KYUNGPOOK Math. J. 57(2017), 163-185
https://doi.org/10.5666/KMJ.2017.57.1.163
pISSN 1225-6951 eISSN 0454-8124

Separating a Chart

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Abstract. In this paper, we shall show a condition for that a chart is C-move equivalent to the product of two charts, the union of two charts Γ^* and Γ^{**} which are contained in disks D^* and D^{**} with $D^* \cap D^{**} = \emptyset$.

1. Introduction

Charts are oriented labeled graphs in a disk which represent surface braids (see [1], [7], and see Section 2 for the precise definition of charts, see [7, Chapter 14] for the definition of surface braids). In a chart there are three kinds of vertices; white vertices, crossings and black vertices. A C-move is a local modification of charts in a disk. The closures of surface braids are embedded closed oriented surfaces in 4-space $\mathbb{R}^4$ (see [7, Chapter 23] for the definition of the closures of surface braids). A C-move between two charts induces an ambient isotopy between the closures of the corresponding two surface braids. In this paper, we shall show a condition for that a chart is C-move equivalent to the product of two charts (Theorem 1).

We will work in the PL category or smooth category. All submanifolds are assumed to be locally flat. In [4] and [15], we investigated minimal charts with exactly four white vertices. In [16], we showed that there is no minimal chart with exactly five vertices. Hasegawa proved that there exists a minimal chart with exactly six white vertices which represents the surface braid whose closure is ambient isotopic to a 2-twist spun trefoil [3]. In [8] and [14], we investigated minimal charts with exactly six white vertices. We show that there is no minimal chart with exactly seven vertices ([9], [10], [11], [12], [13]). Thus the next targets are minimal

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* Corresponding Author.
Received March 27, 2016; revised November 8, 2016; accepted November 17, 2016.
2010 Mathematics Subject Classification: 57Q45, 57Q35.
Key words and phrases: 2-knot, chart, white vertex.
The second author was supported by JSPS KAKENHI Grant Number 23540107.

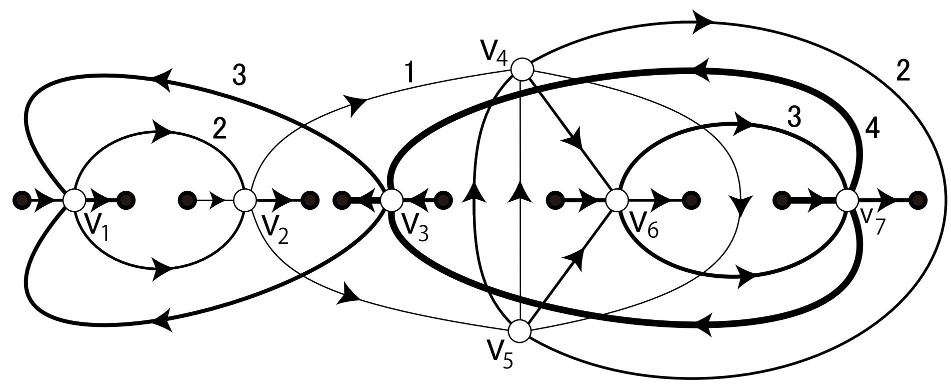

Figure 1: Chart of type $(1;3,2,2)$.

charts with eight or nine white vertices. As an application of Theorem 1, we shall show that there are 12 kinds of types for a minimal chart with eight white vertices (Corollary 2 and Corollary 3), and there are 15 kinds of types for a minimal chart with nine white vertices (Corollary 5).

Let Γ be a chart. For each label m, we denote by Γ_m the 'subgraph' of Γ consisting of all the edges of label m and their vertices.

Two charts are said to be *C-move equivalent* if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

Now we define a type of a chart: Let Γ be a chart, m a label of Γ, and $n_1, n_2, \ldots, n_p$ integers. The chart Γ is said to be of *type* $(m; n_1, n_2, \ldots, n_p)$, or of *type* $(n_1, n_2, \ldots, n_p)$ briefly, if it satisfies the following three conditions:

(i) For each $i = 1, 2, \ldots, p$, the chart Γ contains exactly n_i white vertices in $\Gamma_{m+i-1} \cap \Gamma_{m+i}$.

(ii) If $i < 0$ or $i > p$, then Γ_{m+i} does not contain any white vertices.

(iii) Each of the two subgraphs Γ_m and Γ_{m+p} contains at least one white vertex.

Note that $n_1 \geq 1$ and $n_p \geq 1$ by Condition (iii).

The chart Γ shown in Figure 1 contains exactly seven white vertices, and $\Gamma_1 \cap \Gamma_2 = \{v_2, v_4, v_5\}$, $\Gamma_2 \cap \Gamma_3 = \{v_1, v_6\}$ and $\Gamma_3 \cap \Gamma_4 = \{v_3, v_7\}$. Hence this chart is a chart of type $(1;3,2,2)$. Note that this chart is not a minimal chart.

Two C-move equivalent charts Γ and Γ' are said to be *same C-type* provided that the types of the two charts are same.

For a subset X of a chart, let

$$w(X) = \text{the number of white vertices of the chart contained in } X.$$

A chart Γ is *zero at label* k provided that

(i) $\Gamma_k \cap \Gamma_{k+1} = \emptyset$,

(ii) there exists a label i with $i \leq k$ and $w(\Gamma_i) \neq 0$, and

(iii) there exists a label j with $k < j$ and $w(\Gamma_j) \neq 0$.

Let Γ be an n-chart, and D_1, D_2 disjoint disks with $D_i \cap \Gamma \neq \emptyset$ for $i = 1, 2$, $\partial D_i \cap \Gamma = \emptyset$ for $i = 1, 2$, and $D_1 \cup D_2 \supset \Gamma$. Then we can consider $\Gamma^* = D_1 \cap \Gamma$ and $\Gamma^{**} = D_2 \cap \Gamma$ as n-charts. Then we call the chart Γ the *product of the two charts* Γ^* *and* Γ^{**} [6].

A chart Γ is *separable at label k* if there exist subcharts Γ^*, Γ^{**} such that

(i) Γ is the product of the two charts Γ^* and Γ^{**},

(ii) $w(\Gamma^*) \neq 0$ and $w(\Gamma^{**}) \neq 0$,

(iii) $w(\Gamma^*_i) = 0$ for all label i with $k < i$, and

(iv) $w(\Gamma^{**}_i) = 0$ for all label i with $i \leq k$.

The following is our main theorem:

Theorem 1. *A chart* Γ *is zero at label* k *if and only if there exists a chart* Γ' *with the same C-type of* Γ *such that* Γ' *is separable at label* k.

A chart Γ is *minimal* if it possesses the smallest number of white vertices among the charts C-move equivalent to the chart Γ (cf. [5]).

Corollary 2. *Let* Γ *be a minimal chart with* $w(\Gamma) = 8$. *If* Γ *is zero at some label, then* Γ *is C-move equivalent to one of the following charts:*

(a) *the product of two charts of type* (4).

(b) *the product of two charts of type* $(2, 2)$.

(c) *the product of a chart of type* (4) *and a chart of type* $(2, 2)$.

The chart shown in Figure 2 contains exactly eight white vertices of type $(2; 4, 0, 2, 2)$. This chart is a product of a chart of type a chart of type $(2; 4)$ and a chart of type $(4; 2, 2)$.

Corollary 3. *Let* Γ *be a minimal n-chart with* $w(\Gamma) = 8$ *such that* Γ *is not zero at any label. If necessary we change all the edges of label k to ones of label* $n - k$ *for each* $k = 1, 2, \cdots, n-1$ *simultaneously, then the type of* Γ *is* (8), $(6, 2)$, $(5, 3)$, $(4, 4)$, $(4, 2, 2)$, $(3, 3, 2)$, $(3, 2, 3)$, $(2, 4, 2)$ *or* $(2, 2, 2, 2)$.

The chart shown in Figure 3 is a chart of type $(1; 2, 4, 2)$, and represents a 3-twist spun trefoil [2].

Corollary 4. *If* Γ *is a minimal chart with* $w(\Gamma) = 9$ *or* 11, *then the chart is not zero at any label. Namely if the type of the chart is* $(m; n_1, n_2, \cdots, n_p)$, *then for each* $i = 1, 2, \cdots, p$, *we have* $n_i \neq 0$.

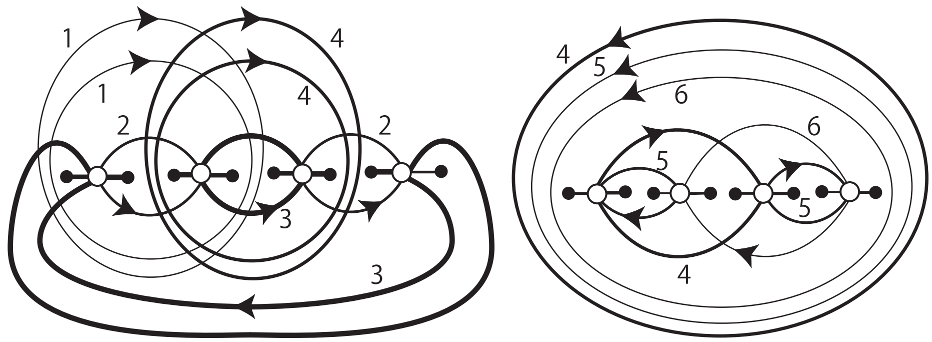

Figure 2: Product of a chart of type $(2; 4)$ and a chart of type $(4; 2, 2)$.

