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On the Ruled Surfaces with L_1-Pointwise 1-Type Gauss Map

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Abstract. In this paper, we study ruled surfaces in 3-dimensional Euclidean and Minkowski space in terms of their Gauss map. We obtain classification theorems for these type of surfaces whose Gauss map G satisfying $\Box G = f(G+C)$ for a constant vector $C \in \mathbb{E}^3$ and a smooth function f, where $\Box$ denotes the Cheng-Yau operator.

1. Introduction

Let M be a hypersurface of the Euclidean space $\mathbb{E}^{n+1}$. A smooth mapping $\phi : M \to \mathbb{E}^N$ is said to be of k-type if it can be expressed as a sum of eigenvectors of Laplace operator Δ corresponding to k distinct eigenvalues of Δ ([7]). If ϕ is an immersion from M into $\mathbb{E}^{n+1}$ is of k-type, then M itself is said to be of k-type ([3]). The study of finite type mappings was summed up in a report by B.-Y. Chen ([4]).

On the other hand, if the Gauss map G of M is of 1-type, then it satisfies

$$\Delta G = \lambda(G + C) \tag{1.1}$$

for a constant $\lambda \in \mathbb{R}$ and a constant vector C. In this case, M is said to have 1-type Gauss map, [8]. However, Gauss map of some important submanifolds such as a

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helicoid and a catenoid in $\mathbb{E}^3$ and several rotational surfaces in $\mathbb{E}^4$ satisfy a very similar equation to (1.1), namely

$$\Delta G = f(G + C) \tag{1.2}$$

for some function $f \in C^\infty(M)$ and a constant vector C, ([11, 12]). These submanifolds whose Gauss map G satisfying (1.2) is said to have pointwise 1-type Gauss map. Submanifolds with pointwise 1-type Gauss map have been worked in several papers (cf. [5, 11, 16, 17, 18, 20, 21]).

In the recent years, the definition of being k-type of an hypersurface is extended in a natural way by replacing Laplace operator Δ with a sequence operators $L_0, L_1, L_2, \ldots, L_k$ such that $L_0 = -\Delta$, where L_k is the linearized operator of the first variation of the $(k+1)$-th mean curvature arising from normal variations of a hypersurface M of the Euclidean space $\mathbb{E}^{n+1}$. For convenience, the notation $\Box$ is used to denote the operator L_1 which is called as the Cheng-Yau operator introduced in [9]. The authors Alías et al. studied an isometric immersion $x : M^n \to R^{n+1}$ satisfying $L_k(x) = Ax + b$ for a constant matrix A and a constant vector b, where k is a positive integer.

In [15], the authors give the following definition.

Definition 1.([15]) An oriented surface M of Euclidean space $\mathbb{E}^3$ is said to have $\Box$-pointwise 1-type Gauss map if its Gauss map satisfies

$$\Box G = f(G + C) \tag{1.3}$$

for a smooth function $f \in C^\infty(M)$ and a constant vector $C \in \mathbb{E}^3$. More precisely, a $\Box$-pointwise 1-type Gauss map is said to be of the first kind if (1.3) is satisfied for $C = 0$; otherwise, it is said to be of the second kind. Moreover, if (1.3) is satisfied for a constant function f, then we say M has $\Box$-(global) 1-type Gauss map.

In the same paper, authors states

Open Problem. Classify surfaces in $\mathbb{E}^3$ with $\Box$-1-type Gauss map.

On the other hand, there are many studies done on rotational surfaces, ruled surfaces and translation surfaces in terms of being finite type or having pointwise 1-type Gauss map. For example, in [5] and [14], the rotational surfaces of the Euclidean 3-space $\mathbb{E}^3$ and the Minkowski 3-space $\mathbb{E}^3_1$ with (Δ-)pointwise 1-type Gauss map have been studied. Also, a classification of ruled surfaces in terms of their Gauss map was studied in [6] and [16].

In this paper, we study rotational surfaces, ruled surfaces and translation surfaces in $\mathbb{E}^3$ and $\mathbb{E}^3_1$ with $\Box$-pointwise 1-type Gauss map.

