ON HODGE STRUCTURES OF QUASITORIC ORBIFOLDS

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Abstract. We give Hodge structures on quasitoric orbifolds. We define orbifold Hodge numbers and show a correspondence of orbifold Hodge numbers for crepant resolutions of quasitoric orbifolds. In short we extend Hodge structures to a non almost complex setting.

1. Introduction

The purpose of this paper is to provide an example of Hodge structures in a non almost complex setting. First we provide a canonical Hodge structure to quasitoric orbifolds. We compute Hodge numbers. Then as an application we define orbifold Hodge numbers and establish a correspondence of these numbers for crepant resolutions.

Physicists believe that orbifold string theory is equivalent to ordinary string theory of certain desingularizations. This belief motivated a body of conjectures, collectively referred to as the orbifold string theory conjecture.

A partial crepant resolution $\rho : Y \to X$ of a singular variety X is a partial resolution of singularities such that $\rho^*(K_X) = K_Y$, i.e., the canonical class of X pulls back to the canonical class of Y under ρ. In this paper we are interested in the orbifold Hodge number conjecture which states that for every bi-degree (p, q), the (p, q)-th orbifold Hodge numbers of a Gorenstein (also known as SL) algebraic orbifold X and its partial crepant resolution Y are equal. Note that when Y is smooth, the (p, q)-th orbifold Hodge number of Y is the same as the ordinary (p, q)-th Hodge number of Y.

In the case of complex algebraic orbifolds with certain stratifications and having Gorenstein toroidal singularities, Batyrev and Dais in [2] proved the correspondence for string theoretic Hodge numbers. In case of algebraic quasitoric orbifolds (our case meeting the above criteria) string theoretic Hodge numbers are same as orbifold Hodge numbers and so the above correspondence follows.

Poddar and Lupercio in [11] proved *Mckay-Ruan* correspondence for complete K-equivalent algebraic Gorenstein orbifolds. This means if the canonical bundles of orbifolds X and Y are related in a prescribed manner resulting in a slightly more general condition than crepancy, their orbifold Hodge numbers and orbifold Hodge structures coincide. Yasuda in [16] and [17] has proved independently the orbifold Hodge structure and orbifold Hodge number correspondence for both Gorenstein and non-Gorenstein K-equivalent complete complex algebraic orbifolds.

We generalize the orbifold Hodge number correspondence to a non-algebraic, non-analytic setting. In a nutshell the main contributions of our paper are

(1) Providing canonical Hodge structures to quasitoric orbifolds in Theorems 4.4 and 4.7.
(2) As an application to (1) we prove the orbifold Hodge number correspondence of a crepant resolution of quasitoric quasi-SL orbifolds in Theorem 5.3 and Corollary 5.4.

A quasitoric orbifold $\mathbf{X}$ of dimension $2n$ is a compact differentiable orbifold equipped with a smooth action of the n-dimensional compact torus such that the orbit space is diffeomorphic as manifold with corners to an n-dimensional simple polytope P. An n-dimensional polytope is called simple if every vertex is the intersection of exactly n codimension one faces. Every facet (codimension one face) F of P corresponds to a torus invariant quasitoric suborbifold $\mathbf{X}(F)$ of real codimension 2, which is stabilized by a circle subgroup of the form $\{(e^{2\pi i a_1 t}, \ldots, e^{2\pi i a_n t}) : t \in \mathbb{R}\}$. The vector $(a_1, \ldots, a_n)$ is a primitive integral vector. It is uniquely determined if an orientation or an almost complex structure on the normal bundle of $\mathbf{X}(F)$, compatible with the isotropy circle action, is specified. We call $(a_1, \ldots, a_n)$ the characteristic vector and $\mathbf{X}(F)$ the characteristic suborbifold associated to F. More generally, a codimension k face is the intersection of k codimension one faces and its characteristic set consists of the characteristic vectors of these codimension one faces. The characteristic set of every face is linearly independent over $\mathbb{R}$.

Quasitoric manifolds (and orbifolds, although not in full generality,) were introduced in [6]. They got their present name in [3]. They are generalizations of smooth projective toric varieties. They include manifolds such as $CP^2 \sharp CP^2$ which do not admit an almost complex structure. A broader class of quasitoric orbifolds were defined in [15]. In that paper and the subsequent papers [8] and [9], questions relating to homology, cohomology, almost complex structures and equivariant blowdown maps were addressed. McKay correspondence of Betti numbers of Chen-Ruan cohomology was also established in [7].

Quasitoric orbifolds includes many orbifolds which are neither complex nor almost complex. In this paper we address the question of Hodge structures of these orbifolds. Since $CP^2 \sharp CP^2$ is a quasitoric orbifold it gets a Hodge structure in spite of being non complex. This opens the possibility of Dolbeault-like theory for non-complex, but almost complex quasitorics. The idea of the

proof is to relate the cohomology of a quasitoric orbifold with the cohomology of a projective toric orbifold which is a Kahler orbifold. The complex De Rham cohomology of the two spaces are shown to be isomorphic as a graded vector space and so the Hodge structure of one can be pulled back to the other.

