequentially compact invariant set that attracts a neighborhood of itself and enjoys some maximal properties. Since in a general topological space, compactness and sequential compactness are different matters, our notion of attractors differs significantly from the ones in the literature, although we can show that they coincide under appropriate conditions.

Based on our notion of attractors given here, we then develop a basic attractor theory on topological spaces. First, we discuss fundamental properties of attractors such as maximality and stability. Then we give some existence results. In particular, we show that if there is a closed set M attracting an admissible neighborhood of itself, then M contains an attractor. Finally, the converse Lyapunov theorems and Morse decompositions of attractors are addressed. Our starting point is the convergence of sequences. It is worth mentioning that throughout the paper we only assume the phase space to be *Hausdorff* and *normal*. No other separation axioms and countability axioms will be required. The interested reader is referred to [11] for an application of this attractor theory, in which we proved some linking theorems and mountain pass type results to detect the existence of invariant sets of dynamical systems.

The theory of dynamical systems on topological spaces has a rich background. We refer the reader to [2, 7, 14, 18] etc. for some earlier work in this line.

This paper is organized as follows. Section 2 is concerned with fundamental properties of local semiflows, and Section 3 consists of some results on limit sets.

In Section 4 we introduce the notion of an attractor, discuss basic properties of attractors and establish some existence results. In Section 5 we prove a converse Lyapunov theorem. Section 6 is devoted to the Morse decomposition of attractors.

2. Semiflows on topological spaces

Throughout the paper we always assume that X is a *Hausdorff* topological space. Sometimes we may also require X to be *normal*, so that any two disjoint closed subsets of X can be separated by their disjoint neighborhoods.

Let $A \subset X$. Denote $\overline{A}$, $\operatorname{int} A$ and ∂A the *closure, interior and boundary* of any subset A of X, respectively. A set $U \subset X$ is called a *neighborhood* of A, if $\overline{A} \subset \operatorname{int} U$.

We make a convention that we identify a singleton $\{x\}$ with the point x.

2.1. Definitions and continuity property

Definition ([2])**.** A *local semiflow* Φ on X is a continuous map from an open subset $\mathcal{D}_\Phi$ of $\mathbb{R}^+ \times X$ to X satisfying the following conditions:

(1) For each $x \in X$, there exists $T_x \in (0, \infty]$ such that
$$(t, x) \in \mathcal{D}_\Phi \Longleftrightarrow t \in [0, T_x)\,.$$
(2) $\Phi(0, x) = x$ for all $x \in X$.
(3) If $(t + s, x) \in \mathcal{D}_\Phi$, where $t, s \in \mathbb{R}^+$, then
$$\Phi(t + s, x) = \Phi\left(t,\ \Phi(s, x)\right).$$

The set $\mathcal{D}_\Phi$ and the number T_x in the above definition are called, respectively, the *domain* of Φ and the *escape time* of $\Phi(t, x)$.

A local semiflow Φ is called a *global semiflow*, if $\mathcal{D}_\Phi = \mathbb{R}^+ \times X$.

Let Φ be a given local semiflow on X. From now on we rewrite $\Phi(t, x)$ as $\Phi(t)x$. For any sets $M \subset X$ and $J \subset \mathbb{R}^+$, denote
$$\Phi(J)M = \{\Phi(t)x :\ x \in M,\ t \in J \cap [0, T_x)\}.$$

Definition. Given an interval $I \subset \mathbb{R}^1$, a map $\gamma : I \to X$ is called a *solution* (or *trajectory*) on I, if
$$\gamma(t) = \Phi(t - s)\gamma(s), \qquad \forall\, s, t \in I,\ s \le t.$$
A solution γ on $I = \mathbb{R}^1$ is called a *full solution.*

It is known that a solution is continuous [2]. Let γ be a solution on I. Set
$$\mathcal{T}_\gamma(I) = \{(t, \gamma(t)) :\ t \in I\}.$$
$\mathcal{T}_\gamma(I)$ is called the *trace* of γ on I.

For any $x \in X$ and $I \subset \mathbb{R}^+$, we will write $\mathcal{T}_x(I) = \mathcal{T}_{\gamma_x}(I)$, where $\gamma_x(t) = \Phi(t)x$.

Proposition 2.1. *Let $x \in X$ and $0 \le T < T_x$. Then*

(1) *there exists a neighborhood U_x of x such that*

$$T_y > T, \qquad \forall\, y \in U_x;$$

(2) *for any compact interval $J \subset [0,T]$, $\mathcal{T}_y(J) \to \mathcal{T}_x(J)$ in $J \times X$ as $y \to x$. Specifically, for any neighborhood $\mathcal{V}$ of $\mathcal{T}_x(J)$ in $J \times X$, there exists a neighborhood U of x such that*

$$\mathcal{T}_y(J) \subset \mathcal{V}, \qquad \forall\, y \in U. \tag{2.1}$$

Proof. The first conclusion (1) is implied in [2], Lemma 1.8. To prove the second one, we define a map $G : \mathcal{D}_\Phi \to \mathbb{R}^+ \times X$ as follows:

$$G(t,x) = (t, \Phi(t)x), \qquad (t,x) \in \mathcal{D}_\Phi.$$

Clearly G is continuous. Let U_x be the neighborhood of x in (1), and $\mathcal{V}$ be a neighborhood of $\mathcal{T}_x(J)$ in $J \times X$. Then for each $t \in J$, there exists a cylindrical neighborhood $Q_t := I_t \times U_t$ of (t,x) such that $G(Q_t) \subset \mathcal{V}$, where I_t is an interval relatively open in J, and $U_t \subset U_x$ is a neighborhood of x. Since $J \subset \bigcup_{t\in J} I_t$, there exists a finite number of $I_t's$, say, $I_{t_1}, I_{t_2}, \dots, I_{t_n}$, such that $J \subset \bigcup_{i=1}^n I_{t_i}$. Set $U = \bigcap_{i=1}^n U_{t_i}$. Then U is a neighborhood of x. It can be seen that U fulfills (2.1). □

Remark 2.2. If X is a metric space with metric d, then Proposition 2.1 can be reformulated in a simpler but more specific manner as below:

Proposition 2.3. *Let $x \in X$ and $0 < T < T_x$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\Phi(t)y$ exists on $[0,T]$ for all $y \in B(x,\delta)$. Moreover,*

$$d\,(\Phi(t)y,\ \Phi(t)x) < \varepsilon, \qquad \forall\, t \in [0,T],\ y \in B(x,\delta).$$

2.2. A convergence result of solutions

In this subsection we give a convergence result concerning sequences of solutions.

Let us first recall the concepts of *strong admissibility*, which was first introduced for local semiflows on metric spaces [17].

Let M be a subset of X.