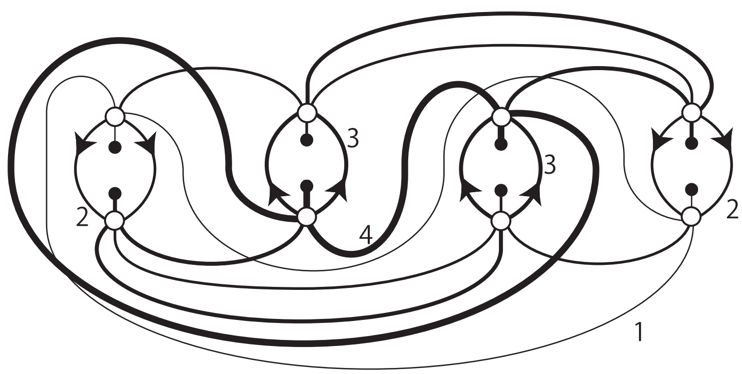

Figure 3: Chart of type $(1; 2, 4, 2)$.

Corollary 5. *Let* Γ *be a minimal* n*-chart with* $w(\Gamma) = 9$*. If necessary we change all the edges of label* k *to ones of label* $n-k$ *for each* $k = 1, 2, \cdots, n-1$ *simultaneously, then the type of* Γ *is* (9), $(7, 2)$, $(6, 3)$, $(5, 4)$, $(5, 2, 2)$, $(4, 3, 2)$, $(4, 2, 3)$, $(4, 1, 4)$, $(3, 4, 2)$, $(3, 3, 3)$, $(2, 5, 2)$, $(4, 1, 2, 2)$, $(3, 2, 2, 2)$, $(2, 3, 2, 2)$ *or* $(2, 2, 1, 2, 2)$.

The paper is organized as follows. In Section 2, we define charts. In Section 3 and Section 4, we prove lemmata that are needed in order to prove Theorem 1. In Section 5, we define ω_k-minimal charts. By using ω_k-minimal charts, we show that if a chart Γ is zero at label k, then there exists a chart Γ^* obtained from Γ by C-I-R2 moves and C-I-M2 moves such that $\Gamma^*_k \supset \Gamma_k$ and all of black vertices and white vertices in $\bigcup_{i=k+1}^{\infty} \Gamma^*_i$ are contained in the same complementary domain of Γ_k. In Section 6, we give a proof of Theorem 1 and proofs of corollaries.

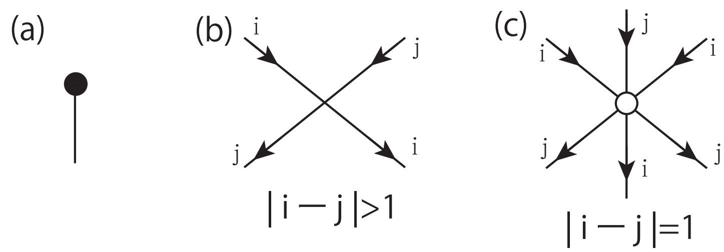

Figure 4: (a) A black vertex. (b) A crossing. (c) A white vertex.

2. Preliminaries

Let n be a positive integer. An *n-chart* is an oriented labeled graph in a disk, which may be empty or have closed edges without vertices, called *hoops*, satisfying the following four conditions:

(i) Every vertex has degree 1, 4, or 6.

(ii) The labels of edges are in $\{1, 2, \dots, n-1\}$.

(iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled i and $i+1$ alternately for some i, where the orientation and the label of each arc are inherited from the edge containing the arc.

(iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i-j| > 1$.

We call a vertex of degree 1 a *black vertex,* a vertex of degree 4 a *crossing*, and a vertex of degree 6 a *white vertex* respectively (see Figure 4).

Let D_1^2, D_2^2 be disks, and $pr_2 : D_1^2 \times D_2^2 \to D_2^2$ the projection defined by $pr_2(x, y) = y$. Let Q_n be a set of n interior points of D_1^2. A *surface braid S* is an oriented surface embedded properly in $D_1^2 \times D_2^2$ such that the map $pr_2|_S : S \to D_2^2$ is a branched covering of degree n and $\partial S = Q_n \times \partial D_2^2$ [7, Chapter 14]. A surface braid can be represented by a motion picture method, a one-parameter family of geometric n-braids $\{b_t\}_{t\in[0,1]}$ except for a finite number of values $t_1, t_2, \cdots, t_m \in [0, 1]$. A motion picture for a white vertex is a motion picture as shown in Figure 5(a) (cf. [7, p. 132, Figure 18.5]). A motion picture for a crossing is a motion picture as shown in Figure 5(b) (cf. [7, p. 131, Figure 18.4]). A motion picture for a black vertex is a motion picture as shown in Figure 5(c) (cf. [7, p. 134, Figure 18.7]). A black vertex is corresponding to a singular point of a branched covering map.

Now *C-moves* are local modifications of charts in a disk as shown in Figure 6 (cf. [1], [7], [17]). We do not use a C-I-M4 move (a tetrahedral move), and we do

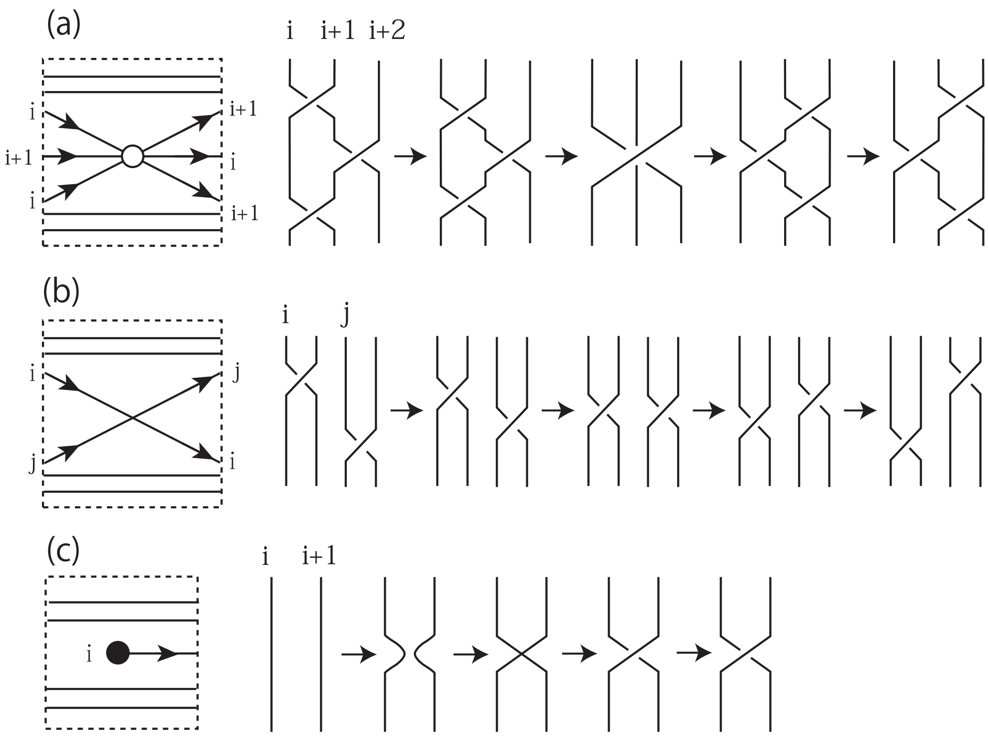

Figure 5:

not use a C-II move and a C-III move. We often use C-I-M2 moves and C-I-R2 moves in this paper.

Let Γ be a chart, and m a label of Γ. The 'subgraph' Γ_m of Γ consists of all the edges of label m and their vertices. An edge of Γ is the closure of a connected component of the set obtained by taking out all white vertices and crossings from Γ. On the other hand, we assume that

> an *edge* of Γ_m is the closure of a connected component of the set obtained by taking out all white vertices from Γ_m.

Thus any vertex of Γ_m is a black vertex or a white vertex. Hence any crossing of Γ is not considered as a vertex of Γ_m.

In this paper for a set X in a space we denote the interior of X, the boundary of X and the closure of X by Int$X$, ∂X and $Cl(X)$ respectively.