Since we have Hodge structures we can define orbifold Hodge numbers and get a correspondence for these numbers under crepant blowdown by imitating Batyrev-Dias correspondence.

A general quasitoric orbifold differs combinatorially from a projective toric orbifold in the following manner. In the case of a projective toric orbifold, the characteristic vector of a codimension one face of the orbit polytope is normal to that face. This enables the characteristic vectors to generate cones that fit together to form a toric fan. In a general quasitoric orbifold the normality condition is relaxed to linear independence of characteristic sets of faces.

The paper is organized in the following manner. In Section 2 we give a combinatorial construction of quasitoric orbifolds. In Section 3 we discuss Betti numbers of quasitoric orbifolds. In Section 4 we define Hodge structures and provide a canonical Hodge structure by showing cohomological vector space isomorphism between a quasitoric and a projective toric. In Section 5 as an application, we define and show orbifold Hodge number correspondence.

2. Quasitoric orbifolds

In this section we describe the combinatorial construction of quasitoric orbifolds. Notations established in this section will be used later. This material can also be found in [9].

Take a copy N of $\mathbb{Z}^n$ and form a torus $T_N := (N \otimes_{\mathbb{Z}} \mathbb{R})/N \cong \mathbb{R}^n/N$.

Take a submodule M of N of rank m and construct the torus $T_M := (M \otimes_{\mathbb{Z}} \mathbb{R})/M$ of dimension m. Define the map $\zeta_M : T_M \to T_N$ the obvious map generated by the inclusion map $M \to N$.

Definition 2.1. We define the image of T_M under the map ζ_M as $T(M)$. If M is a sub-module of rank 1 and λ is the generator, then we call the image $T(\lambda)$.

Definition 2.2. A polytope P is a subset of $\mathbb{R}^n$ which is diffeomorphic as manifolds with corners to a convex hull C of a finite number of points in $\mathbb{R}^n$. The faces of P are images of faces of C.

Definition 2.3. A simple polytope is a polytope where each vertex is an intersection of n co-dimension one faces which are in general position.

Definition 2.4. Codimension one faces of a polytope P are called facets. In a simple polytope every k dimensional face is an intersection of $n - k$ facets. We call $\mathcal{F} = \{F_1, F_2, \ldots, F_M\}$ the set of facets of the simple polytope P.

Definition 2.5. We define a map $\Lambda : \mathcal{F} \to \mathbb{Z}^n$ where F_i is mapped to $\Lambda(F_i)$ and if $F_{i_1} \cdots F_{i_k}$ intersect to form a face of the polytope P, then the corresponding $\Lambda(F_{i_1}) \cdots \Lambda(F_{i_k})$ are linearly independent. From now onwards we call $\Lambda(F_i)$ as λ_i and call it a characteristic vector and Λ the characteristic function.

Remark 2.1. In this article we consider only primitive characteristic vectors and call the corresponding quasitoric orbifolds as primitive quasitoric orbifolds. The codimension of the singular locus of these orbifolds is at least 4.

Definition 2.6. For a face F define $\mathcal{I}(F) = \{i : F \subset F_i, F_i \in \mathcal{F}\}$. The set $\Lambda_F = \{\lambda_i : i \in \mathcal{I}(F)\}$ is called the characteristic set of F. We call $N(F)$ be the sub module generated Λ_F. If $\mathcal{I}(F)$ is empty, $N(F) = 0$.

For any point p in the polytope we denote $F(p)$ the face whose relative interior contains p. We define an equivalence relation in $P \times T_N$ where $(p, t_1) \sim (q, t_2)$ if $p = q$ and $t_2{}^{-1}t_1 \in T(N(F(p)))$ where $N(F(p))$ is the sub module of N generated by integral linear combinations of vectors of $\Lambda_{F(p)}$. The quotient space $X = P \times T_N / \sim$ has a structure of an $2n$ dimensional orbifold and are called quasitoric orbifolds.

The pair (P, Λ) is a model for the above space. If vectors comprising Λ_F are unimodular for all faces F we get a quasitoric manifold. The T_N action on $P \times T_N$ induces a torus action on the quotient space X, of the equivalence relation, and quotient of this action is the polytope P. Let us denote the quotient map by $\pi : X \to P$. $\pi^{-1}(w)$ for a vertex w of P is a fixed point of the above action and we will denote it by w without confusion.

2.1. Orbifold structure

For every vertex w in P consider open set U_w of P the complement of all faces not containing the vertex w. We define

$$(2.1) \qquad X_w = \pi^{-1}(U_w) = U_w \times T_N / \sim .$$

For any face F containing the vertex w there is a natural inclusion of $N(F)$ in $N(w)$ and $T_{N(F)}$ in $T_{N(w)}$. We define another equivalence relation $\sim_w$ on $U_w \times T_{N(w)}$ as follows.