Definition. We say that Φ *does not explode* in M, if $T_x = \infty$ whenever $\Phi([0,T_x))x \subset M$.

Definition. M is said to be *admissible*, if for any sequences $x_n \in M$ and $t_n \to \infty$ with $\Phi([0,t_n])x_n \subset M$ for all n, the sequence $\Phi(t_n)x_n$ has a convergent subsequence.

M is said to be *strongly admissible*, if it is admissible and moreover, Φ does not explode in M.

Proposition 2.4. *Let M be a closed strongly admissible set, and γ_n be a sequence of solutions in M with each γ_n being defined on $[-t_n, t_n]$.*

Suppose $t_n \to \infty$. Then there exist a subsequence γ_{n_k} of γ_n and a full solution γ in M such that for any compact interval $J \subset \mathbb{R}^1$,

$$\mathcal{T}_{\gamma_{n_k}}(J) \to \mathcal{T}_\gamma(J) \text{ in } J \times X.$$

That is, for any neighborhood $\mathcal{V}$ of $\mathcal{T}_\gamma(J)$ in $J \times X$, there exists $k_0 > 0$ such that

$$\mathcal{T}_{\gamma_{n_k}}(J) \subset \mathcal{V}, \qquad \forall\, k > k_0. \tag{2.2}$$

Proof. We may assume $t_n > 1$ for all n. Since $\gamma_n(-1) = \Phi(t_n - 1)\gamma(-t_n)$, by admissibility of M there exists a subsequence γ_{1n} of γ_n such that $\gamma_{1n}(-1)$ converges to a point $x_1 \in M$. Set

$$\sigma_1(t) = \Phi(t+1)x_1, \qquad t \in [-1, T_{x_1} - 1).$$

We claim that σ_1 is a solution on $[-1, \infty)$ contained in M.

Indeed, since $\gamma_{1n}(-1) \to x_1$, by continuity of Φ we deduce that $\gamma_{1n}(t) \to \sigma_1(t)$ for all $t \in [-1, T_{x_1} - 1)$. The closedness of M then implies that $\sigma_1(t) \in M$ for all $t \in [-1, T_{x_1} - 1)$, i.e., $\Phi(t)x_1 \in M$ for all $t \in [0, T_{x_1})$. As Φ does not explode in M, we have $T_{x_1} = \infty$. Thus the claim holds true.

We also infer from Proposition 2.1 that $\mathcal{T}_{\gamma_{1n}}(J)$ converges to $\mathcal{T}_{\sigma_1}(J)$ in $J \times X$ for any compact interval $J \subset [-1, \infty)$.

Repeating the same argument with very minor modifications, one can show that there exist a subsequence γ_{2n} of γ_{1n} and a solution σ_2 on $[-2, \infty)$ contained in M such that $\mathcal{T}_{\gamma_{2n}}(J)$ converges to $\mathcal{T}_{\sigma_2}(J)$ in $J \times X$ for any compact interval $J \subset [-2, \infty)$. Clearly

$$\sigma_2(t) \equiv \sigma_1(t), \qquad t \in [-1, \infty).$$

Continuing the above procedure, we obtain for each k a subsequence γ_{kn} of γ_n and a solution σ_k in M such that

(1) each sequence $\gamma_{(k+1)n}$ is a subsequence of γ_{kn};
(2) σ_k is defined on $[-k, \infty)$, and

$$\sigma_{k+1}(t) \equiv \sigma_k(t), \qquad t \in [-k, \infty); \tag{2.3}$$

(3) for any compact interval $J \subset [-k, \infty)$, $\mathcal{T}_{\gamma_{kn}}(J)$ converges to $\mathcal{T}_{\sigma_k}(J)$ in $J \times X$ as $n \to \infty$ in the sense that for any neighborhood $\mathcal{V}$ of $\mathcal{T}_{\sigma_k}(J)$ in $J \times X$, there exists $n_k > 0$ such that

$$\mathcal{T}_{\gamma_{kn}}(J) \subset \mathcal{V}, \qquad \forall\, n > n_k.$$

Define a full solution γ in M as follows:

$$\gamma(t) = \sigma_k(t), \qquad \text{if } t \in [-k, \infty).$$

By (2.3) it is clear that γ is well defined. Consider the sequence γ_{kk}. By virtue of the classical diagonal procedure it can be easily seen that $\mathcal{T}_{\gamma_{kk}}(J)$ converges to $\mathcal{T}_\gamma(J)$ in $J \times X$ as $k \to \infty$ for any compact interval $J \subset \mathbb{R}^1$. □

3. Invariant sets and limit sets

In this section we talk about some basic facts on invariant sets and limit sets of local semiflows on topological spaces.

Let X be a Hausdorff topological space and Φ be a local semiflow on X.

Definition. The *ω-limit set* $\omega(M)$ of $M \subset X$ is defined as

$$\omega(M) = \{y \in X : \ \exists\, x_n \in M \text{ and } t_n \to +\infty \text{ such that } \Phi(t_n)x_n \to y\}.$$

The *ω-limit set* $\omega(\gamma)$ of a solution γ on (a, ∞) is defined as

$$\omega(\gamma) = \{y \in X : \ \exists\, t_n \to \infty \text{ such that } \gamma(t_n) \to y\}.$$

The *α-limit set* $\alpha(\gamma)$ of a solution γ on $(-\infty, a)$ is defined as

$$\alpha(\gamma) = \{y \in X : \ \exists\, t_n \to -\infty \text{ such that } \gamma(t_n) \to y\}.$$

Remark 3.1. In [15] the ω-limit set $\omega(M)$ of $M \subset X$ is defined as

$$\omega(M) := \bigcap_{t \geq 0} \overline{\Phi([t, \infty))M},$$

which is somewhat different from that of ours here. But if we comeback to the situation of a metric space, then both coincide [20].

To discuss fundamental properties of limit sets, we need to introduce several notions on invariance and attraction.

Let M be a subset of X.

Definition. M is said to be *positively* (*resp. negatively*) *invariant* if

$$\Phi(t)M \subset M \ \text{ (resp. } M \subset \Phi(t)M), \qquad \forall\, t \geq 0.$$

M is said to be *invariant*, if it is both negatively and positively invariant.