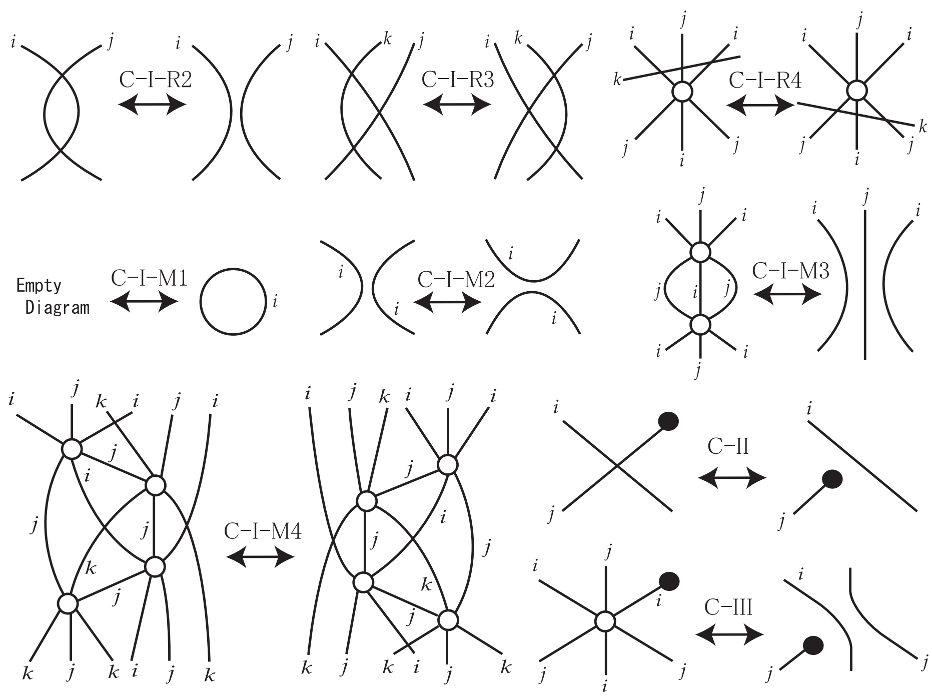

Figure 6: For the C-III move, the black vertex in the left figure is not contained in a middle edge of three consecutive edges oriented inward or outward.

3. Separating Systems

In this paper we assume that *every chart is in the plane.*

For a subset X of a chart Γ, let

$$\mathbb{BW}(X) = \text{the set of all the black and white vertices of } \Gamma \text{ in } X.$$

A disk D is *in general position* with respect to a chart Γ provided that

(i) ∂D does not intersect the set of crossings nor $\mathbb{BW}(\Gamma)$, and

(ii) if an edge of Γ intersects ∂D, then the edge intersects ∂D transversely.

Let D be a disk. A simple arc ℓ is called a *proper arc* of D provided that $\ell \cap \partial D = \partial \ell$. Let L be a simple arc on ∂D. A proper arc ℓ of D is called a *(D, L)-arc* provided that $\partial \ell \subset L$.

Let Γ be a chart, and k a label of Γ. A simple arc in an edge of Γ_k is called an *arc of label k*.

Let D be a disk in general position with respect to a chart Γ, and L a simple arc on ∂D. A (D, L)-arc ℓ of label k is said to be *reducible* if for the subarc L' of L with $\partial L' = \partial \ell$ we have $\mathrm{Int}L' \cap (\Gamma_{k-1} \cup \Gamma_k \cup \Gamma_{k+1}) = \emptyset$.

Let k be a positive integer. Let D^- and D^+ be disks in general position with respect to a chart Γ such that $J = D^- \cap D^+$ is an arc. The triplet (D^-, D^+, J) is called a *separating system at label k* for the chart Γ provided that

(i) $D^- \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma_i) = \emptyset$,

(ii) $D^+ \cap \mathbb{BW}(\bigcup_{i=1}^{k} \Gamma_i) = \emptyset$,

(iii) $J \cap \Gamma_k = \emptyset$, and

(iv) $Cl(\partial D^+ - J) \cap \Gamma = \emptyset$.

Lemma 6. *Let (D^-, D^+, J) be a separating system at label k for a chart Γ. Then the following hold:*

(a) *If there exists a (D^+, J)-arc of label less than k, then there exists a reducible (D^+, J)-arc of label less than k.*

(b) *If there exists a (D^-, J)-arc of label greater than k, then there exists a reducible (D^-, J)-arc of label greater than k.*

Proof. We show Statement (a). Suppose that there exists a (D^+, J)-arc of label less than k. Let $\mathbb{S}$ be the set of all (D^+, J)-arcs of label less than k. For each $\ell \in \mathbb{S}$ let P_ℓ be the subarc of J with $\partial P_\ell = \partial \ell$. Let

$$m(\ell) = \text{the number of points in } \mathrm{Int}(P_\ell) \cap (\textstyle\bigcup_{i=1}^{k} \Gamma_i).$$

Let ℓ_0 be an element in $\mathbb{S}$ with

$$m(\ell_0) = \min\{\, m(\ell) \mid \ell \in \mathbb{S} \,\}.$$

Let j be the label of ℓ_0. We have $j \leq k-1$. Let

$$s = \text{the number of points in } \mathrm{Int}(P_{\ell_0}) \cap (\Gamma_{j-1} \cup \Gamma_j \cup \Gamma_{j+1}).$$

We show that $s = 0$ by contradiction. Suppose that $s > 0$. Let D_0 be the disk in D^+ bounded by $\ell_0 \cup P_{\ell_0}$. Let ℓ_1 be a connected component of $D_0 \cap (\Gamma_{j-1} \cup \Gamma_j \cup \Gamma_{j+1})$ different from ℓ_0. Since $j+1 \leq (k-1)+1 = k$ and since there do not exist any black vertices nor white vertices of $\bigcup_{i=1}^{k} \Gamma_i$ in D^+ by Condition (ii) of a separating sysytem, the arc ℓ_1 is a proper arc of D_0. Since $\ell_0 \cap \ell_1 = \emptyset$, the arc ℓ_1 is a (D_0, P_{ℓ_0})-arc. Hence ℓ_1 is a (D^+, J)-arc. Let p be the label of ℓ_1. Now $J \cap \Gamma_k = \emptyset$ implies $p \neq k$. Hence we have $p < k$. Thus $\ell_1 \in \mathbb{S}$. Since $P_{\ell_1} \cap \ell_0 = \emptyset$, we have

$$m(\ell_1) \leq m(\ell_0) - 2.$$

This contradicts that $m(\ell_0)$ is minimal. Hence we have $s = 0$. Therefore

$$\mathrm{Int}(P_{\ell_0}) \cap (\Gamma_{j-1} \cup \Gamma_j \cup \Gamma_{j+1}) = \emptyset.$$

This means that ℓ_0 is a desired reducible (D^+, J)-arc. This proves Statement (a).
Similarly we can show Statement (b). $\square$

Let Γ be a chart, and e_1 and e_2 edges of label m (possibly $e_1 = e_2$). Let α be an arc such that

(i) $\partial\alpha$ consists of a point in e_1 and a point in e_2, and

(ii) $\mathrm{Int}(\alpha)$ transversely intersects edges of Γ (see Figure 7(a)).

Let D be a regular neighborhood of the arc α. Let $\gamma_1 = e_1 \cap D$ and $\gamma_2 = e_2 \cap D$. Then γ_1 and γ_2 are proper arcs of D and they split the disk D into three disks. Let E be the one of the three disks with $E \supset \alpha$ (see Figure 7(b)). A chart Γ' *is obtained from* Γ *by a surgery along* α provided that

(iii) $\Gamma'_m = (\Gamma_m - (\gamma_1 \cup \gamma_2)) \cup Cl(\partial E - (\gamma_1 \cup \gamma_2))$, and

(iv) $\Gamma'_i = \Gamma_i$ $(i \neq m)$ (see Figure 7(c)).

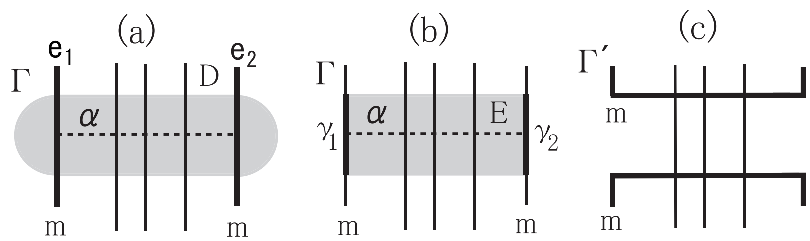

Figure 7:

Let k be a positive integer. Let Γ and Γ^* be charts. We write $\Gamma^* \overset{k}{\sim} \Gamma$ provided that

(i) the chart Γ^* is obtained from Γ by applying C-I-M2 moves, C-I-R2 moves and ambient isotopies of the plane, and

(ii) Γ_k is a subset of Γ^*_k.

For a positive integer k and a chart Γ, let

$$Fix(\Gamma_k; \Gamma) = \{\ \Gamma^* \mid \Gamma^* \overset{k}{\sim} \Gamma\ \}.$$

Remark. The relation $\Gamma^* \overset{k}{\sim} \Gamma$ implies that Γ^* and Γ are same C-type.