2. Prelimineries

Let $q \in \{0, 1\}$ and $\mathbb{E}^3_q$ denote the 3-dimensional semi-Euclidean space with the canonical semi-Euclidean metric tensor of index q given by

$$g = \langle\ ,\ \rangle = (-1)^q dx_1^2 + dx_2^2 + dx_3^2.$$

A non-zero vector u in the Minkowski space $\mathbb{E}^3_1$ is called space-like (resp., time-like or light-like) if $\langle u,u\rangle > 0$ (resp., $\langle u,u\rangle < 0$ or $\langle u,u\rangle = 0$). Furthermore, a curve β is called space-like (resp., time-like or light-like) if its tangent vector β' is space-like (resp., time-like or light-like) at every point.

On the other hand, a two dimensional subspace U of $\mathbb{E}^3_1$ is called non-degenerate if $U \cap U^{\perp} = \{0\}$ and a non-degenerate subspace U of index r is called space-like (resp., time-like) if $r = 0$ (resp., $r = 1$). Eventually, a surface M in $\mathbb{E}^3_1$ is called non-degenerate, (resp., degenerate, space-like or time-like) if its tangent space T_pM is non-degenerate, (resp., degenerate, space-like or time-like) at every point $p \in M$.

The following lemmas are well-known and useful (see, for instance [13]):

Lemma 2.1. *Let u, v be two orthogonal vectors in $\mathbb{E}^3_1$. If u is time-like, then v is space-like.*

Lemma 2.2. *Two light-like vectors are orthogonal if and only if they are linearly dependent.*

Lemma 2.3. *A two dimensional subspace U of $\mathbb{E}^3_1$ is a time-like space if and only if it contains two linearly independent lightlike vectors.*

2.1. Surfaces in 3-dimensional Euclidean and Minkowski spaces

Let M be an oriented surface in $\mathbb{E}^3_q$. We denote the Levi-Civita connections of $\mathbb{E}^3_1$ and M by $\widetilde{\nabla}$ and ∇, respectively and D stands for the normal connection of M. We put

$$\varepsilon = \begin{cases} \ \ 1 & \text{if } M \text{ is space-like} \\ -1 & \text{if } M \text{ is time-like} \end{cases}.$$

Then, we have $\langle N, N\rangle = (-1)^q \varepsilon$, where N is the unit normal vector field associated with the orientation of M. The mapping $G : M \to \mathbb{E}^3_q$ which assigns every point p to $N(p)$ is called the Gauss map of M.

The well-known Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\widetilde{\nabla}_X N = -S(X) \tag{2.2}$$

for tangent vector fields $X,\ Y$ of M, where h is the second fundamental form and S is the shape operator of M. The covariant derivative of h is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

Then, the Codazzi equation is given by

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z) \tag{2.3}$$

for tangent vector fields $X,\ Y,\ Z$ of M. Note that S and h satisfy $\langle S(X), Y\rangle = \langle h(X,Y), N\rangle$.

The functions $\mathcal{Q}$, H and K defined by $\mathcal{Q}(\lambda) = \det(S - \lambda I) = \lambda^2 - 2H\lambda + K$ are called the characteristic polynomial of S, the mean curvature of M and the Gaussian curvature of M, respectively. M is said to be minimal (resp., flat) if H (resp., K) vanishes identically. Sometimes the (complex valued) functions λ_1 and λ_2 satisfying $\mathcal{Q}(\lambda_i) = 0,\ i = 1, 2$ are called the principal curvatures of M.

The Gauss equation is given by

$$R(e_1, e_2, e_2, e_1) = K, \tag{2.4}$$

where R is the curvature tensor associated with connection ∇ and e_1, e_2 are orthonormal vector fields on M.

We will use $\chi(M)$ to denote the space of all smooth functions from M into $\mathbb{E}^3_q$ and $C^\infty(M)$ the space of all smooth functions defined on M. Let $\mathbb{B} = \{e_1, e_2, e_3\}$ be an orthonormal frame field defined on M, i.e., $\langle e_1, e_1\rangle = \varepsilon$, $\langle e_2, e_2\rangle = 1$, $e_3 = N$ and $\langle e_i, e_j\rangle = 0$ for $i \neq j$ $(i, j = 1, 2, 3)$. If $X \in \chi(M)$ is tangent to M, its divergence $\mathrm{div}X$ is defined by $\mathrm{div}X = \varepsilon\langle\nabla_{e_1}X, e_1\rangle + \langle\nabla_{e_2}X, e_2\rangle$. On the other hand, the gradient of a function $f \in C^\infty(M)$ is given by $\nabla f = \varepsilon e_1(f)e_1 + e_2(f)e_2$ and the Laplace operator acting on M is given as $\Delta = -\varepsilon\nabla_{e_1}e_1 - \nabla_{e_2}e_2 + e_1e_1 + e_2e_2$.