Proposition 3.2 ([2])**.** *If M is positively invariant, then so is $\overline{M}$.*

Remark 3.3. We do not know whether a similar result holds true for negative invariance. But by a very standard argument one can easily verify that if M is negatively invariant, then for each $y \in M$ there exists a solution γ on $(-\infty, 0]$ contained in M such that $\gamma(0) = y$. Further noting that $y = \Phi(n)\gamma(-n)$ for all n, by the definition of $\omega(M)$ one concludes that $y \in \omega(M)$. Hence

$$M \subset \omega(M).$$

Definition. We say that M *attracts* $B \subset X$, if $T_x = \infty$ for all $x \in B$ and moreover, for any neighborhood V of M, there exists $t_0 > 0$ such that

$$\Phi(t)B \subset V, \qquad \forall\, t > t_0.$$

Proposition 3.4. *Suppose $\Phi(\mathbb{R}^+)M$ is contained in a closed strongly admissible set N. Then $\omega(M)$ is a nonempty invariant set that attracts M.*

Proof. The nonemptyness of $\omega(M)$ is a simple consequence of the strong admissibility of N. Clearly $\omega(M) \subset N$. We show that $\omega(M)$ is invariant.

Let $y \in \omega(M)$. Then there exist sequences $x_n \in M$ and $t_n \to \infty$ such that $y_n := \Phi(t_n)x_n \to y$. We claim that $\Phi(t)y \in \omega(M)$ for all $t \in [0, T_y)$, hence $\omega(M)$ is positively invariant. Indeed, for any $0 < t < T_y$ we have

$$\Phi(t)y = \lim_{n\to\infty} \Phi(t)y_n = \lim_{n\to\infty} \Phi(t)\Phi(t_n)x_n = \lim_{n\to\infty} \Phi(t + t_n)x_n,$$

from which one immediately deduces that $\Phi(t)y \in \omega(M)$.

We also show that for any $t \geq 0$, there exists $z \in \omega(M)$ such that $\Phi(t)z = y$, thus $\omega(M)$ is negatively invariant. We may assume $t_n \geq t$ for all n. Let $z_n = \Phi(t_n - t)x_n$. By admissibility of N there exists a subsequence of z_n (still denoted by z_n) converging to a point z. Clearly $z \in \omega(M)$. We have

$$y = \lim_{n\to\infty} \Phi(t_n)x_n = \lim_{n\to\infty} \Phi(t)\Phi(t_n - t)x_n = \lim_{n\to\infty} \Phi(t)z_n = \Phi(t)z.$$

The verification of the attraction property is trivial. We omit the details. □

Proposition 3.5. *Let γ be a solution on $I := (a, \infty)$ (resp. $(-\infty, a)$). Suppose $\gamma(I)$ is contained in a closed strongly admissible set N. Then $\omega(\gamma)$ (resp. $\alpha(\gamma)$) is a nonempty invariant set.*

Proof. If γ is a solution on (a, ∞), then it can be easily seen that $\omega(\gamma) = \omega(\gamma(t_0))$, where $t_0 > a$. Thus the conclusion on $\omega(\gamma)$ immediately follows from Proposition 3.4.

Now let γ be a solution on (a, ∞) and consider the α-limit set $\alpha(\gamma)$. For convenience in statement, we may assume $a = 0$. We first check that $\alpha(\gamma) \neq \emptyset$. Take a sequence $0 < t_n \to \infty$. Then $\gamma(-t_n) = \Phi(t_n)\gamma(-2t_n)$. By admissibility of N we deduce that $\gamma(-t_n)$ has a subsequence converging to a point z. Clearly $z \in \alpha(\gamma)$.

As N is closed, we have $\alpha(\gamma) \subset N$. Let $y \in \alpha(\gamma)$, and let $t_n \to \infty$ be such that $y_n := \gamma(-t_n) \to y$. By a similar argument as in the verification of positive invariance of $\omega(M)$ in Proposition 3.4, it can be shown that $\Phi(t)y \in \alpha(\gamma)$ for all $t \in [0, T_y)$. Therefore $\alpha(\gamma)$ is positively invariant. The verification of negative invariance of $\alpha(\gamma)$ can also be performed in a similar manner as in the case of $\omega(M)$ in Proposition 3.4, and is thus omitted. □

4. Attractors

In this section we introduce the notion of attractors and discuss basic properties of attractors. Some existence results will also be given.

Let X be a Hausdorff space and Φ be a given local semiflow on X.

4.1. Definition and basic properties

We first recall that a set $M \subset X$ is said to be *sequentially compact* (*s-compact* in short), if each sequence x_n in M has a subsequence converging to a point $x \in M$.

Definition. A nonempty s-compact invariant set $\mathscr{A}$ is called an *attractor* of Φ, if there is a neighborhood N of $\mathscr{A}$ such that

(1) $\mathscr{A}$ attracts N; and
(2) $\mathscr{A}$ is the maximal s-compact invariant set in N.

Given an attractor $\mathscr{A}$, define

$$\Omega(\mathscr{A}) = \{x \in X : \mathscr{A} \text{ attracts } x\}.$$

$\Omega(\mathscr{A})$ is called the *region of attraction* of $\mathscr{A}$. If $\Omega(\mathscr{A}) = X$, then $\mathscr{A}$ is simply called a *global attractor.*

Theorem 4.1. *Let $\mathscr{A}$ be an attractor of Φ. Then the following assertions hold.*

(1) *$\Omega(\mathscr{A})$ is open, and for each compact set $K \subset \Omega(\mathscr{A})$, $\mathscr{A}$ attracts a neighborhood U of K.*
(2) *$\mathscr{A}$ is the maximal s-compact invariant set in $\Omega(\mathscr{A})$.*
(3) *If X is normal, then for any closed admissible neighborhood V of $\mathscr{A}$ with $V \subset \Omega(\mathscr{A})$, $\mathscr{A}$ is the maximal invariant set in V.*

Proof. (1) By definition there is a neighborhood N of $\mathscr{A}$ such that $\mathscr{A}$ attracts N, moreover, $\mathscr{A}$ is the maximal s-compact invariant set in N.

Take a $\tau > 0$ such that

$$\Phi(t)N \subset \text{int}N, \qquad t \geq \tau. \tag{4.1}$$

Let $x \in \Omega(\mathscr{A})$. Then the escape time $T_x = \infty$. Furthermore, there exists $T > 0$ such that

$$\Phi(T)x \in \text{int}N.$$

By virtue of Proposition 2.1, there is a neighborhood U_x of x such that $T_y > T$ and $\Phi(T)y \in \text{int}N$ for all $y \in U_x$, from which it can be easily seen that $\mathscr{A}$ attracts each point in U_x. Hence $\Omega(\mathscr{A})$ is open.

Let K be a compact subset of $\Omega(\mathscr{A})$. For each $x \in K$, pick a $t_x > 0$ such that $\Phi(t_x)x \in \text{int}N$. Then by continuity there exists an open neighborhood U_x of x such that $\Phi(t_x)U_x \subset \text{int}N$. Combining this with (4.1) it yields

$$\Phi(t)U_x \subset \text{int}N, \qquad t \geq \tau + t_x := \tau_x. \tag{4.2}$$

Since K is compact, there exists a finite number of points in K, say, $x_1, x_2, \ldots$, x_n such that

$$K \subset \bigcup_{1 \leq i \leq n} U_{x_i} := U.$$

Let $\tau_0 = \max\{\tau_{x_i} : 1 \leq i \leq n\}$. Then (4.2) implies that $\Phi(\tau_0)U \subset N$. Hence we see that $\mathscr{A}$ attracts U.