Lemma 7. *Let* (D^-, D^+, J) *be a separating system at label* k *for a chart* Γ*. Then there exists a chart* $\Gamma' \in Fix(\Gamma_k; \Gamma)$ *such that*

(a) *the chart* Γ' *is obtained from* Γ *by applying surgeries along subarcs of* J*, and*

(b) *the chart* Γ' *does not possess any* (D^-, J)*-arcs of label greater than* k *nor* (D^+, J)*-arcs of label less than* k*.*

Proof. Let $\mathbb{S}$ be the set of all charts obtained from Γ by applying surgeries along subarcs of J. Since $J \cap \Gamma_k = \emptyset$ by Condition (ii) of a separating system, we have that $\Gamma^*_k = \Gamma_k$ for each chart $\Gamma^* \in \mathbb{S}$. Thus for each chart $\Gamma^* \in \mathbb{S}$ we have that $\Gamma^* \in Fix(\Gamma_k;\Gamma)$ and that (D^-, D^+, J) is a separating system at label k for Γ^*. For each chart $\Gamma^* \in \mathbb{S}$, let

$$\begin{array}{rcl} n(\Gamma^*) & = & \text{the number of } (D^+, J)\text{-arcs of label less than } k \\ & + & \text{the number of } (D^-, J)\text{-arcs of label greater than } k. \end{array}$$

Let Γ' be a chart in $\mathbb{S}$ such that

$$n(\Gamma') = min\{\ n(\Gamma^*) \mid \Gamma^* \in \mathbb{S}\ \}.$$

We show that $n(\Gamma') = 0$ by contradiction. Suppose that $n(\Gamma') > 0$. Then by Lemma 6 the chart Γ' possesses a reducible (D^-, J)-arc of label greater than k or a reducible (D^+, J)-arc of label less than k, say ℓ. Let P_ℓ be the subarc of J with $\partial P_\ell = \partial \ell$. Let Γ'' be a chart obtained from Γ' by applying a surgery along the subarc P_ℓ. Then we have $\Gamma'' \in \mathbb{S}$ and

$$n(\Gamma'') \leq n(\Gamma') - 1.$$

This contradicts that $n(\Gamma')$ is minimal. Therefore $n(\Gamma') = 0$. The chart Γ' is a desired chart. □

4. Movable Disks

Let D be a disk in general position with respect to a chart Γ. The disk D is called a *movable disk* at label k with respect to the chart Γ provided that

(i) $D \cap \mathbb{BW}(\bigcup_{i=1}^{k} \Gamma_i) = \emptyset$, and

(ii) $\partial D \cap (\bigcup_{i=1}^{k+1} \Gamma_i) = \emptyset$.

Let Γ be a chart, and m a label of Γ. A *hoop* is a closed edge of Γ without vertices (hence without crossings, neither). A *ring* is a closed edge of Γ_m containing a crossing but not containing any white vertices.

Lemma 8. *Let (D^-, D^+, J) be a separating system at label k for a chart Γ. If $\partial(D^- \cup D^+) \cap \Gamma_{k+1} = \emptyset$, then there exists a chart $\Gamma' \in Fix(\Gamma_k;\Gamma)$ such that*

(a) *the chart Γ' is obtained from Γ by applying surgeries along subarcs of J, and*

(b) *D^+ is a movable disk at label k with respect to Γ'.*

Proof. By Lemma 7 there exists a chart $\Gamma' \in Fix(\Gamma_k;\Gamma)$ obtained from Γ by applying surgeries along subarcs of J such that

(1) Γ' does not possess any (D^-, J)-arcs of label greater than k nor (D^+, J)-arcs of label less than k.

Thus by Condition (ii), (iii) and (iv) of a separating system,

(2) $D^+ \cap \mathbb{BW}(\bigcup_{i=1}^k \Gamma'_i) = D^+ \cap \mathbb{BW}(\bigcup_{i=1}^k \Gamma_i) = \emptyset$,

(3) $J \cap \Gamma'_k = J \cap \Gamma_k = \emptyset$,

(4) $Cl(\partial D^+ - J) \cap \Gamma' = Cl(\partial D^+ - J) \cap \Gamma = \emptyset$.

Hence any connected component of $D^+ \cap (\bigcup_{i=1}^k \Gamma'_i)$ is a hoop, a ring or a (D^+, J)-arc of label less than k. Since Γ' does not possess any (D^+, J)-arcs of label less than k by (1), we have

(5) $J \cap (\bigcup_{i=1}^k \Gamma'_i) = \emptyset$.

Since the chart Γ' is obtained from Γ by applying surgeries along subarcs of J, we have

(6) $\partial(D^- \cup D^+) \cap \Gamma'_{k+1} = \partial(D^- \cup D^+) \cap \Gamma_{k+1} = \emptyset$, and

(7) $D^- \cap \mathbb{BW}(\bigcup_{i=k+1}^\infty \Gamma'_i) = D^- \cap \mathbb{BW}(\bigcup_{i=k+1}^\infty \Gamma_i) = \emptyset$ by Condition (i) of a separating system.

Hence any connected component of $D^- \cap \Gamma'_{k+1}$ is a hoop, a ring or a (D^-, J)-arc. Since Γ' does not possess any (D^-, J)-arcs of label greater than k by (1), we have $J \cap \Gamma'_{k+1} = \emptyset$. Thus (4) and (5) imply

$$\partial D^+ \cap (\bigcup_{i=1}^{k+1} \Gamma'_i) = \emptyset.$$

Therefore D^+ is a movable disk at label k with respect to Γ'. □

Let E be a disk in general position with respect to a chart Γ. The disk E is called a *c-disk at label* k with respect to Γ provided that in Int$E$ there exist mutually disjoint movable disks $D_1, D_2, \cdots, D_s$ at label k with respect to the chart Γ and a connected component W of $E - (\Gamma_k - \bigcup_{i=1}^s D_i)$ (see Figure 8) such that

(i) $W \supset \partial E$,

(ii) $W \supset \bigcup_{i=1}^s D_i$, and

(iii) $W \supset E \cap \mathbb{BW}(\bigcup_{i=k+1}^\infty \Gamma_i)$.

We call W the *principal domain* of the c-disk E. We also call the movable disks $D_1, D_2, \cdots, D_s$ *associated movable disks* of the c-disk E.

Lemma 9. *Let E be a c-disk at label k with respect to a chart Γ. Let $p \in \partial E - \Gamma$. Then there exists a separating system (D^-, D^+, J) at label k for the chart Γ with $E = D^- \cup D^+$ and $p \in \partial D^+ - J$.*

Proof. Let W be the principal domain of the c-disk E and $D_1, D_2, \cdots, D_s$ associated movable disks of E. Suppose that

$$(W - \bigcup_{i=1}^s D_i) \cap \mathbb{BW}(\bigcup_{i=k+1}^\infty \Gamma_i) = \{ w_{s+1}, w_{s+2}, \ldots, w_t \}.$$

For each $i = s+1, s+2, \ldots, t$, let D_i be a regular neighbourhood of w_i in W. Then by Condition (iii) of a c-disk we have

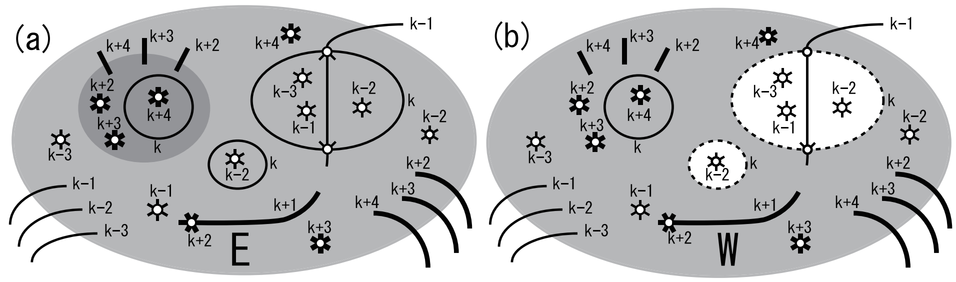

Figure 8: (a) The dark disk is a movable disk in a c-disk. (b) The gray 'disk with two holes' is the principal domain of the c-disk.

(1) $E \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma_i) \subset \bigcup_{i=1}^{t} D_i$.

Since W is a connected component of $E - (\Gamma_k - \bigcup_{i=1}^{s} D_i)$ by the condition of a c-disk, we have

$$(W - \textstyle\bigcup_{i=1}^{s} D_i) \cap \Gamma_k = \emptyset.$$

Hence there exist mutually disjoint simple arcs $\ell_1, \ell_2, \cdots, \ell_t$ in W (see Figure 9(a)) such that

(2) for each $i = 1, 2, \cdots, t$, $\ell_i \cap \Gamma_k = \emptyset$, and $\partial \ell_i \cap \Gamma = \emptyset$,

(3) for each $i = 1, 2, \cdots, t$, the arc ℓ_i does not contain any white vertices, black vertices nor crossings of Γ, and the arc ℓ_i intersects edges of Γ transversely,

(4) $(\bigcup_{i=1}^{t} \mathrm{Int}(\ell_i)) \cap (\bigcup_{i=1}^{t} D_i) = \emptyset$,

(5) for each $i = 1, 2, \cdots, t-1$, the arc ℓ_i connects a point on ∂D_i and a point on ∂D_{i+1}, and

(6) the arc ℓ_t connects a point on ∂D_t and the point p.

Let D^+ be a regular neighbourhood of $\bigcup_{i=1}^{t}(D_i \cup \ell_i)$ in E and $D^- = Cl(E - D^+)$. Then D^- and D^+ are disks with $E = D^- \cup D^+$. Since $E \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma_i) \subset \bigcup_{i=1}^{t} D_i \subset D^+$ by (1), we have

$$D^- \cap \mathbb{BW}(\textstyle\bigcup_{i=k+1}^{\infty} \Gamma_i) = \emptyset \text{ (see Figure 9(b))}.$$

Since $w_{s+1}, w_{s+2}, \ldots, w_t \in W - \bigcup_{i=1}^{s} D_i \subset E - \Gamma_k$, we have $w_{s+1}, w_{s+2}, \ldots, w_t \notin \Gamma_k$. Thus

(7) none of $D_{s+1}, D_{s+2}, \cdots, D_t$ intersect $\bigcup_{i=1}^{k} \Gamma_i$.