2.2. Surfaces with $\Box$-pointwise 1-type Gauss map

Let M be a surface in $\mathbb{E}^3_q$ and P_0, P_1 the Newton transformations given by $P_0 = I$, $P_1 = 2HI - S$, where I is the identity operator acting on the tangent bundle of M. Then, the second order differential operators $L_k : C^\infty(M) \to C^\infty(M)$ associated with P_k are given by $L_k(f) = tr(P_k \circ \nabla^2 f),\ k = 1, 2$. Note that we have $L_0 = -\Delta$ and $L_1 = \Box$, where $\Box$ is the Cheng-Yau operator introduced in [9]. As a matter of fact, it turns out to be

$$L_k f = \mathrm{div}\left(P_k\left(\nabla f\right)\right) \tag{2.5}$$

for $f \in C^\infty(M)$ ([2]).

We will use following lemma and theorems in [15].

Lemma 2.4.([15]) *Let M be an oriented surface in $\mathbb{E}^3$ with Gaussian curvature K and mean curvature H. Then, the Gauss map G of M satisfies*

$$\Box G = -\nabla K - 2HKG. \tag{2.6}$$

Theorem 2.5.([15]) *An oriented surface M in $\mathbb{E}^3$ has $\Box$-harmonic Gauss map if and only if it is flat, i.e, its Gaussian curvature vanishes identically.*

Theorem 2.6.([15]) *An oriented surface M in $\mathbb{E}^3$ has $\Box$-pointwise 1-type Gauss map of the first kind if and only if it has constant Gaussian curvature.*

Theorem 2.7.([15]) *An oriented minimal surface M in $\mathbb{E}^3$ has $\Box$-pointwise 1-type Gauss map if and only if it is an open part of a plane.*

3. Ruled Surfaces in $\mathbb{E}^3$

Let M be a ruled surface in $\mathbb{E}^3$ given by (2.5). Then, as a surface, we have $x_t = \beta \neq 0$ and $x(s,t) = \alpha + \tilde{t}\tilde{\beta}$ where $\tilde{t} = t\langle \beta, \beta\rangle^{1/2}$ and $\tilde{\beta} = \beta/\langle \beta, \beta\rangle^{1/2}$. Thus, without loss of generality, we may assume $\langle \beta, \beta\rangle = 1$. By re-defining s appropriately, we also suppose that $\langle \beta', \beta'\rangle = 1$. Moreover, for another base curve $\bar{\alpha}$ of M given by $\bar{\alpha}(s) = \alpha(s) + g(s)\beta(s)$ with $g'(s) + \langle \alpha'(s), \beta(s)\rangle = 0$ we have $\langle \bar{\alpha}', \beta\rangle = 0$. Hence, without loss of generality, we may also assume $\langle \alpha', \beta\rangle = 0$.

Because of these assumptions, we have

$$\alpha' = a\beta' + b\beta \wedge \beta' \tag{3.1}$$

and

$$\beta'' = -\beta + c\beta \wedge \beta'$$

for some smooth functions $a = a(s)$, $b = b(s)$ and $c = c(s)$.

We choose an orthonormal frame field as

$$e_1 = \frac{1}{E}\partial_s, \tag{3.2a}$$

$$e_2 = \partial_t, \tag{3.2b}$$

$$G = \frac{1}{E}(b\beta' - (a+t)\beta \wedge \beta') \tag{3.2c}$$

where

$$E = \sqrt{b^2 + (a+t)^2}. \tag{3.3}$$

By a direct calculation, we obtain the connection form ω_1^2 as

$$\omega_1^2 = w\theta_1, \quad w = -\frac{a+t}{E^2}, \tag{3.4}$$

where $\{\theta_1,\ \theta_2\}$ is the dual base of $\{e_1,\ e_2\}$. Moreover, the Gaussian curvature K and the mean curvature H are given by

$$K = -h_2^2, \tag{3.5}$$

$$H = h_1/2, \tag{3.6}$$

where

$$h_1 = \langle h(e_1, e_1), G\rangle = \frac{b(a' - bc) - b'(a+t) - c(a+t)^2}{E^3}, \tag{3.7}$$