(2) Let M be an s-compact invariant set in $\Omega(\mathscr{A})$. We need to prove that $M \subset \mathscr{A}$. Since $\mathscr{A}$ is the maximal s-compact invariant set of Φ in N, for this purpose it suffices to check that $M \subset N$.

We argue by contradiction and suppose the contrary. Then there would exist $y \in M \backslash N$. Let γ be a solution on $(-\infty, 0]$ contained in M with $\gamma(0) = y$. As M

is s-compact, using some similar argument as in the proof of Proposition 3.5 one can easily verify that $\alpha(\gamma)$ is a nonempty invariant set with $\alpha(\gamma) \subset M \subset \Omega(\mathscr{A})$.

Now if $\alpha(\gamma) \cap \mathrm{int}N \neq \emptyset$, one can pick a $z \in \alpha(\gamma) \cap \mathrm{int}N$. Take a sequence $t_n \to \infty$ such that $\gamma(-t_n) \to z$. Then $\gamma(-t_n) \in \mathrm{int}N$ for t_n sufficiently large. Fix a $t_n > \tau$ such that $\gamma(-t_n) \in \mathrm{int}N$. By (4.1) we find that

$$y = \gamma(0) = \Phi(t_n)\gamma(-t_n) \in N,$$

a contradiction! On the other hand, if $\alpha(\gamma) \cap \mathrm{int}N = \emptyset$, then by invariance of $\alpha(\gamma)$ we necessarily have $\alpha(\gamma) \cap \Omega(\mathscr{A}) = \emptyset$, which again leads to a contradiction.

(3) Assume X is normal. Let $V \subset \Omega(\mathscr{A})$ be a closed admissible neighborhood of $\mathscr{A}$. Then one can find a closed neighborhood O of $\mathscr{A}$ with $O \subset V \cap N$. Clearly $\mathscr{A}$ attracts O. Hence there exists $t_0 > 0$ such that

$$\Phi([t_0, \infty))O \subset O. \tag{4.3}$$

By (4.3) and Proposition 3.4 we deduce that $\omega(O)$ is a nonempty invariant set with $\omega(O) \subset O \subset N$. We claim that $\omega(O)$ is s-compact. Indeed, let $y_n \in \omega(O)$ be a sequence. Then by invariance of $\omega(O)$ there exists a sequence $x_n \in \omega(O)$ such that $y_n = \Phi(n)x_n$. The admissibility of O implies that y_n has a subsequence y_{n_k} converging to a point y. As $x_n \in \omega(O) \subset O$, by the definition of w-limit set we have $y \in \omega(O)$, which completes the proof of the claim.

Because $\mathscr{A}$ is the maximal s-compact invariant set in N and $\omega(O) \subset N$, we have

$$\omega(O) \subset \mathscr{A}. \tag{4.4}$$

Now if K is an invariant set in O, then $K \subset \omega(O)$. Hence by (4.4) we have $K \subset \mathscr{A}$. It follows that $\mathscr{A}$ is the maximal invariant set in O.

To show that $\mathscr{A}$ is the maximal invariant set in V, it now suffices to check that if K is an invariant set in V, then $K \subset O$. We argue by contradiction and suppose $K \setminus O \neq \emptyset$. Pick a $y \in K \setminus O$. Then there is a solution γ on $(-\infty, 0]$ contained in K such that $\gamma(0) = y$. Since $K \subset V$, by Proposition 3.5 $\alpha(\gamma)$ is a nonempty invariant set. Using some similar argument as in the proof of (2) (with N therein replaced by O), one immediately obtains a contradiction. □

Remark 4.2. We infer from Theorem 4.1(1) that the global attractor, if exists, is necessarily unique.

In [6] a global attractor is defined to be a compact invariant set that attracts each compact set. By Theorem 4.1(1) we see that if a global attractor in our terminology is compact, then it is a global attractor in the terminology in [6].

Now we turn our attention to stability of attractors.

Definition. A set $M \subset X$ is called *stable*, if for any neighborhood U of M, there is a neighborhood V of M such that

$$\Phi(\mathbb{R}^+)V \subset U. \tag{4.5}$$

Theorem 4.3. *Let $\mathscr{A}$ be an attractor of Φ. Then $\mathscr{A}$ is stable.*

Proof. Let U be an open neighborhood of $\mathscr{A}$. We need to prove that there exists an open neighborhood V of $\mathscr{A}$ such that (4.5) holds.

By definition $\mathscr{A}$ attracts a neighborhood N of $\mathscr{A}$. Fix a $T > 0$ so that

$$\Phi(t)N \subset U, \qquad \forall\, t > T. \tag{4.6}$$

In what follows we argue by contradiction and suppose (4.5) fails to be true. Then for any open neighborhood $V \subset N$ of $\mathscr{A}$, there exist $y \in V$ and $t_y \geq 0$ such that

$$\Phi(t_y)y \notin U. \tag{4.7}$$

In view of (4.6), we necessarily have $t_y \leq T$.

We claim that there exists $x_0 \in \overline{\mathscr{A}}$ such that for any open neighborhood $O \subset N$ of x_0, one can find a $z \in O$ and $t_z \in [0, T]$ such that

$$\Phi(t_z)z \notin U. \tag{4.8}$$

Indeed, if this was false, each point $x \in \overline{\mathscr{A}}$ would have an open neighborhood $U_x \subset N$ such that $\Phi([0,T])U_x \subset U$. Set $V = \bigcup_{x\in\overline{\mathscr{A}}} U_x$. Obviously V is an open neighborhood of $\mathscr{A}$. However, $\Phi([0,T])V \subset U$, which contradicts (4.7) and proves our claim.

On the other hand, we infer from Proposition 3.2 that $\overline{\mathscr{A}}$ is positively invariant. Hence $\Phi([0,\, T])x_0 \subset \overline{\mathscr{A}} \subset U$. As U is open, by virtue of Proposition 2.1 there is an open neighborhood O of x_0 such that $\Phi([0,\, T])O \subset U$. But this contradicts (4.8).

The proof of the theorem is finished. □

4.2. Existence of attractors

Now we prove some existence results on attractors. First, we have:

Theorem 4.4. *Suppose X is normal. Let $M \subset X$ be closed. Assume M attracts an admissible neighborhood N of itself. Then $\mathscr{A} = \omega(N)$ is an attractor.*

Proof. Since X is normal, it can be assumed that N is closed. By the definition of attraction we have $T_x = \infty$ for all $x \in N$. Furthermore, there exists $T > 0$ such that

$$\Phi([T,\infty))N \subset N.$$

Proposition 3.4 then asserts that $\mathscr{A} = \omega(N)$ is a nonempty invariant set attracting N.