For $p \in U_w$, let F be the face which contains p in its relative interior, by definition F contains w. We define the relation as $(p, t_1) \sim_w (q, t_2)$, if $p = q$ and $t_2{}^{-1}t_1 \in T_{N(F)}$. We define

$$(2.2) \qquad \tilde{X}_w = U_w \times T_{N(w)} / \sim_w .$$

The above space is equivariantly diffeomorphic to an open set in C^n with the standard torus action on C^n and $T_{N(w)}$ action on $\tilde{X}_w$. The diffeomorphism will be clear from the subsequent discussion. The map $\zeta_{N(w)} : T_{N(w)} \to T_N$ induces a map from $\zeta_w : \tilde{X}_w \to X_w$ in the following way

$$(2.3) \qquad \zeta_w((p, t) \sim_w) = (p, \zeta_{N(w)}(t)) \sim .$$

The kernel of the map $\zeta_{N(w)}$ is $G_w = N/N(w)$ a subgroup of $T_{N(w)}$ and has a smooth action on $\tilde{X}_w$ and the quotient of this action is X_w. This action is not free and so X_w is an orbifold and the uniformizing chart of X_w is $(\tilde{X}_w, G_w, \zeta_w)$.

We define a homeomorphism $\phi(w) : \tilde{X}_w \to \mathbb{R}^{2n}$ as follows. Assume without loss of generality $F_1, F_2, \ldots, F_n$ are the facets containing w and $p_i(w) = 0$ is

the facet F_i and in U_w $p_i's$ have non-negative values with positive in interiors of U_w. Let Λ_w be the corresponding set of characteristic vectors represented as follows

$$(2.4) \qquad \Lambda_w = [\lambda_1 \cdots \lambda_n].$$

If $q(w)$ be the representation of the angular coordinates of T_N in the basis with respect to $\lambda_1 \cdots \lambda_n$ of $N \otimes_\mathbb{Z} \mathbb{R}$. Then the standard coordinates q are related in the following manner to $q(w)$

$$(2.5) \qquad q = \Lambda_w q(w).$$

The homeomorphism $\phi(w) : \tilde{X}_w \to \mathbb{R}^{2n}$ is

$$(2.6) \quad x_i = x_i(w) := \sqrt{p_i(w)} \cos(2\pi q_i(w)), \ y_i = y_i(w) := \sqrt{p_i(w)} \sin(2\pi q_i(w))$$

for $i = 1, \ldots, n$. We write

$$(2.7) \qquad z_i = x_i + \sqrt{-1}y_i, \quad \text{and} \quad z_i(w) = x_i(w) + \sqrt{-1}y_i(w).$$

Now consider the action of $G_w = N/N(w)$ on $\tilde{X}_w$. An element g of G_w is represented by a vector $\sum_{i=1}^n a_i\lambda_i$ in N where each $a_i \in \mathbb{Q}$. The action of g transforms the coordinates $q_i(w)$ to $q_i(w) + a_i$. Therefore

$$(2.8) \qquad g \cdot (z_1, \ldots, z_n) = (e^{2\pi\sqrt{-1}a_1}z_1, \ldots, e^{2\pi\sqrt{-1}a_n}z_n).$$

We define

$$(2.9) \qquad G_F := ((N(F) \otimes_\mathbb{Z} \mathbb{Q}) \cap N)/N(F).$$

We denote the space X with the above orbifold structure by $\mathbf{X}$.

2.2. Invariant suborbifolds

The T_N invariant subset $\pi^{-1}(F)$ where F is a face of P is a quasitoric orbifold. The face F acts as the polytope of $\mathbf{X(F)}$ and the characteristic vectors are obtained by projecting characteristic vectors of $\mathbf{X}$ to $N/\tilde{N}(F)$ where $\tilde{N(F)} = N(F) \otimes_\mathbb{Z} \mathbb{Q} \cap N$. With this structure $\mathbf{X(F)}$ is a suborbifold of $\mathbf{X}$. The suborbifolds corresponding to the facets are called characteristic suborbifolds. We denote the interior of a face by F°. The interior of a vertex w° is w.

2.3. Orientation

Quasitoric orbifolds are oriented. For more detailed discussion see Section 2.8 of [9]. A choice of orientation of T_N and a choice of orientation of the polytope P induces an orientation of the quasitoric orbifold $\mathbf{X}$.

2.4. Omniorientation

A choice of orientations of the normal bundles of the orbifolds corresponding to the facets (which we named as characteristic suborbifolds) is termed as fixing an omniorientation. This is equivalent to fixing the sign of the characteristic vector associated to the facet (note: we call co-dimension one faces as facets). A quasitoric orbifold with a fixed omniorientation is called an *omnioriented quasitoric orbifold*. A quasitoric orbifold is *positively omnioriented* if it has an omniorientation such that for every vertex w, Λ_w has a positive determinant. For more detailed discussion see Section 2.9 of [9].

3. Betti numbers of a quasitoric orbifold

Poddar and Sarkar computed the $\mathbb{Q}$ homology and cohomology of quasitoric orbifolds in [15]. In particular the computation of homology in Section 4 of [15] gives a strong connection between the combinatorics of the polytope P and the Betti numbers. We discuss the connection in following proposition.