Since none of the movable disks at label k nor $D_{s+1}, D_{s+2}, \cdots, D_t$ intersect $\mathbb{BW}(\bigcup_{i=1}^{k} \Gamma_i)$, we have

$$D^+ \cap \mathbb{BW}(\textstyle\bigcup_{i=1}^{k} \Gamma_i) = \emptyset.$$

Let $L = D^+ \cap \partial E$. Then L is a regular neighbourhood of p in ∂E. Since $p \in \partial E - \Gamma$, we have $L \cap \Gamma = \emptyset$. Let $J = D^- \cap D^+$. Then $L = Cl(\partial D^+ - J)$. Thus $Cl(\partial D^+ - J) \cap \Gamma = L \cap \Gamma = \emptyset$. Since for each $i = 1, 2, \cdots, t$, $\partial D_i \cap \Gamma_k = \emptyset$ by (2), (7) and Condition (ii) of a movable disk, we have $J \cap \Gamma_k = \emptyset$. Therefore the triplet (D^-, D^+, J) is a desired separating system at label k for Γ. $\square$

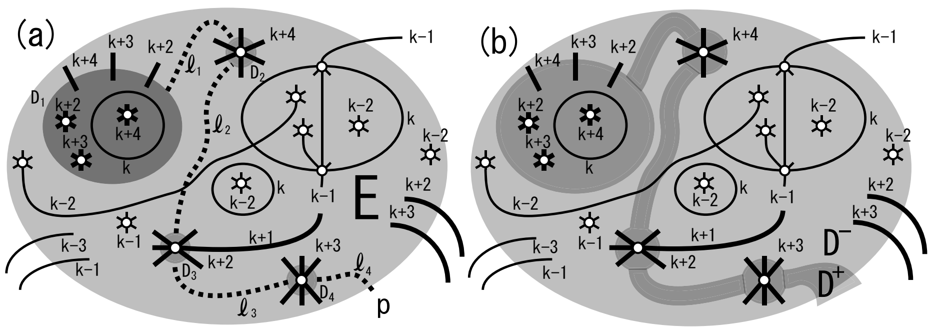

Figure 9: (a) The dark disk is a movable disk.

5. ω_k-minimal Charts

Let Γ be a chart. Let Γ^* be a chart in $Fix(\Gamma_k; \Gamma)$. We denote $\Gamma^* \overset{k}{\approx} \Gamma$ provided that

> for any complementary domain U of Γ_k, the domain U contains mutually disjoint movable disks $D_1, D_2, \cdots, D_s$ at label k with respect to Γ^* with $U \cap (\Gamma^*_k - \Gamma_k) \subset \bigcup_{i=1}^{s} D_i$.

The movable disks $D_1, D_2, \cdots, D_s$ are called *basic movable disks at label* k with respect to U and Γ^*. Let

$$\Omega(\Gamma_k; \Gamma) = \{ \ \Gamma^* \mid \Gamma^* \overset{k}{\approx} \Gamma \ \}.$$

Since $\Gamma \in \Omega(\Gamma_k; \Gamma)$, we have $\Omega(\Gamma_k; \Gamma) \neq \emptyset$.

Let Γ be a chart, k a label of Γ, and $\Gamma' \in \Omega(\Gamma_k; \Gamma)$. Let U be a complementary domain of Γ_k , and D a movable disk at label k in U with respect to Γ'. Let $F = Cl(U)$. Let $p \in \partial F - \bigcup_{i \neq k} \Gamma'_i$ and $q \in \partial D$. Suppose that *there exists a simple arc* α *in* F *connecting the two points* p *and* q *with* $\alpha \cap \Gamma' = p$ *and* $\alpha \cap D = q$ (see Figure 10(a)). Then we can shift the movable disk D at label k to another complementary domain of Γ_k as follows (see Figure 10): Let N_1 be a regular neighbourhood of $D \cup \alpha$ in F. Let N_2 be a regular neighbourhood of N_1 in F and N_3 a regular neighbourhood of N_2 in F. For each $i = 1, 2, 3$ let $\beta_i = N_i \cap \partial F$ and $\gamma_i = Cl(\partial N_i - \beta_i)$ (see Figure 10(b)). Let Γ'' be a chart with

$$\Gamma''_j = \begin{cases} (\Gamma'_k - \beta_3) \cup (\gamma_3 \cup \partial N_1) & \text{if } j = k, \\ \Gamma'_j & \text{otherwise.} \end{cases}$$

Then Γ' is C-move equivalent to Γ'' by modifying ∂F by C-I-R2 moves along γ_2 and a C-I-M2 move (see Figure 10(c)). Let N be a regular neighbourhood of N_1. Then N is a movable disk at label k with respect to Γ''. In a regular neighbourhood of N_3 we can modify the arc γ_3 to the arc β_3 by an ambient isotopy keeping $\partial\gamma_3$ fixed. Let Γ^* be the resulting chart and D^* the disk modified from the movable disk N (see Figure 10(d)). Then Γ^* is in $Fix(\Gamma_k;\Gamma)$ and D^* is a movable disk at label k with respect to Γ^*. Now Γ^*_k is the union of Γ'_k and a ring in the movable disk D^*. Thus $\Gamma^* \overset{k}{\approx} \Gamma$. We say that Γ^* is *obtained from* Γ' *by shifting the movable disk* D *to the outside of* $Cl(U)$ *along the arc* α and that D^* is a *movable disk induced from the movable disk* D *circled by a ring of label* k.

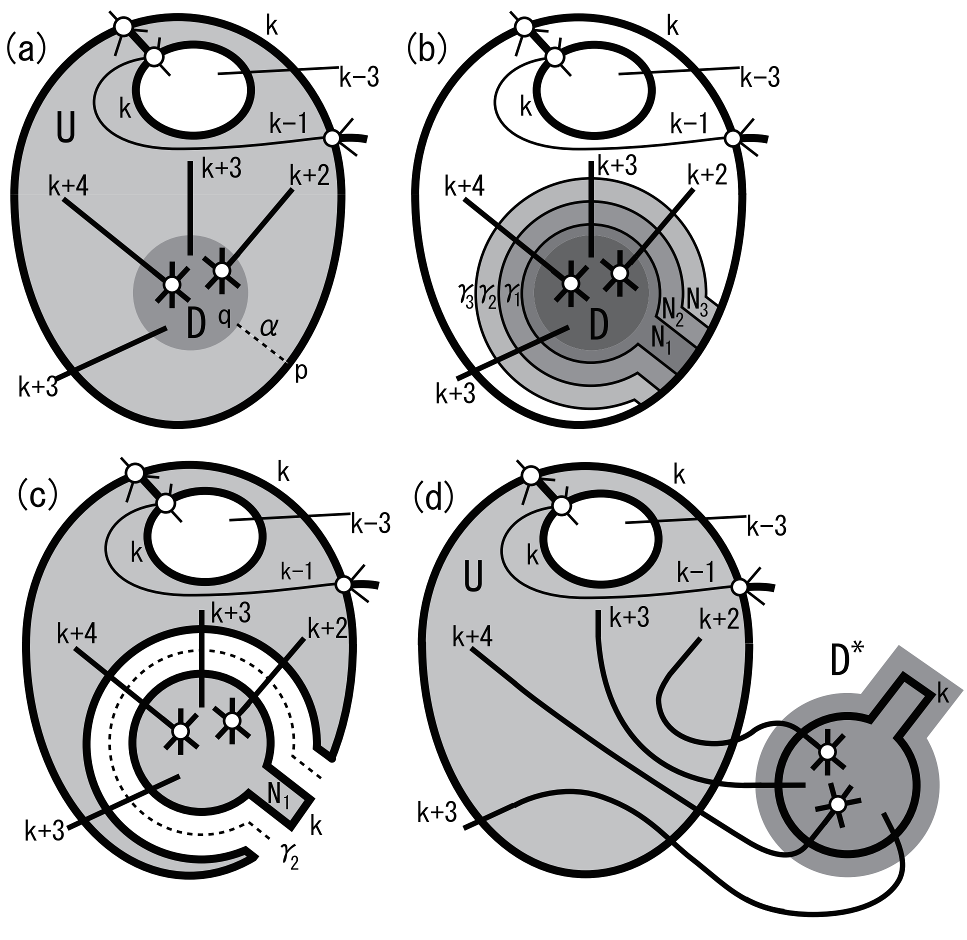

Figure 10: Shifting a movable disk.

Let Γ be a chart and k a label of Γ. Let T be a maximal tree of the dual graph of Γ_k. Namely

(i) each vertex v of the tree T corresponds to a complementary domain U_v of Γ_k, and

(ii) each edge e of the tree T with $\partial e = \{v_1, v_2\}$ corresponds to an edge e_Γ of Γ_k with $e_\Gamma \subset Cl(U_{v_1}) \cap Cl(U_{v_2})$.

The tree T is called a *dual tree with respect to* Γ_k.