We claim that $\mathscr{A} \subset M$. Indeed, if $y \in \mathscr{A} \setminus M \neq \emptyset$, then by closedness of M there exist open neighborhoods U of M and V of y such that $U \cap V = \emptyset$. Let $x_n \in N$ and $t_n \to \infty$ be such that $\Phi(t_n)x_n \to y$. Then $\Phi(t_n)x_n \in V$ for n sufficiently large. On the other hand, since M attracts N, one should have $\Phi(t_n)x_n \in U$ for n sufficiently large, which leads to a contradiction. Hence the claim holds true.

Let $K \subset N$ be an invariant set. Then

$$K \subset \omega(K) \subset \omega(N) = \mathscr{A}.$$

Thus we deduce that $\mathscr{A}$ is the maximal invariant set in N.

In what follows we show that $\mathscr{A}$ is s-compact, hence it is an attractor. Let y_n be a sequence in $\mathscr{A}$. Then for each n there is an $x_n \in \mathscr{A}$ such that $y_n = \Phi(n)x_n$. Further by admissibility of N we deduce that y_n has a convergent subsequence y_{n_k}. Let $y_{n_k} \to y$ as $k \to \infty$. We observe that

$$y = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} \Phi(n_k)x_{n_k} \in \omega(N) = \mathscr{A}.$$

Therefore one concludes that $\mathscr{A}$ is s-compact. $\square$

Theorem 4.5. *Suppose X is normal. Let M be a closed subset of X. Assume M is stable, and that there exists a strongly admissible neighborhood N of M such that M attracts each point $x \in N$.*

Then Φ has an attractor $\mathscr{A} \subset M$ with $N \subset \Omega(\mathscr{A})$.

Proof. Take a closed neighborhood F of M with $F \subset N$. Then by stability of M there is a closed neighborhood W of M such that $\Phi([0,\infty))W \subset F$. Thus by Proposition 3.4 we deduce that $\omega(W)$ is an invariant set that attracts W. Clearly $\omega(W) \subset F$. We show that

$$\omega(W) \subset M, \tag{4.9}$$

hence M attracts W. It then immediately follows by Theorem 4.4 that $\mathscr{A} = \omega(W)$ is an attractor.

We argue by contradiction and suppose the contrary. Then $\omega(W) \setminus M \neq \emptyset$. Pick a $y \in \omega(W) \setminus M$. One can find a neighborhood V of M such that $y \notin V$. Take an open neighborhood U of M such that

$$\Phi(t)U \subset V, \qquad \forall\, t \geq 0.$$

Let γ be a solution on $(-\infty, 0]$ contained in $\omega(W)$ with $\gamma(0) = y$. If $\gamma(\tau) \in U$ for some $\tau < 0$, then

$$y = \gamma(0) = \Phi(-\tau)\gamma(\tau) \in V,$$

which leads to a contradiction. Thus we have

$$\gamma(t) \notin U, \qquad \forall\, t < 0. \tag{4.10}$$

Noticing that $\gamma((-\infty,0]) \subset \omega(W) \subset F$, by Proposition 3.5 we see that $\alpha(\gamma)$ is a nonempty invariant set. Clearly $\alpha(\gamma) \subset F$. We claim that

$$\alpha(\gamma) \cap U = \emptyset. \tag{4.11}$$

Indeed, if this was false, there would exist $z \in \alpha(\gamma) \cap U$. Take a sequence $t_n \to -\infty$ such that $\gamma(t_n) \to z$. Then $\gamma(t_n) \in U$ for t_n sufficiently large, which contradicts (4.10) and proves (4.11).

Now since $\alpha(\gamma)$ is invariant, (4.11) implies that M does not attract any point $x \in \alpha(\gamma)$, which contradicts the attraction assumption on M (recall that $\alpha(\gamma) \subset F \subset N$) and completes the proof of (4.9).

To complete the proof of the theorem, there remains to check that $N \subset \Omega(\mathscr{A})$. Let $x \in N$. As M attracts x and W is a neighborhood of M, there exists t_0 such that $\Phi(t_0)x \in W$. As $\mathscr{A}$ attracts W, it immediately follows that $x \in \Omega(\mathscr{A})$. □

The following result is a simple consequence of Theorem 4.5. It can be seen as a converse theorem of Theorem 4.3.

Corollary 4.6. *Let $\mathscr{A}$ be a closed invariant set. If $\mathscr{A}$ is stable and attracts each point in a strongly admissible neighborhood of itself, then it is an attractor.*

5. Lyapunov functions of attractors

It is well known that Lyapunov functions play crucial roles in many aspects of stability analysis. In this section we prove a converse Lyapunov theorem for attractors of local semiflows on topological spaces.

Let Φ be a given local semiflow on Hausdorff space X and $\mathscr{A}$ be an attractor of Φ with the region of attraction $\Omega = \Omega(\mathscr{A})$.

Definition. A nonnegative function $\zeta \in C(\Omega)$ is called a $\mathcal{K}_0$ *function* of $\mathscr{A}$, if

$$\zeta(x) = 0 \Longleftrightarrow x \in \mathscr{A}.$$

Definition. A $\mathcal{K}_0$ function L of $\mathscr{A}$ is called a *Lyapunov function* of $\mathscr{A}$, if

$$L(\Phi(t)x) < L(x), \qquad \forall\, x \in \Omega \setminus \mathscr{A},\ t > 0.$$

Theorem 5.1. *Assume that $\mathscr{A}$ is closed and has a $\mathcal{K}_0$ function ζ on Ω. Then $\mathscr{A}$ has a Lyapunov L on Ω.*

If we assume, in addition, that X is normal, then for any closed subset K of X with $K \subset \Omega \setminus \mathscr{A}$, there exists a Lyapunov function L of $\mathscr{A}$ such that

$$L(x) \geq 1, \qquad \forall\, x \in K. \tag{5.1}$$

Proof. The construction of L is the same as in [9, p. 226]. See also [12]. We give the details for the reader's convenience.