Proposition 3.1. *Quasitoric orbifolds with combinatorially equivalent polytopes have same Betti numbers.*

Proof. A brief discussion of the homology computation in [15] is required to establish the above proposition. The computation depends on defining a continuous height function on the polytope P with following properties.

(1) Distinguishes vertices.
(2) Strictly increases or decreases on edges.
(3) Each face has a unique maximum and minimum vertex.
(4) The maximum vertex is the unique vertex of the face where all the edges of the face meeting the vertex has a maximum on the vertex.
(5) The minimum vertex is the unique vertex of the face where all the edges of the face meeting the vertex has a minimum on the vertex.

A vertex distinguishing linear functional of $\mathbb{R}^n$ does the job. Here we assume P is embedded in $\mathbb{R}^n$. Once we have such a function we orient the edges of the polytope in increasing direction of the height function and arrange the *vertices in increasing order of height*. We define index i_w of a vertex w as the number of incoming edges. The smallest face containing these incoming edges is the largest face F_w which has w as the maximum vertex. Now start attaching $2i_w$ q-cells following the increasing order of vertices. The q-cell covers the entire inverse image of F_w in the orbifold. For definition and description of q-cells and the attaching maps we ask the *reader to consult* [15].

Now each face has a unique maximum vertex w and interior of the face will be contained in F_w by points 3 and 4 above. So each face gets covered and each point in the orbifold is in the interior of exactly one q-cell. Considering the polytope as a face there will be exactly one 0 q-cell and one $2n$ q-cell. Thus we get a q cellular decomposition of the quasitoric orbifold.

Now it is shown in [15] that the $2k$ Betti numbers depends on the number of vertices with index k while the odd Betti numbers are zero. Now if we have two quasitoric orbifolds with two combinatorially equivalent polytopes (which means they are diffeomorphic as manifold with corners) the height function of one composed with the diffeomorphism gives a height function of the other with identical vertex indices. Thus their Betti numbers will be same by what is done in [15]. $\qquad\square$

Corollary 3.2. *The dimension of each degree of singular cohomology with coefficients in $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ of quasitoric orbifolds $\mathbf{X}$ and $\mathbf{X}'$ with combinatorially equivalent polytopes are same.*

Proof. By Universal Coefficient Theorem, Corollary 3.2 follows. $\qquad\square$

Corollary 3.3. *The dimension of each degree of singular cohomology with coefficients in $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ of a quasitoric orbifold $\mathbf{X}$ is identical with a projective toric orbifold X'.*

Proof. Take a quasitoric orbifold $\mathbf{X}$. A slight perturbation makes the polytope P associated with the orbifold into a rational polytope (see Section 5.1.3 in [3]) without changing its combinatorial class, and with suitable dilations makes it into an integral polytope P' which is combinatorially equivalent to P. Now from P' taking normal fan we get a projective toric orbifold X' (the analytic structure determines the orbifold structure so we do not use the bold notation) with polytope P'. Since polytope P and polytope P' are combinatorially equivalent by (3.2) the above holds. $\qquad\square$

Corollary 3.4. *Each degree of the cohomologies of the two spaces $\mathbf{X}$ and X' are isomorphic.*

Proof. Since the combinatorial equivalence map defines a map between the q-cells and since the homology and cohomology depends on q-cells of a given dimension we get vector space isomorphisms J_k of kth degree cohomology of the two spaces. $\qquad\square$

4. Hodge structure

Definition 4.1. A pure Hodge structure of weight n consists of an Abelian group H_K and a decomposition of its complexification into complex subspaces $H^{p,q}$ where $p + q = n$ with the property conjugate of $H^{p,q}$ is $H^{q,p}$.

$$(4.1) \qquad H_C = H_K \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$$

and

$$(4.2) \qquad \overline{H^{p,q}} = H^{q,p}.$$

Definition 4.2. By a Hodge structure on a compact space we mean the singular cohomology group of degree k has a pure Hodge structure of weight k for all k.

Proposition 4.1. *Kahler compact orbifolds have a canonical Hodge structure.*

Proof. By Baily's Hodge decomposition see [1], Proposition 4.1 follows. $\quad\square$

Proposition 4.2. *Projective toric orbifolds coming from integral simple polytopes are Kahler.*

Proof. By Theorems 8.1, 9.1 and 9.2 in [10], Proposition 4.2 follows. $\quad\square$

Corollary 4.3. *Projective toric orbifolds coming from integral simple polytopes have a canonical Hodge structure.*

Definition 4.3. Let $H^{p,q}$ be the (p,q) Hodge component of the canonical Hodge structure on Projective toric orbifolds $X^{'}$ coming from integral simple polytopes. We define

$$(4.3) \qquad\qquad H^{p,q}(X^{'}) = H^{p,q}$$

and

$$(4.4) \qquad\qquad h^{p,q}(X^{'}) = \dim(H^{p,q}(X^{'})).$$

Let $\mathbf{X}$ be a quasitoric orbifold and $X^{'}$ be the projective toric orbifold whose integral simple polytope $P^{'}$ is combinatorially equivalent to the polytope P of $\mathbf{X}$. We assign

$$(4.5) \qquad\qquad H^{p,q}(\mathbf{X}) = J_k(H^{p,q}(X^{'})),$$

where $p + q = k$ and J_k is the isomorphisms of the degree k cohomologies defined in (3.4).