Let Γ be a chart and k a label of Γ. Let T be a dual tree with respect to Γ_k. Let v_0 be the vertex of T which corresponds to the unbounded complementary domain of Γ_k. Let $V(T)$ be the set of all vertices of T. For each vertex $v \in V(T)$, let $T(v, v_0)$ be the path in T connecting v and v_0. Let

$$length(v; T) = \text{the number of edges in } T(v, v_0).$$

We have $length(v_0; T) = 0$. For each chart $\Gamma^* \in \Omega(\Gamma_k; \Gamma)$ and $v \in V(T)$, let

$$weight_k(v; \Gamma^*) = \begin{cases} 0 & \text{if } U_v \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma_i^*) = \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

$$\omega_k(\Gamma^*, T) = \sum_{v \in V(T)} weight_k(v; \Gamma^*) \times length(v; T).$$

A chart $\Gamma' \in \Omega(\Gamma_k; \Gamma)$ is ω_k*-minimal* if there exists a dual tree T with respect to Γ_k such that $\omega_k(\Gamma', T) = \min\{\ \omega_k(\Gamma^*, T) \mid \Gamma^* \in \Omega(\Gamma_k; \Gamma)\ \}$.

Lemma 10. *Let Γ be a chart which is zero at label k. If a chart $\Gamma' \in \Omega(\Gamma_k; \Gamma)$ is ω_k-minimal, then there exists a dual tree T with respect to Γ_k with $\omega_k(\Gamma', T) = 0$.*

Proof. Suppose that for a dual tree T with respect to Γ_k, we have

$$\omega_k(\Gamma', T) = \min\{\ \omega_k(\Gamma^*, T) \mid \Gamma^* \in \Omega(\Gamma_k; \Gamma)\ \}.$$

We shall show $\omega_k(\Gamma', T) = 0$ by contradiction. Suppose that $\omega_k(\Gamma', T) > 0$. Let $V(T)$ be the set of all vertices of T and

$$\mathbb{P} = \{\ v \in V(T) \mid weight_k(v; \Gamma') \times length(v; T) > 0\ \}.$$

Then $\omega_k(\Gamma', T) > 0$ implies $\mathbb{P} \neq \emptyset$. Let v_1 be a vertex in $\mathbb{P}$ such that

(1) $length(v_1; T) = \max\{\ length(v; T) \mid v \in \mathbb{P}\ \}$.

Then $v_1 \in \mathbb{P}$ implies

(2) $0 < weight_k(v_1; \Gamma') \times length(v_1; T) = length(v_1; T)$.

Let V_1 be a complementary domain of Γ_k corresponding to v_1. Since $\Gamma' \in \Omega(\Gamma_k; \Gamma)$, there exist basic movable disks $D_1, D_2, \cdots, D_s$ at label k with respect to V_1 and Γ' (see Figure 11(a)) such that

(3) $V_1 \cap (\Gamma'_k - \Gamma_k) \subset \bigcup_{i=1}^{s} D_i$.

Let δ be a connected component of ∂V_1 such that a bounded complementary domain of δ contains V_1 (see Figure 11(b)). Let $N(\delta)$ be a regular neighbourhood of δ and $W = Cl(V_1 - N(\delta))$ (see Figure 11(c)). Let ℓ be a connected component of ∂W such that W is contained in the disk E bounded by ℓ (see Figure 11(c) and (d)). Then we have

$$W \supset \partial E.$$

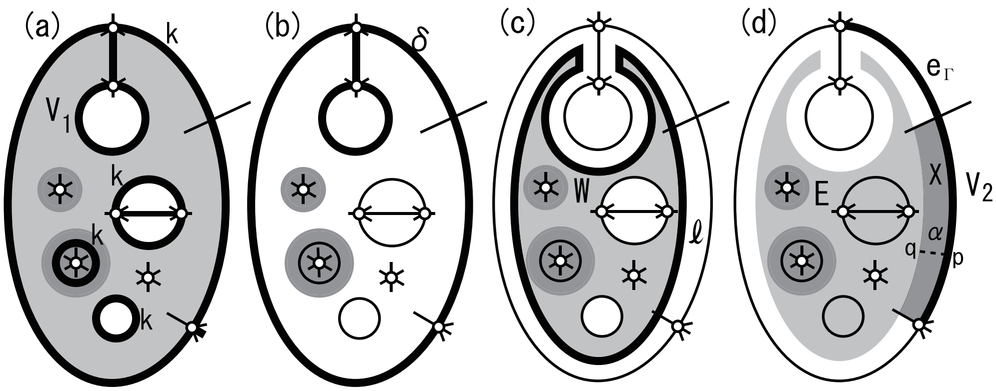

Figure 11: Dark disks are movable disks at label k. (a) The thicken curves are of label k. (b) The thicken curve is the set δ. (c) The thicken curve is the simple closed curve ℓ. (d) The thicken curve is the edge e_Γ. The set X is a disk with $\text{Int}X \cap \text{Int}E = \emptyset$.

Claim. The disk E is a c-disk.

Proof of Claim. Since $\Gamma' \overset{k}{\approx} \Gamma$, the chart Γ' is zero at label k. Hence $\delta \subset \Gamma_k \subset \Gamma'_k$ implies that $\delta \cap \Gamma'_{k+1} = \emptyset$. Thus

(4) $\partial E \cap (\Gamma'_k \cup \Gamma'_{k+1}) = \emptyset$.

Now $\delta \cap (\bigcup_{i=1}^s D_i) = \emptyset$ implies $N(\delta) \cap (\bigcup_{i=1}^s D_i) = \emptyset$. Thus $V_1 \supset \bigcup_{i=1}^s D_i$ implies

$$W = Cl(V_1 - N(\delta)) \supset \textstyle\bigcup_{i=1}^s D_i.$$

Let $\mathbb{S}$ be the set of all connected components of $E - (\Gamma_k - \bigcup_{i=1}^s D_i)$ different from W. Then we have

(5) $E = W \cup (\bigcup\{\ Cl(U) \mid U \in \mathbb{S}\ \})$.

For each domain $U \in \mathbb{S}$, let v_U be the vertex of T corresponding to U. Since U is surrounded by V_1, we have $v_1 \in T(v_U, v_0)$ where v_0 is the vertex of T which corresponds to the unbounded complementary domain of Γ_k. Hence

$$length(v_U;T) \geq length(v_1;T) + 1 \text{ and}$$

$$weight_k(v_U; \Gamma') = 0$$

by the property (1) of the vertex v_1. This means

$$U \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma'_i) = \emptyset.$$

Since $\Gamma' \overset{k}{\approx} \Gamma$, the chart Γ' is zero at label k. Thus $\Gamma'_k \cap \Gamma'_{k+1} = \emptyset$. Hence $\partial U \subset \Gamma_k \subset \Gamma'_k$ implies

$$Cl(U) \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma'_i) = \emptyset.$$

Thus (5) implies that

$$E \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma'_i) \subset W.$$

Therefore the disk E is a c-disk at label k with respect to Γ'. Hence Claim holds.

Let $T(v_1, v_0)$ be the path in the tree T connecting v_1 and v_0. Let e be the edge of $T(v_1, v_0)$ with $e \ni v_1$. Let v_2 be the vertex of e different from v_1. Now we have

(6) $length(v_2; T) = length(v_1; T) - 1.$

Let e_Γ be the edge of Γ_k corresponding to the edge e which connects the two vertices v_1 and v_2. Let V_2 be the complementary domain of Γ_k corresponding to v_2. Then we have $e_\Gamma \subset Cl(V_1) \cap Cl(V_2)$. Let p be a point in $\text{Int}(e_\Gamma) - \bigcup_{i \neq k} \Gamma'_i$ and X the closure of a connected component of $(V_1 - E) - \Gamma'$ with $X \ni p$. Let q be a point in $\partial E \cap X$ and α a proper simple arc in X with $\alpha \cap \Gamma' = p$ and $\alpha \cap E = q$ (see Figure 11(d)).

By Lemma 9 there exists a separating system (D^-, D^+, J) at label k for Γ' with $E = D^- \cup D^+$ and $q \in D^+ \cap \partial E$.

Now (4) implies

$$\partial(D^- \cup D^+) \cap \Gamma'_{k+1} = \partial E \cap \Gamma'_{k+1} = \emptyset.$$

Thus by Lemma 8 there exists a chart $\Gamma'' \in Fix(\Gamma_k; \Gamma)$ obtained from Γ' by applying surgeries along subarcs of J such that D^+ is a movable disk at label k with respect to Γ''.