Set

$$\xi(x) = \sup_{t \geq 0} \zeta(\Phi(t)x), \qquad x \in \Omega. \tag{5.2}$$

We show that ξ is continuous. For this purpose, it suffices to check that for any fixed $x_0 \in \Omega$ and $\varepsilon > 0$, there is a neighborhood U of x_0 with $U \subset \Omega$ such that

$$|\xi(x) - \xi(x_0)| < \varepsilon, \qquad \forall\, x \in U. \tag{5.3}$$

Let $\xi_0 = \xi(x_0)$. First, by the definition of ξ there exists $t_0 \geq 0$ such that

$$\zeta(\Phi(t_0)x_0) > \xi_0 - \varepsilon/2.$$

Hence by continuity one can find a neighborhood V of x with $V \subset \Omega$ such that

$$\zeta(\Phi(t_0)x) > \xi_0 - \varepsilon, \qquad \forall\, x \in V.$$

Therefore

$$\xi(x) \geq \zeta(\Phi(t_0)x) > \xi_0 - \varepsilon, \qquad \forall\, x \in V. \tag{5.4}$$

We also check that there is a neighborhood U of x_0 with $U \subset V$ such that

$$\xi(x) < \xi_0 + \varepsilon, \qquad \forall\, x \in U, \tag{5.5}$$

thus proving (5.3).

Let $W = \zeta^{-1}([0,\, \xi_0 + \varepsilon/2))$. Then W is an open neighborhood of $\mathscr{A}$. By definition $\mathscr{A}$ attracts a neighborhood N of itself. It can be assumed that N is open. Fix a $T_1 > 0$ such that

$$\Phi(t)N \subset W, \qquad \forall\, t \geq T_1. \tag{5.6}$$

Take a $T_2 > 0$ such that $\Phi(T_2)x_0 \in N$. Then by continuity of Φ one can find a neighborhood U' of x_0 with $U' \subset V$ such that $\Phi(T_2)U' \subset N$. Combining this with (5.6) it yields

$$\Phi(t)U' \subset W, \qquad \forall\, t > T_1 + T_2 := T.$$

Hence by the definition of W we have

$$\zeta(\Phi(t)x) < \xi_0 + \varepsilon/2, \qquad \forall\, x \in U',\ t > T. \tag{5.7}$$

Recalling that $\zeta(\Phi(t)x_0) \leq \xi(x_0) = \xi_0$ for $t \geq 0$, we have

$$\Phi([0, T])x_0 \subset W.$$

Thus by Proposition 2.1 one easily deduces that there is a neighborhood U of x_0 with $U \subset U' \subset V$ such that $\Phi([0,\, T])U \subset W$. It follows that

$$\zeta(\Phi(t)x) < \xi_0 + \varepsilon/2, \qquad \forall\, x \in U,\ t \in [0,\, T]. \tag{5.8}$$

(5.7) and (5.8) assure that

$$\zeta(\Phi(t)x) < \xi_0 + \varepsilon/2, \qquad \forall x \in U,\ t \geq 0.$$

Thereby

$$\xi(x) = \sup_{t \geq 0} \zeta(\Phi(t)x) \leq \xi_0 + \varepsilon/2 < \xi_0 + \varepsilon, \qquad \forall\, x \in U.$$

This completes the proof of (5.5).

Clearly $\xi(x) \equiv 0$ on $\mathscr{A}$. A basic property of ξ is that it is decreasing along each solution of Φ in Ω. Indeed, for any $x \in \Omega$ and $t \geq 0$, we have

$$\xi(\Phi(t)x) = \sup_{s \geq 0} \zeta\left(\Phi(s)\Phi(t)x\right) = \sup_{\tau \geq t} \zeta(\Phi(\tau)x) \leq \xi(x).$$

Define

$$L(x) = \xi(x) + \int_0^\infty e^{-t}\xi(\Phi(t)x)dt, \qquad x \in \Omega.$$

We show that L is precisely a Lyapunov function of $\mathscr{A}$.

Let $x \in \Omega \setminus \mathscr{A}$ and $s > 0$. Then $\xi(x) > 0$. Because ξ is decreasing along each solution in Ω, we see that

$$\xi(\Phi(t)\Phi(s)x) = \xi\left(\Phi(s)\Phi(t)x\right) \leq \xi\left(\Phi(t)x\right), \qquad \forall t \geq 0.$$

We claim that there is at least one point $t \geq 0$ such that

$$\xi\left(\Phi(t)\Phi(s)x\right) < \xi\left(\Phi(t)x\right).$$

Indeed, if this was false, one should have

$$\xi\left(\Phi(t)x\right) \equiv \xi\left(\Phi(t)\Phi(s)x\right) = \xi\left(\Phi(t+s)x\right), \qquad t \geq 0.$$

Hence $\xi\left(\Phi(t)x\right)$ is an s-periodic function. This contradicts to the fact that $\xi\left(\Phi(t)x\right) \to 0$ as $t \to \infty$.

Now since both $\xi\left(\Phi(t)\Phi(s)x\right)$ and $\xi\left(\Phi(t)x\right)$ are continuous in t, we have

$$\begin{aligned} L(\Phi(s)x) &= \xi(\Phi(s)x) + \int_0^\infty e^{-t}\xi(\Phi(t)\Phi(s)x)dt \\ &\leq \xi(x) + \int_0^\infty e^{-t}\xi(\Phi(t)\Phi(s)x)dt \\ &< \xi(x) + \int_0^\infty e^{-t}\xi(\Phi(t)x)dt = L(x). \end{aligned}$$

The other properties of L simply follow from that of ξ. We omit the details of the argument.

If X is a normal Hausdorff space, then for any closed subset K of X with $K \subset \Omega \setminus \mathscr{A}$, there exists a nonnegative continuous function ψ on X such that

$$\psi(x) \equiv 0 \text{ (on } \mathscr{A}), \quad \text{ and } \psi(x) \equiv 1 \text{ (on } K).$$

Set $\eta(x) = \zeta(x) + \psi(x)$. Clearly $\zeta(x)$ is a $\mathcal{K}_0$ function of $\mathscr{A}$ on Ω. If we replace the function ζ by η in the above argument, one immediately obtains a Lyapunov function of $\mathscr{A}$ satisfying (5.1). □

Remark 5.2. To guarantee the existence of a Lyapunov function of an attractor, we have assumed in Theorem 5.1 the existence of a $\mathcal{K}_0$ function. In the case of a general topological space, we do not know whether such a function does exit. However, in many cases the space on which we are working may be the quotient space of a pair (N, E) of closed subsets of a metric space X. For such a space a $\mathcal{K}_0$ function can be directly formulated by using the metric d of X; see, e.g., [11].

6. Morse decompositions

In this section we always assume X is a normal Hausdorff space.

Let Φ be a given local semiflow on X, and $\mathscr{A}$ an attractor of Φ. Since $\mathscr{A}$ is invariant, the restriction $\Phi|_{\mathscr{A}}$ of Φ is a global semiflow on $\mathscr{A}$.

A set $A \subset \mathscr{A}$ is called an *attractor* of Φ in $\mathscr{A}$, this means that it is an attractor of $\Phi|_{\mathscr{A}}$. We have the following fundamental result which generalizes the corresponding one for semiflows on metric spaces [9].