Theorem 4.4. *The above assignment defines a Hodge structure on $\mathbf{X}$ depending on J_k. For independence of J_k see Theorem 4.7.*

Proof. By Corollary 3.3, Theorem 4.4 follows. $\quad\square$

Theorem 4.5. *The Hodge numbers $h^{p,q}$ does not depend on $X^{'}$.*

Proof. To show the above we must understand the E-polynomial. Let Y be an algebraic variety over $\mathbb{C}$ which is not necessarily compact or smooth. Denote by $h^{p,q}(H_c^k(Y))$ the dimension of the (p,q) Hodge component of the k-th cohomology with compact supports. This is a generalization of the Hodge structures discussed on the above class of compact projective toric orbifolds and are called mixed Hodge structures. For more detailed discussion we ask the reader to consult [12].

We define

$$(4.6) \qquad\qquad e^{p,q}(Y) = \Sigma_{k \geq 0}(-1)^k h^{p,q}(H_c^k(Y)).$$

The polynomial

$$(4.7) \qquad E(Y; u, v) := \Sigma_{p,q} e^{p,q}(Y) u^p v^q$$

is called E-polynomial of Y. When we have a pure Hodge structure like the above class of compact projective toric orbifold $X^{'}$,

$$(4.8) \qquad e^{p,q}(X^{'}) = (-1)^{p+q} h^{p,q}(X^{'}).$$

Now if we have a stratification of an algebraic variety Y by disjoint locally closed sub-varieties Y_i (i.e., $Y_i \subset Y$ and $Y = \cup_i Y_i$) by Proposition 3.4 of [2]

$$(4.9) \qquad E(Y; u, v) = \Sigma_i E(Y_i; u, v).$$

Now in a projective toric orbifold coming from a integral simple polytope as in our case we have a stratification by algebraic tori corresponding to the interior of each face. Let $X^{'}$ be the concerned orbifold and F_i be a k dimensional face of the corresponding polytope then $\pi^{-1}(F_i^{\circ})$ is a k dimensional algebraic tori which we denote $X_i^{'}$. So by (4.9) we have

$$(4.10) \qquad E(X^{'}; u, v) = \Sigma_i E(X_i^{'}; u, v).$$

Here i runs over all the faces. Now if we have two projective toric orbifolds $X^{'}$ and $X^{''}$ both having combiantorially equivalent polytopes with that of $\mathbf{X}$ by (4.10) we claim they have the same E-polynomial. This is because since they have combinatorially equivalent polytopes, number of faces of a given dimension will be same for each polytope. So the sum on the right hand side of (4.10) can be partitioned into E-polynomial of k dimensional algebraic tori with a multiplicity of number of faces of dimension k, where k runs from 0 to dimension of the polytopes. Since same dimensional algebraic tori have same E-polynomial the above claim holds.

Thus the Hodge numbers of the two projective toric orbifolds will be same by (4.8). So the theorem holds. $\qquad \square$

Theorem 4.6. *The Hodge numbers of a quasitoric orbifold are as follows* $h^{p,q}(\mathbf{X}) = 0$ *if* $p \neq q$ *and* $h^{p,p}(\mathbf{X}) = \dim(H^{2p}(\mathbf{X}, \mathbb{C}))$.

Proof. We show this for projective toric orbifolds coming from integral simple polytope. We know that the E-polynomial of a k-dimensional algebraic torus is $(uv - 1)^k$. Since by (4.10) the E-polynomial of the projective toric orbifold decomposes into sum of E-polynomial of algebraic tori and since E-polynomial of the algebraic tori have only terms of the form $(uv)^l$, implies that coefficient of $u^p v^q$ is zero if $p \neq$ q in the E-polynomial of the projective toric orbifold. Since these projective toric orbifolds have a pure Hodge structure the claim of the theorem is true. $\qquad \square$

Theorem 4.7. *The above Hodge structure does not depend on J_k and is canonical.*

Proof. Since there is only one non-zero Hodge number for a given degree of cohomology, we can define the same Hodge decomposition on J_k. $\qquad \square$

Remark 4.8. The above proof of Hodge numbers of quasitoric orbifolds is also a proof for Hodge numbers of projective toric orbifolds coming from Deligne's mixed hodge structures. We have not seen this proof in literature before.

4.1. Example

We compute the Hodge structure for $CP^2 \sharp CP^2$ which does not have an almost complex structure. We take a projective toric orbifold $X^{'}$ with a combinatorially equivalent polytope $P^{'}$. Since the polytope of P is a four sided polygon (see Example 1.19 [6]) it will have four vertices, four edges and one 2-face.

$$(4.11) \qquad E(X^{'}; u, v) = (uv - 1)^2 + 4(uv - 1) + 4.$$

$$(4.12) \qquad E(X^{'}; u, v) = u^2 v^2 + 2uv + 1.$$

This tallies with the cohomology of $CP^2 \sharp CP^2$ and so we have the decomposition $h^{2,2} = 1$, $h^{1,1} = 2$ and $h^{0,0} = 1$.