Let Γ^* be a chart obtained from Γ'' by shifting the movable disk D^+ to the outside of $Cl(V_1)$ along the arc α and let D^* be a movable disk induced from the movable disk D^+ circled by a ring of label k. Then D^* is a new movable disk in V_2 disjoint from the old movable disks in V_2. Hence we have $\Gamma^* \in \Omega(\Gamma_k; \Gamma)$. Now
$weight_k(v_1; \Gamma') \times length(v_1; T) = length(v_1; T) > 0$ by (2),
$weight_k(v_1; \Gamma^*) \times length(v_1; T) = 0,$
$weight_k(v_2; \Gamma^*) \times length(v_2; T) = length(v_2; T) = length(v_1; T) - 1$ by (6), and
$weight_k(v; \Gamma^*) \times length(v; T) = weight_k(v; \Gamma') \times length(v; T) \quad (v \neq v_1, v_2)$
imply that

$$\omega_k(\Gamma^*, T) \leq \omega_k(\Gamma', T) - 1.$$

This contradicts that Γ' is ω_k-minimal. Therefore $\omega_k(\Gamma', T) = 0$. □

6. Proof of Main Theorem

Proof of Theorem 1. Suppose that a chart Γ is zero at label k. Then there exists a chart $\Gamma' \in \Omega(\Gamma_k; \Gamma)$ such that Γ' is ω_k-minimal. Since Γ is zero at label k, the chart Γ' is zero at label k, too. Let E be a disk containing the chart Γ' in its inside, i.e. $\mathrm{Int}E \supset \Gamma'$. By Lemma 10 there exists a dual tree T with respect to Γ_k with $\omega_k(\Gamma', T) = 0$. Namely $\mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma'_i)$ is contained in the unbounded complementary domain U of Γ_k. Now $U \cap \Gamma_k = \emptyset$. Since $\Gamma' \in \Omega(\Gamma_k; \Gamma)$, the set $U \cap (\Gamma'_k - \Gamma_k)$ is contained in the union of basic movable disks of label k with respect to U and Γ'. Furthermore, since $E \cap U$ is connected, by the same way as the one in Lemma 9 there exists a disk D^+ in E such that

(1) D^+ is in general position with respect to Γ',

(2) $D^+ \cap \partial E$ is an arc,

(3) $\partial D^+ \cap \Gamma'_k = \emptyset$, and

(4) $\mathbb{BW}(\Gamma') \cap D^+ = \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma'_i)$.

Let $D^- = Cl(E - D^+)$ and $J = D^- \cap D^+$. Then we have

(5) $D^- \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma'_i) = \emptyset$ and $D^+ \cap \mathbb{BW}(\bigcup_{i=1}^{k} \Gamma'_i) = \emptyset$ by (4),

(6) $Cl(\partial D^+ - J) \cap \Gamma' \subset \partial E \cap \Gamma' = \emptyset$ by $\mathrm{Int}E \supset \Gamma'$,

(7) $J \cap \Gamma'_k \subset \partial D^+ \cap \Gamma'_k = \emptyset$ by (3).

Hence the triplet (D^-, D^+, J) is a separating system at label k for Γ'. By Lemma 7 there exists a chart $\Gamma'' \in Fix(\Gamma_k; \Gamma)$ obtained from Γ' by applying surgeries along subarcs of J such that

(8) $\Gamma''_k = \Gamma'_k$,

(9) Γ'' does not possess any (D^+, J)-arcs of label less than k nor (D^-, J)-arcs of label greater than k,

(10) $D^- \cap \mathbb{BW}(\bigcup_{i=k+1}^{\infty} \Gamma''_i) = \emptyset$ and $D^+ \cap \mathbb{BW}(\bigcup_{i=1}^{k} \Gamma''_i) = \emptyset$ by (5).

Now $\Gamma'' \in Fix(\Gamma_k; \Gamma)$ implies that

(11) Γ'' is zero at label k.

Hence $\Gamma''_k \cap \Gamma''_{k+1} = \emptyset$. Thus (10) and $\partial(D^- \cup D^+) \cap \Gamma'' = \partial E \cap \Gamma'' = \emptyset$ imply that

(12) for each label i with $i \leq k$ if a connected component of $\Gamma''_i \cap D^+$ intersects J, then the component is a (D^+, J)-arc,

(13) for each label i with $k < i$ if a connected component of $\Gamma''_i \cap D^-$ intersects J, then the component is a (D^-, J)-arc.

Since $J \cap \Gamma''_k = J \cap \Gamma'_k = \emptyset$ by (7) and (8), we have $J \cap \Gamma'' = \emptyset$ by (9), (12) and (13). Thus $(\partial D^+ - J) \subset \partial E$ implies

$$\partial D^+ \cap \Gamma'' = \emptyset.$$

Let $\Gamma^* = D^- \cap \Gamma''$ and $\Gamma^{**} = D^+ \cap \Gamma''$. Then Γ'' is the product of Γ^* and Γ^{**}. Since Γ'' is zero at label k by (11), there exist two labels i and j with $i \leq k < j$, $w(\Gamma''_i) \neq 0$ and $w(\Gamma''_j) \neq 0$. Namely $w(\Gamma^*) > 0$ and $w(\Gamma^{**}) > 0$. Now (10) implies

$w(\Gamma^*_i) = 0$ for all label i with $k < i$, and

$w(\Gamma^{**}_i) = 0$ for all label i with $i \leq k$.

Therefore Γ'' is separable at label k. It is clear that Γ'' and Γ are same C-type.

Conversely if Γ is separable at label k, then the chart Γ is clearly zero at label k. Thus we have done. □

Lemma 11.([10, Lemma 6.1, Proposition 6.6 and Proposition 6.11]) *Let Γ be a minimal chart of type $(n_1, n_2, \cdots, n_p)$. Then we have the following:*

(a) $n_1 > 1$ *and* $n_p > 1$.

(b) *If* $n_1 = 2$ (*resp.* $n_p = 2$), *then* $n_2 > 1$ (*resp.* $n_{p-1} > 1$).

(c) *If* $n_1 = 3$ (*resp.* $n_p = 3$), *then* $n_2 > 1$ (*resp.* $n_{p-1} > 1$).

Proof of Corollary 2. Let Γ be a minimal chart with $w(\Gamma) = 8$. Suppose that Γ is zero at label k. Then by Theorem 1 there exists a chart Γ' with the same C-type of Γ which is separable at label k. Here $w(\Gamma') = w(\Gamma)$. Hence there exist two subcharts Γ^*, Γ^{**} such that

(1) Γ' is the product of Γ^* and Γ^{**},

(2) $w(\Gamma^*) \neq 0$ and $w(\Gamma^{**}) \neq 0$.

Then we have

(3) $w(\Gamma^*) + w(\Gamma^{**}) = w(\Gamma') = 8$.

If either Γ^* or Γ^{**} is not minimal, then Γ' is C-move equivalent to a chart Γ'' with $w(\Gamma'') < w(\Gamma') = w(\Gamma)$. Namely the chart Γ is C-move equivalent to Γ''. This contradicts the fact that Γ is minimal. Hence the two charts Γ^* and Γ^{**} are minimal. Since there does not exist a minimal chart with at most three white vertices, we have $w(\Gamma^*) \geq 4$ and $w(\Gamma^{**}) \geq 4$. Thus by (3), we have $w(\Gamma^*) = 4$ and $w(\Gamma^{**}) = 4$. By Lemma 11(a) and (b), each of Γ^* and Γ^{**} is of type (4) or $(2, 2)$. Therefore we complete the proof of Corollary 2. □

Proof of Corollary 3. Let Γ be a minimal n-chart with $w(\Gamma) = 8$ of type $(n_1, n_2, \cdots, n_p)$ such that Γ is not zero at any label. Then

(1) $n_1 + n_2 + \cdots + n_p = w(\Gamma) = 8$,

(2) $n_i \geq 1$ for each i $(1 \leq i \leq p)$.

By Lemma 11(a), we have $n_1 \geq 2$ and $n_p \geq 2$. If necessary we change all the edges of label k to ones of label $n-k$ for each $k = 1, 2, \cdots, n-1$ simultaneously, we can assume that

(3) $n_1 \geq n_p \geq 2$.

There are four cases: (i) $p = 1$ or 2, (ii) $p = 3$, (iii) $p = 4$, (iv) $p \geq 5$.
Case (i). If $p = 1$, then Γ is of type (8). If $p = 2$, then by (1) and (3) we have that the chart Γ is of type $(6, 2)$, $(5, 3)$ or $(4, 4)$.
Case (ii). Suppose $p = 3$. If $n_3 = 2$ or 3, then $n_2 = n_{p-1} \geq 2$ by Lemma 11(b) and (c). Thus by (1) and (3) we have that the chart Γ is of type $(4, 2, 2)$, $(3, 3, 2)$, $(2, 4, 2)$ or $(3, 2, 3)$.

If $n_3 \geq 4$, then by (2) and (3) we have that $w(\Gamma) = n_1 + n_2 + n_3 \geq 4+1+4 = 9$. This contradicts the fact $w(\Gamma) = 8$.
Case (iii). Suppose $p = 4$. If $n_4 = 2$ or 3, then $n_3 = n_{p-1} \geq 2$ by Lemma 11(b) and (c). Thus by (1), (2) and (3) we have that the chart Γ is of type $(2, 2, 2, 2)$, $(2, 1, 3, 2)$ or $(3, 1, 2, 2)$. However if $n_1 = 2$ or 3, then $n_2 \geq 2$ by Lemma 11(b) and (c). Hence Γ is of type $(2, 2, 2, 2)$.

If $n_4 \geq 4$, then by (2) and (3) we have that $w(\Gamma) = n_1 + n_2 + n_3 + n_4 \geq 4 + 1 + 1 + 4 = 10$. This contradicts the fact $w(\Gamma) = 8$.
Case (iv). Suppose $p \geq 5$. There are two cases: (iv-1) $n_p = 2$ or 3, (iv-2) $n_p \geq 4$.
Cases (iv-1). We have $n_{p-1} \geq 2$ by Lemma 11(b) and (c). Thus by (2), (3) we have that $w(\Gamma) = n_1 + n_2 + \cdots + n_{p-1} + n_p \geq n_1 + n_2 + n_3 + n_{p-1} + n_p \geq n_1 + n_2 + 1 + 2 + 2 = n_1 + n_2 + 5$. Hence

$$w(\Gamma) \geq n_1 + n_2 + 5.$$

If $n_1 = 2$, then $n_2 \geq 2$ by Lemma 11(b). Thus $w(\Gamma) \geq 2 + 2 + 5 = 9$. This contradicts the fact $w(\Gamma) = 9$.