$$h_2 = \langle h(e_1, e_2), G\rangle = \frac{b}{E^2}. \tag{3.8}$$

On the other hand, from the Codazzi equation (2.3) and the Gauss equation (2.4) we obtain

$$w_t = w^2 + K, \tag{3.9}$$

$$h_{1,t} = \frac{h_{2,s}}{E} + wh_1, \tag{3.10}$$

$$h_{2,t} = 2wh_2. \tag{3.11}$$

By using (3.5) and (3.11), we obtain

$$K_t = 4wK. \tag{3.12}$$

3.1. Ruled surfaces with □-pointwise 1-type Gauss map of the first kind

We first give the following theorem:

Theorem 3.1. *Let M be a non-cylindrical ruled surface whose position vector given by* (2.5)*. Then, the following statements are equivalent:*

(1) *M has □-pointwise 1-type Gauss map of the first kind.*

(2) *M has □-harmonic Gauss map.*

(3) *$\alpha' = a\beta'$ for a smooth function a.*

Proof. (1) ⇔ (2) : $K_t = 0$ implies $Kw = 0$ because of (3.12). Thus, if $K_t = 0$ and $K \neq 0$ at a point p of M, then there exists a neighborhood $\mathcal{N}_p$ of p such that $\omega\Big|_{\mathcal{N}_p} = 0$ which is not possible because of (3.4). Therefore, we have if K is constant, then $K = 0$. Hence, from Theorem 2.5 and Theorem 2.6 we obtain (1) ⇔ (2).

(2) ⇔ (3) : Because of (3.5) and (3.8), M is flat if and only if $b \equiv 0$ which is equivalent to $\alpha' = a\beta'$ because of (3.1). □

3.2. Ruled surfaces with □-pointwise 1-type Gauss map of the second kind

Theorem 3.2. *A ruled surface in $\mathbb{E}^3$ has □-pointwise 1-type Gauss map of the second kind if and only if M is flat.*

Proof. Let M be a ruled surface in $\mathbb{E}^3$ given by (2.5) with □-pointwise 1-type Gauss map of the second kind. Then, there exist a function f and a vector $C = C_1e_1 + C_2e_2 + C_3G$ such that

$$fC_1 = -\frac{K_s}{E}, \tag{3.13a}$$

$$fC_2 = -K_t, \tag{3.13b}$$

$$f(C_3 + 1) = -2KH. \tag{3.13c}$$

Note that from (3.12) and (3.13b) we obtain

$$fC_2 = -4Kw. \tag{3.14}$$

On the other hand, by using Gauss and Weingarten formulas, we obtain

$$\nabla_{\partial_t} C = (C_{1,t})e_1 + (C_{2,t})e_2 + (C_{3,t})G + C_1h_2G - C_3h_2e_1.$$

Thus, $\nabla_{\partial_t} C = 0$ implies

$$C_{1,t} = h_2 C_3, \tag{3.15a}$$
$$C_{2,t} = 0, \tag{3.15b}$$
$$C_{3,t} = -h_2 C_1. \tag{3.15c}$$

Now, we assume towards a contradiction that M is not flat, i.e., the open subset $\mathcal{M} = \{p \in M | K(p) \neq 0\}$ of M is not empty. By multiplying both sides of (3.13c) by C_2 and using (3.6) and (3.14), we obtain $K\left(4w(C_3+1) - h_1 C_2\right) = 0$ from which we get

$$4w(C_3+1) = h_1 C_2$$

on $\mathcal{M}$. By taking derivative of this equation and using (3.9), (3.10), we obtain

$$4(w^2+K)(C_3+1) - 4wh_2C_1 = \left(\frac{h_{2,s}}{E} + wh_1\right) C_2$$

on $\mathcal{M}$. Next, we multiply both sides of this equation by f and use (3.13a), (3.13c) and (3.14) to obtain $K\left(h_2^2 h_1 E + 3w\frac{\partial h_2}{\partial s}\right) = 0$ from which we get

$$h_2^2 h_1 E + 3w\frac{\partial h_2}{\partial s} = 0 \tag{3.16}$$

on $\mathcal{M}$.