5. An application–orbifold Hodge numbers and a correspondence

5.1. Orbifold Hodge numbers

Orbifold Hodge numbers for closed global quotient orbifold was defined in [18] and [2] and for Kahler orbifolds in [14]. They are the dimensions of the Dolbeaut orbifold cohomology (see [14, Section 2.2]). They depend on the twisted sectors of the orbifold. The twisted sector for toric variety was computed in [13]. The determination of twisted sectors of quasitoric orbifolds are similar in essence. Let $\mathbf{X}$ be an omnioriented quasitoric orbifold (i.e., the signs of characteristic vectors are fixed). Consider an element g belonging to the group G_F defined in equation (2.9). Then g may be represented by the vector $\sum_{j \in \mathcal{I}(F)} a_j \lambda_j$ where a_j is restricted to $[0, 1) \cap \mathbb{Q}$ and λ_j belongs to the characteristic set of F. We define the degree shifting number or *age* as

$$(5.1) \qquad \iota(g) = \sum a_j.$$

For faces F and H of P we write $F \leq H$ if F is a sub-face of H, and $F < H$ if it is a proper sub-face. If $F \leq H$ we have a natural inclusion of G_H into G_F induced by the inclusion of $N(H)$ into $N(F)$. Therefore we may regard G_H as a subgroup of G_F. Define the set

$$(5.2) \qquad G_F^{\circ} = G_F - \bigcup_{F < H} G_H.$$

Note that $G_F^{\circ} = \{\sum_{j \in \mathcal{I}(F)} a_j \lambda_j \mid 0 < a_j < 1\} \cap N$, and $G_P^{\circ} = G_P = \{0\}$.

Definition 5.1. We define the *orbifold Dolbeault cohomology* groups of an omnioriented quasitoric orbifold $\mathbf{X}$ to be

$$H_{orb}^{p,q}(\mathbf{X}) = \bigoplus_{F \leq P} \bigoplus_{g \in G_F^{\circ}} H^{p-\iota(g), q-\iota(g)}(X(F)).$$

Here $H^{p-\iota(g),q-\iota(g)}(X(F))$ refers to the components of the Hodge structures defined above ,when $X(F)$ is considered as a quasitoric orbifold $\mathbf{X}(F)$. The pairs $(X(F),g)$ where $F < P$ and $g \in G_F^\circ$ are called twisted sectors of $\mathbf{X}$. The pair $(X(P),1)$, i.e., the underlying space X, is called the untwisted sector.

Definition 5.2. We define *orbifold Hodge numbers* as

$$h_{orb}^{p,q}(\mathbf{X}) = \dim(H_{orb}^{p,q}(\mathbf{X})).$$

Now we introduce some notation. Consider a co-dimension k face $F = F_1 \cap \cdots \cap F_k$ of P where $k \geq 1$. Define a k-dimensional cone C_F in $N \otimes \mathbb{R}$ as follows,

$$(5.3) \qquad C_F = \left\{ \sum_{j=1}^k a_j \lambda_j : a_j \geq 0 \right\}.$$

The group G_F can be identified with the subset Box_F of C_F, where

$$(5.4) \qquad Box_F := \left\{ \sum_{j=1}^k a_j \lambda_j : 0 \leq a_j < 1 \right\} \cap N.$$

Consequently the set G_F° is identified with the subset

$$(5.5) \qquad Box_F^\circ := \left\{ \sum_{j=1}^k a_j \lambda_j : 0 < a_j < 1 \right\} \cap N$$

of the interior of C_F. We define $Box_P = Box_P^\circ = \{0\}$.

Suppose $w = F_1 \cap \cdots \cap F_n$ is a vertex of P. Then $Box_w = \bigsqcup_{w \leq F} Box_F^\circ$. This implies

$$(5.6) \qquad G_w = \bigsqcup_{w \in F} G_F^\circ.$$

An almost complex orbifold is SL if the linearization of each g is in $SL(n,\mathbb{C})$. This is equivalent to $\iota(g)$ being integral for every twisted sector. Therefore, to suit our purposes, we make the following definition.

Definition 5.3. An omnioriented quasitoric orbifold is said to be quasi-SL if the age of every twisted sector is an integer.

5.2. Blowdowns

In order to get a blow up along a face we replace the face by a facet with a new characteristic vector. Suppose F is a face of P. We choose a hyperplane $H = \{\widehat{p}_0 = 0\}$ such that $\widehat{p}_0$ is negative on F and $\widehat{P} := \{\widehat{p}_0 > 0\} \cap P$ is a simple polytope having one more facet than P. Suppose $F_1, \ldots, F_m$ are the facets of P. Denote the facets $F_i \cap \widehat{P}$ by F_i without confusion. Denote the extra facet $H \cap P$ by F_0.