Theorem 6.1. *Suppose $\mathscr{A}$ has an admissible neighborhood N_1. Then any attractor A in $\mathscr{A}$ is also an attractor in X.*

Proof. By definition $\mathscr{A}$ attracts a neighborhood N_2 of $\mathscr{A}$. As X is normal, one can find a closed neighborhood N of $\mathscr{A}$ with

$$N \subset N_1 \cap N_2.$$

N is strongly admissible. By stability of $\mathscr{A}$ there exists a closed neighborhood M of $\mathscr{A}$ such that

$$\Phi(\mathbb{R}^+)M \subset N. \tag{6.1}$$

Pick a closed neighborhood $\mathcal{O}'$ of A in $\mathscr{A}$ with $\mathcal{O}' \subset M$ such that A attracts $\mathcal{O}'$ (hence $\mathcal{O}' \subset \Omega_{\mathscr{A}}(A)$, where $\Omega_{\mathscr{A}}(A)$ is the region of attraction of A in $\mathscr{A}$). By the basic knowledge on general topology, one can find a closed neighborhood $\mathcal{O}$ of A in X such that

$$\mathcal{O}' = \mathcal{O} \cap \mathscr{A}.$$

We can also assume $\mathcal{O} \subset M$. We show that A attracts a neighborhood $W \subset \mathcal{O}$ of A, from which one immediately concludes that A is an attractor in X.

Let us first check that there is a neighborhood $W \subset \mathcal{O}$ of A such that

$$\Phi(\mathbb{R}^+)W \subset \mathcal{O}. \tag{6.2}$$

Let $V = \text{int}\mathcal{O}$. Then V is an open neighborhood of $\overline{A}$. For each integer $n > 0$, since $\Phi(\mathbb{R}^+)x \subset \overline{A} \subset V$ for any fixed $x \in \overline{A}$ (see Proposition 3.2), by Lemma 2.1 there exists a neighborhood $U_x = U_x(n)$ of x such that $\Phi([0,n])U_x \subset V$. Set $U_n = \bigcup_{x\in\overline{A}} U_x$. Clearly U_n is a neighborhood of A for each n, and

$$\Phi([0,n])U_n \subset V = \text{int}\mathcal{O}. \tag{6.3}$$

We show that

$$\Phi(\mathbb{R}^+)U_n \subset \mathcal{O} \tag{6.4}$$

for some $n > 0$, thus proving (6.2).

Suppose the contrary. Then for each n there exists $x_n \in U_n$ such that $\Phi(t)x_n \notin \mathcal{O}$ for some $t > 0$. Let

$$t_n = \min\{t > 0 : \ \Phi(t)x_n \notin \mathcal{O}\}.$$

Then

$$\Phi(t_n)x_n \in \partial\mathcal{O}, \quad \Phi([0,t_n])x_n \subset \mathcal{O}. \tag{6.5}$$

(6.3) and the first relation in (6.5) imply that $t_n > n$. Thus by admissibility of $\mathcal{O}$ we deduce that the sequence $\Phi(t_n)x_n$ has a subsequence (still denoted by $\Phi(t_n)x_n$) converging to a point $x \in \partial\mathcal{O}$. For each n, let

$$\gamma_n(t) = \Phi(t_n + t)x_n, \qquad t \in [-t_n, \infty).$$

Recalling that $\mathcal{O} \subset M$, by (6.1) we have

$$\gamma_n(t) \in N, \qquad t \in [-t_n, \infty)$$

for all n. Thanks to Proposition 2.4, the sequence γ_n has a subsequence converging uniformly on any compact interval (in the sense given in Proposition 2.4) to a complete trajectory γ. It is obvious that $\gamma(0) = x \in \partial\mathcal{O}$. Furthermore,

$$\gamma(t) \in \mathcal{O}, \qquad t \in (-\infty, 0]. \tag{6.6}$$

Because $\mathcal{O} \subset M$, we infer from (6.1) and (6.6) that γ is contained in N. Thus by Theorem 4.1(3) one concludes that $\gamma(\mathbb{R}) \subset \mathscr{A}$. (6.6) then asserts that

$$\gamma(t) \in \mathcal{O} \cap \mathscr{A} = \mathcal{O}' \subset \Omega_{\mathscr{A}}(A), \qquad t \in (-\infty, 0]. \tag{6.7}$$

Hence we actually have $\gamma(\mathbb{R}) \subset \Omega_{\mathscr{A}}(A)$.

In the following we further prove that $\gamma(\mathbb{R}) \subset A$. For this purpose, we first verify that $\alpha(\gamma) \subset A$.

By (6.7) we see that $\alpha(\gamma) \subset \mathcal{O}'$. If $\alpha(\gamma) \setminus A \neq \emptyset$, then there is a $y \in \alpha(\gamma) \setminus A$. Pick a neighborhood F of A such that $y \notin F$. As A attracts $\mathcal{O}'$, there exists $T > 0$ such that

$$\Phi(t)\mathcal{O}' \subset F, \qquad \forall t > T. \tag{6.8}$$

On the other hand, by invariance of $\alpha(\gamma)$, for any $t > T$ there exists $z \in \alpha(\gamma)$ such that $\Phi(t)z = y$. Hence by (6.8) one deduces that $y \in F$, which leads to a contradiction! Thus $\alpha(\gamma) \subset A$.

Now we show that $\gamma(\mathbb{R}) \subset \mathcal{O}'$. It then follows by Theorem 4.1(3) that $\gamma(\mathbb{R}) \subset A$, which contradicts the fact that $\gamma(0) = x \in \partial\mathcal{O}$ and completes the proof of what we desired in (6.4).

By stability of A, one can find a neighborhood H of A in $\mathscr{A}$ with $H \subset \mathcal{O}'$ such that

$$\Phi(\mathbb{R}^+)H \subset \mathcal{O}'. \tag{6.9}$$

Since $\alpha(\gamma) \subset A$, we deduce that there is a $\tau < 0$ such that $\gamma(t) \in H$ for all $t < \tau$. Fix a $t_0 < \tau$. Then by (6.9) one has $\gamma(t) \in \mathcal{O}'$ for all $t \geq t_0$. Hence $\gamma(\mathbb{R}) \subset \mathcal{O}'$.