If $n_1 \geq 3$, then by (2) we have $w(\Gamma) \geq 3 + 1 + 5 = 9$. This contradicts the fact $w(\Gamma) = 9$.
Cases (iv-2). Since $n_p \geq 4$, by (2) and (3) we have that $w(\Gamma) = n_1 + n_2 + \cdots + n_{p-1} + n_p \geq n_1 + n_2 + n_3 + n_{p-1} + n_p \geq 4 + 1 + 1 + 1 + 4 = 11$. This contradicts the fact $w(\Gamma) = 9$. □

A chart Γ belongs to *the first class* provided that

(i) $w(\Gamma)$ is odd, and

(ii) there does not exist a minimal chart Γ' such that $w(\Gamma')$ is odd and less than $w(\Gamma)$.

Corollary 12. *If a minimal chart belongs to the first class, then the chart is not zero at any label. Namely if the type of the chart is $(m; n_1, n_2, \cdots, n_p)$, then for each $i = 1, 2, \cdots, p$, we have $n_i \neq 0$.*

Proof. Let Γ be a minimal chart belonging to the first class. Suppose that the chart is zero at label k. Then by Theorem 1 there exists a chart Γ' with the same C-type of Γ which is separable at label k. Thus there exist subcharts Γ^* and Γ^{**} such that

(1) Γ' is the product of the two charts Γ^* and Γ^{**},

(2) $w(\Gamma^*) \neq 0$ and $w(\Gamma^{**}) \neq 0$.

Then we have

(3) $w(\Gamma^*) + w(\Gamma^{**}) = w(\Gamma')$.

Since Γ' and Γ are same C-type, we have $w(\Gamma') = w(\Gamma)$. Since Γ belongs to the first class, we have that $w(\Gamma')(= w(\Gamma))$ is odd. Thus $w(\Gamma^*)$ or $w(\Gamma^{**})$ is odd less than $w(\Gamma)$. Since Γ belongs to the first class, either Γ^* or Γ^{**} is not a minimal chart. Hence Γ' is C-move equivalent to a chart Γ'' with $w(\Gamma'') < w(\Gamma') = w(\Gamma)$. Namely the chart Γ is C-move equivalent to Γ''. This contradicts the fact that Γ is minimal. Therefore Γ is not zero at any label. □

Proof of Corollary 4. There does not exist a minimal chart Γ with $w(\Gamma) = 1, 2$, or 3. Further there does not exist a minimal chart Γ with $w(\Gamma) = 5$ ([16]). Furthermore there does not exist a minimal chart Γ with $w(\Gamma) = 7$ ([9], [10], [11], [12], [13]). Hence any chart with nine white vertices belongs to the first class. Thus any minimal chart with nine white vertices is not zero at any label by Corollary 12.

Let Γ be a minimal chart with $w(\Gamma) = 11$. Suppose that Γ is zero at label k. Then by Theorem 1 there exists a chart Γ' with the same C-type of Γ which is separable at label k. Here $w(\Gamma') = w(\Gamma)$. Hence there exist two subcharts Γ^*, Γ^{**} such that

(1) Γ' is the product of the two charts Γ^* and Γ^{**},

(2) $w(\Gamma^*) \neq 0$ and $w(\Gamma^{**}) \neq 0$.

Then we have

(3) $w(\Gamma^*) + w(\Gamma^{**}) = w(\Gamma') = 11$.

There are five cases:

(i) one of $w(\Gamma^*)$ and $w(\Gamma^{**})$ equals 1 and the other equals 10,

(ii) one of $w(\Gamma^*)$ and $w(\Gamma^{**})$ equals 2 and the other equals 9,

(iii) one of $w(\Gamma^*)$ and $w(\Gamma^{**})$ equals 3 and the other equals 8,

(iv) one of $w(\Gamma^*)$ and $w(\Gamma^{**})$ equals 4 and the other equals 7, and

(v) one of $w(\Gamma^*)$ and $w(\Gamma^{**})$ equals 5 the other equals 6.

But in any case, either Γ^* or Γ^{**} is not minimal. Hence Γ' is C-move equivalent to a chart Γ'' with $w(\Gamma'') < w(\Gamma') = w(\Gamma)$. Namely the chart Γ is C-move equivalent to Γ''. This contradicts the fact that Γ is minimal. Therefore any minimal chart Γ with $w(\Gamma) = 11$ is not zero at any label. □

Proof of Corollary 5. Let Γ be a minimal n-chart with $w(\Gamma) = 9$ of type $(n_1, n_2, \cdots, n_p)$. Then

(1) $n_1 + n_2 + \cdots + n_p = w(\Gamma) = 9$.

By Corollary 4, we have

(2) $n_i \geq 1$ for each i $(1 \leq i \leq p)$.

By Lemma 11(a), we have $n_1 \geq 2$ and $n_p \geq 2$. If necessary we change all the edges of label k to ones of label $n-k$ for each $k = 1, 2, \cdots, n-1$ simultaneously, we can assume that

(3) $n_1 \geq n_p \geq 2$, and

(4) if $n_1 = n_p$ and $p \geq 4$, then $n_2 \geq n_{p-1}$.

If $p = 1$, then Γ is of type (9). If $p = 2$, then by (1) and (3) the chart Γ is of type $(7,2)$, $(6,3)$ or $(5,4)$.

Suppose $p \geq 3$. There are two cases: (i) $n_p = 2$ or 3, (ii) $n_p \geq 4$.

Case (i). By Lemma 11(b) and (c), we have $n_{p-1} \geq 2$. There are four cases: (i-1) $p = 3$, (i-2) $p = 4$, (i-3) $p = 5$, (i-4) $p \geq 6$.

Case (i-1). Suppose $p = 3$. By (1), (3) and $n_2 = n_{p-1} \geq 2$, the chart Γ is of type $(5,2,2)$, $(4,3,2)$, $(3,4,2)$, $(2,5,2)$, $(4,2,3)$ or $(3,3,3)$.

Case (i-2). Suppose $p = 4$. If $n_1 = 2$ or 3, then $n_2 \geq 2$ by Lemma 11(b) and (c). Thus by (1), (3), (4) and $n_3 = n_{p-1} \geq 2$, the chart Γ is of type $(2,3,2,2)$ or $(3,2,2,2)$.

If $n_1 = 4$, by (1), (2), (3) and $n_3 = n_{p-1} \geq 2$, then Γ is of type $(4,1,2,2)$

If $n_1 \geq 5$, then by (2), (3) and $n_3 = n_{p-1} \geq 2$, we have $w(\Gamma) = n_1+n_2+n_3+n_4 \geq 5+1+2+2 = 10$. This contradicts the fact $w(\Gamma) = 9$.

Case (i-3). Suppose $p = 5$. If $n_1 = 2$, then $n_2 \geq 2$ by Lemma 11(b). Thus by (1), (2), (3) and $n_4 = n_{p-1} \geq 2$, we have that the chart Γ is of type $(2,2,1,2,2)$.

If $n_1 = 3$, then $n_2 \geq 2$ by Lemma 11(c). Thus by (2), (3) and $n_4 = n_{p-1} \geq 2$ we have $w(\Gamma) = n_1 + n_2 + n_3 + n_4 + n_5 \geq 3+2+1+2+2 = 10$. This contradicts the fact $w(\Gamma) = 9$.

If $n_1 \geq 4$, then by (2), (3) and $n_4 = n_{p-1} \geq 2$ we have $w(\Gamma) = n_1 + n_2 + n_3 + n_4 + n_5 \geq 4+1+1+2+2 = 10$. This contradicts the fact $w(\Gamma) = 9$.

Case (i-4). Suppose $p \geq 6$. If $n_1 = 2$ or 3, then $n_2 \geq 2$ by Lemma 11(b) and (c). Thus by (2), (3) and $n_{p-1} \geq 2$, we have $w(\Gamma) = n_1 + n_2 + \cdots + n_{p-1} + n_p \geq n_1 + n_2 + n_3 + n_4 + n_{p-1} + n_p \geq 2+2+1+1+2+2 = 10$. This contradicts the fact $w(\Gamma) = 9$.

If $n_1 \geq 4$, then by (2), (3) and $n_{p-1} \geq 2$ we have $w(\Gamma) = n_1 + n_2 + \cdots + n_{p-1} + n_p \geq n_1 + n_2 + n_3 + n_4 + n_{p-1} + n_p \geq 4+1+1+1+2+2 = 11$. This contradicts the fact $w(\Gamma) = 9$.

Case (ii). If $p = 3$, then by (1), (2) and (3) we have Γ is of type $(4,1,4)$.

If $p \geq 4$, then by (2) and (3) we have $w(\Gamma) = n_1 + n_2 + \cdots + n_{p-1} + n_p \geq n_1 + n_2 + n_3 + n_p \geq 4+1+1+4 = 10$. This contradicts the fact $w(\Gamma) = 9$. □

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