Without loss of generality let $F = \bigcap_{j=1}^{k} F_j$. Suppose there exists a primitive vector $\lambda_0 \in N$ such that

$$(5.7) \qquad\qquad \lambda_0 = \sum_{j=1}^{k} b_j \lambda_j, \ b_j > 0 \,\forall\, j.$$

Then the assignment $F_0 \mapsto \lambda_0$ extends the characteristic function of P to a characteristic function $\widehat{\Lambda}$ on $\widehat{P}$. Denote the omnioriented quasitoric orbifold derived from the model $(\widehat{P}, \widehat{\Lambda})$ by $\mathbf{Y}$.

Definition 5.4. We define blowdown a map $\mathbf{Y} \mapsto \mathbf{X}$ which is inverse of a blow-up. Such maps have been constructed in [9].

Lemma 5.1 ([9, Lemma 4.2]). *If $\mathbf{X}$ is positively omnioriented, then so is a blowup $\mathbf{Y}$.*

Definition 5.5. A blowdown or blow up is called crepant if $\sum b_j = 1$.

Lemma 5.2 ([9, Lemma 8.2]). *The crepant blowup of a quasi-SL quasitoric orbifold is quasi-SL.*

5.3. Correspondence of orbifold Hodge numbers

The statement of the theorem we are going to prove is as follows

Theorem 5.3. *For crepant blowdowns (or blowups) orbifold Hodge numbers of quasi-SL quasitoric orbifolds do not change.*

Corollary 5.4. *For crepant resolution orbifold Hodge numbers of quasi-SL quasitoric orbifolds do not change.*

We admit the proof is similar to the proof of Mckay Correspondence of Betti- numbers of Chen-Ruan cohomology in the author's previous paper [7] and motivated by Strong Mckay correspondence proof [2], but still we give a detailed argument for the convenience of the reader.

5.4. Singularity and lattice polyhedron

Following the discussion in Section 5.1, a singularity of a face F is defined by a cone C_F formed by positive linear combinations of vectors in its characteristic set $\lambda_1, \ldots, \lambda_d$ where d is the co-dimension of the face in the polytope. The elements of the local group G_F are of the form $g = diag(e^{2\pi\sqrt{-1}\alpha_1}, \ldots, e^{2\pi\sqrt{-1}\alpha_d})$, where $\sum_{i=1}^{d} \alpha_i \lambda_i \in N$, and $0 \leq \alpha_i < 1$. Recall that the age

$$(5.8) \qquad\qquad \iota(g) = \alpha_1 + \cdots + \alpha_d$$

is integral in quasi-SL case by Definition 5.3.

The singularity along the normal bundle of the sub-orbifold corresponding to interior of F is of the form $\mathbb{C}^d/G_F$. These singularities are same as Gorenstein *toric* quotient singularities in complex algebraic geometry. This means they are toric (coming from a cone) *SL orbifold singularity* (*SL* means linearization of

a group element is SL,which in our case implies $\iota(g)$ is integral). Now let N_w be the lattice formed by $\{\lambda_1, \ldots, \lambda_n\}$, the characteristic vectors at a vertex w contained in the face F. Let m_w be the element in the dual lattice of N_w such that its evaluation on each λ_i is one. Now from Lemma 9.2 of [5] we know that the cone C_w contains an integral basis, say $e_1, \ldots, e_n$. Suppose $e_i = \sum a_{ij}\lambda_j$. By (5.4) e_i corresponds to an element of G_w, and since the singularity is qausi-SL, $\sum a_{ij}$ is integral. Hence m_w evaluated on each e_j is integral. So m_w an element of the dual of the integral lattice N.

Consider the $(n-1)$-dimensional lattice polyhedron Δ_w defined as $\{x \in C_w \mid \langle x, m_w \rangle = 1\}$. Note that $\Delta_w = \{\sum_{i=1}^{n} a_i\lambda_i \mid a_i \geq 0, \ \sum a_i = 1\}$. For any face F containing w we define $\Delta_F = \Delta_w \cap C_F$. If $\{\lambda_i, \ldots, \lambda_d\}$ denote the characteristic set of F, then $\Delta_F = \{\sum_{i=1}^{d} a_i\lambda_i \mid a_i \geq 0, \ \sum a_i = 1\}$. Hence Δ_F is independent of the choice of w.

Remark 5.5. An element $g \in G$ of an SL orbifold singularity can be diagonalized to the form g$= diag(e^{2\pi\sqrt{-1}\alpha_1}, \ldots, e^{2\pi\sqrt{-1}\alpha_d})$, where $0 \leq \alpha_i < 1$ and $\iota(g) = \alpha_1 + \cdots + \alpha_d$ is integral.

We make some definitions following [2].

Definition 5.6. Let G be a finite subgroup of $SL(d, \mathbb{C})$. Denote by $\psi_i(G)$ the number of the conjugacy classes of G having $\iota(g) = i$. Define

$$(5.9) \qquad W(G; uv) = \psi_0(G) + \psi_1(G)uv + \cdots + \psi_{d-1}(G)(uv)^{d-1}.$$

Definition 5.7. We define $height(g) = \mathrm{rank}(g - I)$.