By using (3.3), (3.4), (3.7) and (3.8) in (3.16), we obtain

$$b^3(a' - bc) + 2b^2 b'(a+t) + (-cb^2 + 6a'b)(a+t)^2 - 3b'(a+t)^3 = 0 \tag{3.17}$$

on $\mathcal{M}$, from which, we obtain b is a constant and

$$b(a' - bc) = 0, \tag{3.18a}$$
$$b(6a' - bc) = 0. \tag{3.18b}$$

Note that if $b = 0$, then (3.1) implies $\alpha' = a\beta'$ and from Theorem 3.1 we have $\mathcal{M}$ is flat which yields a contradiction. Thus, we have $b \neq 0$.

From (3.18) we have $a' = c = 0$. Therefore, (3.6) and (3.7) imply that $\mathcal{M}$ is minimal. However, Theorem 2.7 implies that $\mathcal{M}$ is an open part of a plane which is contradiction since $K \neq 0$ on $\mathcal{M}$. Hence we have $\mathcal{M}$ is an empty set, i. e., M is flat.

The converse is obvious. □

3.3. Ruled surfaces with $\Box G = AG$ for a matrix $A \in \mathbb{R}^{3x3}$

In this section, we suppose that M is a ruled surface whose Gauss map satisfies $\Box G = AG$ for some matrix $A \in \mathbb{R}^{3x3}$ with real entities. From this equation and (2.6) we obtain

$$-AG = e_1(K)e_1 + e_2(K)e_2 + 2KHG.$$

By taking covariant derivative of this equation on the direction e_2 we have

$$-\tilde{\nabla}_{e_2}(AG) = ASe_2 = (e_2e_1(K) - 2KHh_2)\, e_1 + e_2e_2(K)e_2 + (h_2e_1(K) + 2e_2(KH))\, G$$

as $\nabla_{e_2}e_1 = \nabla_{e_2}e_2 = 0$ and $h(e_2, e_2) = 0$. From this equation we obtain

$$h_2\langle Ae_1, e_2\rangle = K_{tt}. \tag{3.19}$$

Note that, by using (3.9) and (3.12), one can obtain $K_{tt} = 20w^2K + 4K^2$. On the other hand, by using (3.2) we obtain

$$\langle Ae_1, e_2\rangle = \frac{1}{E}\left((a+t)\langle A\beta', \beta\rangle + b\langle A(\beta \wedge \beta'), \beta\rangle\right).$$

From this equation and (3.19) we have

$$\langle A\beta', \beta\rangle E^5(a+t) + b\langle A(\beta \wedge \beta'), \beta\rangle E^5 + 20b(a+t) - 4b^3 = 0$$

which implies $b = 0$, i. e., M is flat. Hence, we have

Theorem 3.3. *The Gauss map G of a ruled surface in $\mathbb{E}^3$ satisfies $\Box G = AG$ for a matrix $A \in \mathbb{R}^{3x3}$ if and only if M is flat.*

Combining Theorem 3.2 and Theorem 3.3, we obtain

Theorem 3.4. *Let M be a non-cylindrical ruled surface whose position vector given by* (2.5)*. Then, the following statements are equivalent:*

(1) *M has $\Box$-pointwise 1-type Gauss map of the second kind.*

(2) *The Gauss map G of M satisfies $\Box G = AG$ for a matrix $A \in \mathbb{R}^{3x3}$.*

(3) *M is flat.*

4. Null scrolls in $\mathbb{E}^3_1$

A non-degenerate ruled surface M in $\mathbb{E}^3_1$ given by (2.5) is called a null scroll if $\langle\beta, \beta\rangle = \langle\alpha', \alpha'\rangle = 0$ and $\langle\alpha', \beta\rangle \neq 0$. In this case, without loss of generality we may assume that $\langle\alpha', \beta\rangle = 1$. Furthermore, we may choose an appropriate parameter s in such a way that $\langle\alpha', \beta'\rangle = 0$, which is possible if the base curve α is chosen as a null geodesic of M.

On the other hand, if $\langle\beta', \beta'\rangle = 0$ at an open subset $\mathcal{M}$ of M, then there exists a function a such that $\beta' = a\beta$ which implies $\beta = (\beta_1, \beta_2, \beta_3)$ is $\beta = \beta_1 c_0$ for a constant light-like vector $c_0 \in \mathbb{E}^3_1$. Hence, we may assume $\beta = c_0$ which implies that $\mathcal{M}$ is cylindirical. Therefore, we may locally assume $\langle\beta', \beta'\rangle = E^2 > 0$.