Now let W be the neighborhood of A in (6.2). Then $\omega(W) \subset \mathcal{O}$. Therefore we deduce by invariance of $\omega(W)$ and Theorem 4.1(3) that $\omega(W) \subset \mathscr{A}$. Hence

$$\omega(W) \subset \mathcal{O} \cap \mathscr{A} = \mathcal{O}'.$$

Recalling that $\mathcal{O}' \subset \Omega_{\mathscr{A}}(A)$, again by Theorem 4.1(3) one concludes that $\omega(W) \subset A$. Thereby A attracts W. $\square$

Let A be an attractor of Φ in $\mathscr{A}$. Define

$$A^* = \mathscr{A} \setminus \Omega_{\mathscr{A}}(A). \tag{6.10}$$

It is trivial to check that A^* is a nonempty s-compact invariant set. A^* is called the *repeller* of Φ in $\mathscr{A}$ dual to A, and (A, A^*) an *attractor-repeller pair* in $\mathscr{A}$.

Proposition 6.2. *Let A be an attractor of Φ in $\mathscr{A}$, and let $\gamma : \mathbb{R} \to \mathscr{A}$ be a complete trajectory through $x \in \mathscr{A}$. Then the following properties hold.*

(1) *If $\omega(\gamma) \cap A^* \neq \emptyset$, then $\gamma(\mathbb{R}) \subset A^*$.*

(2) *If* $\alpha(\gamma)\cap\overline{A}^{\mathscr{A}}\neq\emptyset$, *then* $\gamma(\mathbb{R})\subset A$. *Here* $\overline{A}^{\mathscr{A}}$ *is the closure of* A *in* $\mathscr{A}$.
(3) *If* $x\in\mathscr{A}\setminus(A\cup A^*)$, *then*

$$\alpha(\gamma)\subset A^*,\quad \omega(\gamma)\subset A.$$

Proof. (1) Suppose $\gamma(\tau)\notin A^*$ for some $\tau\in\mathbb{R}$. Then by the definition of A^* we have $\gamma(\tau)\in\Omega_{\mathscr{A}}(A)$. Since A attracts $\gamma(\tau)$, we have $\omega(\gamma)\subset\overline{A}^{\mathscr{A}}$. Take a closed neighborhood W of A in $\mathscr{A}$ with $W\subset\Omega_{\mathscr{A}}(A)$. Then by s-compactness of $\mathscr{A}$ we see that W is admissible. Thus by Theorem 4.1(3) one deduces that $\omega(\gamma)\subset A$. But this leads to a contradiction. Hence $\gamma(\mathbb{R})\subset A^*$.

(2) We first show that $\gamma(\mathbb{R})\subset\overline{A}^{\mathscr{A}}$. Suppose the contrary. Then $y=\gamma(\tau)\notin\overline{A}^{\mathscr{A}}$ for some $\tau\in\mathbb{R}$. Take a neighborhood V of $\overline{A}^{\mathscr{A}}$ such that $y\notin V$. By stability of A in $\mathscr{A}$ there is a neighborhood W of A in $\mathscr{A}$ such that $\Phi(\mathbb{R}^+)W\subset V$. On the other hand, since $\alpha(\gamma)\cap\overline{A}^{\mathscr{A}}\neq\emptyset$, one can easily find a $t_0<\tau$ such that $\gamma(t_0)\in W$. It then follows that $\gamma(t)\in V$ for all $t\geq t_0$. In particular, $y=\gamma(\tau)\in V$, a contradiction!

Now observe that $M=\gamma(\mathbb{R})\cup A$ is an invariant set in $\overline{A}^{\mathscr{A}}$. Using a similar argument as in (1) one immediately concludes that $M\subset A$. Thereby $\gamma(\mathbb{R})\subset A$.

(3) As $x\notin A^*$, we have $x\in\Omega_{\mathscr{A}}(A)$. Thus the same argument in (1) applies to show that $\omega(\gamma)\subset A$.

In the following we verify that $\alpha(\gamma)\subset A^*$. First, we infer from (2) that $\alpha(\gamma)\cap\overline{A}^{\mathscr{A}}=\emptyset$. Thus if $\alpha(\gamma)\not\subset A^*$, then there is a point $y\in\alpha(\gamma)$ such that $y\in\Omega_{\mathscr{A}}(A)\setminus\overline{A}^{\mathscr{A}}$. Pick a neighborhood U of y in $\mathscr{A}$ with $U\subset\Omega_{\mathscr{A}}(A)$ and a neighborhood V of A in $\mathscr{A}$ such that $U\cap V=\emptyset$. Then by attraction property of A there exist a neighborhood O of y in $\mathscr{A}$ with $O\subset U$ and a $T>0$ such that

$$\Phi(t)O\subset V,\qquad t>T. \tag{6.11}$$

On the other hand, we infer from the definition of α-limit set that there is a sequence $0<t_n\to\infty$ such that $\gamma(-t_n)\in O$ for all n. Noticing that $y=\Phi(t_n)\gamma(-t_n)$, by (6.11) one finds that $y\in V$, which leads to a contradiction. □

Definition. Let $\mathscr{A}$ be an attractor. An ordered collection $\mathcal{M}=\{M_1,\dots,M_n\}$ of subsets of $\mathscr{A}$ is called a *Morse decomposition* of $\mathscr{A}$, if there is an increasing sequence $\emptyset=A_0\subset A_1\subset\cdots\subset A_n=\mathscr{A}$ of attractors of Φ in $\mathscr{A}$ such that

$$M_k=A_k\cap A^*_{k-1},\qquad 1\leq k\leq n. \tag{6.12}$$

The sets M_k $(1\leq k\leq n)$ in (6.12) will be referred to as *Morse sets* of $\mathscr{A}$.

Theorem 6.3. *Let* $\mathcal{M}=\{M_1,\dots,M_n\}$ *be a Morse decomposition of* $\mathscr{A}$ *with the corresponding attractor sequence* $\emptyset=A_0\subset A_1\subset\cdots\subset A_n=\mathscr{A}$. *Then the following assertions hold.*

(1) *For each* k, (A_{k-1},M_k) *is an attractor-repeller pair in* A_k.
(2) M_k $(1\leq k\leq n)$ *are pair-wise disjoint* s-*compact invariant sets.*

(3) *If γ is a complete trajectory, then either $\gamma(\mathbb{R}) \subset M_k$ for some Morse set M_k, or else there are indices $i < j$ such that $\alpha(\gamma) \subset M_j$ and $\omega(\gamma) \subset M_i$.*

(4) *The attractors A_k's are uniquely determined by the Morse sets. Specifically,*

$$A_k = \bigcup_{1 \le i \le k} W^u(M_i), \qquad 1 \le k \le n,$$

where $W^u(M_i)$ is the unstable manifold of M_i which is defined as

$$W^u(M_i) = \{x \mid \text{there is a trajectory } \gamma : \mathbb{R} \to \mathscr{A} \text{ through } x \text{ with } \alpha(\gamma) \subset M_i\}.$$

Proof. The proof of the theorem is the same as that of [17], Chapter III, Theorem 1.7, and is thus omitted. □

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