Definition 5.8. Let G be a finite subgroup of $SL(d, \mathbb{C})$. Denote by $\tilde{\psi}_i(G)$ the number of the conjugacy classes of G having the $height(g) = d$ and $\iota(g) = i$. Define

$$(5.10) \qquad \widetilde{W}(G; uv) = \tilde{\psi}_0(G) + \tilde{\psi}_1(G)uv + \cdots + \tilde{\psi}_{d-1}(G)(uv)^{d-1}.$$

Definition 5.9. For a lattice polyhedron Δ_F defining a SL singularity $\mathbb{C}^d/G_F$, we define the following:

$$(5.11) \qquad\qquad W(\Delta_F; uv) = W(G_F; uv).$$

$$(5.12) \qquad\qquad \psi_i(\Delta_F) = \psi_i(G_F).$$

$$(5.13) \qquad\qquad \widetilde{W}(\Delta_F; uv) = \widetilde{W}(G_F; uv).$$

$$(5.14) \qquad\qquad \tilde{\psi}_i(\Delta_F) = \tilde{\psi}_i(G_F).$$

5.5. *E*-polynomial for quastiotoric orbifold

Definition 5.10. We define the E-polynomial of a quasitoric orbifold $\mathbf{X}$ as follows

$$(5.15) \qquad E_{quas}(\mathbf{X} : u, v) = \Sigma_{p,q}(-1)^{p+q} h^{p,q}(\mathbf{X}) u^p v^q.$$

If X_i is the stratification of the quasitoric orbifold by inverse image of the quotient map on interior of faces F_i. *Here i runs over all the faces.*

Theorem 5.6.

$$(5.16) \qquad E_{quas}(\mathbf{X}:u,v) = \Sigma_i E(X_i:u,v).$$

Proof. Let $X^{'}$ be a projective toric orbifold whose Hodge structure has been pulled backed to $\mathbf{X}$. The by [2, Proposition 3.4] we have

$$(5.17) \qquad E_{quas}(\mathbf{X}:u,v) = E(X^{'}:u,v) = \Sigma_i E(X_i^{'}:u,v).$$

Where is X'_i is stratification by inverse images of interiors of faces of the polytope of $X^{'}$. Since the two orbifolds have combinatorially equivalent polytopes number of faces of a given dimension is same. And since the stratas are algebraic tori of dimension equal to its corresponding face, we can replace X'_i by the corresponding X_i in the right hand most sum. The identification of X'_i with X_i is by the combinatorial equivalence map. $\square$

Definition 5.11. We define

$$(5.18) \qquad E_{orb}(\mathbf{X}:u,v) = \Sigma_{p,q}(-1)^{p+q}h_{orb}^{p,q}(\mathbf{X})u^p v^q.$$

From the above discussions and since each G_F is Abelian, it is easy to prove

$$(5.19) \qquad E_{orb}(\mathbf{X}:u,v) = \Sigma_i E_{quas}(\overline{X_i}:u,v)\widetilde{W}(\Delta_{F_i},uv).$$

The following can also be seen from what has been discussed in the previous subsection

$$(5.20) \qquad W(\Delta_{F_i},uv) = \Sigma_{X_j \geq X_i}\widetilde{W}(\Delta_{F_j},uv),$$

$$(5.21) \qquad E_{orb}(\mathbf{X}:u,v) = \Sigma_i E(X_i,u,v)W(\Delta_{F_i},uv),$$

where $X_j \geq X_i$ means $X_i \subset \overline{X_j}$ and $\mathbf{X}$ is a quasi-SL quasitoric orbifold.

We generalize E_{st} defined in [2, 6.7] to quasitorics as it has similar stratification in to X_i's

$$(5.22) \qquad E_{st}(\mathbf{X}:u,v) = \Sigma_i E(X_i,u,v)W(\Delta_{F_i},uv).$$

Comparing our E_{orb} with their E_{st} we have

$$(5.23) \qquad E_{st}(\mathbf{X}:u,v) = E_{orb}(\mathbf{X}:u,v).$$

5.6. Proof of the main theorem

We state the theorem again for the reader's convenience.

Theorem 5.7. *For crepant blowdowns (or blowups) orbifold Hodge numbers of quasi-SL quasitoric orbifold do not change.*

Proof. Let $\rho : \hat{\mathbf{X}} \to \mathbf{X}$ be a crepant blowdown of omnioriented quasi-SL quasitoric orbifolds. We set $\hat{X}_i := \rho^{-1}(X_i)$. Then $\hat{X}_i$ has a natural stratification it is enough to prove

$$(5.24) \qquad\qquad E_{st}(\hat{\mathbf{X}}) = E_{st}(\mathbf{X}).$$

But since quasi-SL quasitoric orbifold have Gorenstein torodial singularity defined in [2] and a blow up effects only singularity cone of the face which is blowed up and neighboring cones, where things are toric and since no global patching is required, the proof of Batyrev-Dias can be imitated here (see [2, Theorem 6.2]). $\qquad\square$

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