The tangent vector fields $e_1 = -\partial_s + (t^2E^2/2)\partial_t$ and $e_2 = \partial_t$ form a pseudo-orthonormal frame field and the unit nomal vector field is $N = -E^{-1}\beta' + tE\beta$. By a direct calculation, we obtain

$$Se_1 = -Ee_1 + be_2, \qquad Se_2 = -Ee_2, \tag{4.1a}$$

$$\widetilde{\nabla}_{e_1}e_2 = -tE^2e_2 + EN, \qquad \widetilde{\nabla}_{e_2}e_2 = 0 \tag{4.1b}$$

for a non-vanishing function b. From (4.1a) we have $H = E$ which implies

$$P_1 = -2EI - S. \tag{4.2}$$

4.1. Gauss map of null scrolls

In the next lemma, we obtain $\Box G$ for a null scroll in $\mathbb{E}^3_1$.

Lemma 4.1. *Let M be a null scroll in $\mathbb{E}^3_1$. Then, the Gauss map G of M satisfies*

$$\Box G = -2EE'e_2 + 2E^3 G. \tag{4.3}$$

Proof. Let C be a constant vector in $\mathbb{E}^3_1$. By a direct computation, we obtain

$$\nabla\langle G, C\rangle = -E\langle e_2, C\rangle e_1 - E\langle e_1, C\rangle e_2 + b\langle e_2, C\rangle e_2.$$

By considering (4.2), we get

$$P_1(\nabla\langle G, C\rangle) = E^2\langle e_2, C\rangle e_1 + E^2\langle e_1, C\rangle e_2.$$

By using this equation and (2.5), we obtain

$$\begin{aligned} \langle \Box G, C\rangle &= -\langle \nabla_{e_1}\big(E^2\langle e_2, C\rangle e_1 + E^2\langle e_1, C\rangle e_2\big), e_2\rangle \\ &- \langle \nabla_{e_2}\big(E^2\langle e_2, C\rangle e_1 + E^2E^2\langle e_1, C\rangle e_2\big), e_1\rangle \\ &= e_1(E^2)\langle e_2, C\rangle + e_2(E^2)\langle e_1, C\rangle + 2E^2\langle h(e_1, e_2), C\rangle \\ &= \langle -2EE'e_2 + 2E^3G, C\rangle. \end{aligned} \tag{4.4}$$

Thus, we have (4.3). □

Example 1. If $\alpha(s)$ is a null curve in $\mathbb{E}^3_1$ with the Cartan frame $\{A, B, C\}$ such that $\langle A, A\rangle = \langle B, B\rangle = 0$, $\langle A, B\rangle = -1$, $\langle A, C\rangle = \langle B, C\rangle = 0$ and $\langle C, C\rangle = 1$ with $\alpha' = A$, $A' = k_1(s)C$ and $B' = k_2C$ for a constant k_2 and a smooth function k_1 and $\beta(s) = B(s)$, then the null scroll given by (2.5) is said to be a B-scroll. It is well-known that a null scroll M is a B-scroll if and only if E is a constant (see [19]). In this case, the Gauss map of M satisfies

$$\Box G = 2E^3 G$$

because of (4.3) which implies M has $\Box$-1-type Gauss map of the first kind.

Next, we want to give classification of null scrolls in $\mathbb{E}^3_1$ with $\Box$-pointwise 1-type Gauss map.

Proposition 4.2. *A null scroll in $\mathbb{E}^3_1$ has $\Box$-pointwise 1-type Gauss map if and only if it is a B-scroll.*

Proof. Let M be a null scroll in $\mathbb{E}^3_1$ with $\Box$-pointwise 1-type Gauss map. Then, the Gauss map G of M satisfies

$$-2EE'e_2 + 2E^3G = f(G + C) \tag{4.5}$$

for a constant vector C and a smooth function f. From (4.5) we have

$$f\langle C, e_2\rangle = 0. \tag{4.6}$$

Now, we consider the open subset $\mathcal{M} = \{p \in M | f(p) \neq 0\}$ of M on which $\langle C, e_2\rangle = 0$ is satisfied. From this equation we get

$$e_1\big(\langle e_2, C\rangle\big) = 0 \tag{4.7}$$

on $\mathcal{M}$. By a further calculation taking into account of Gauss formula (2.1), (4.6) and (4.7), we obtain

$$\langle G, C\rangle = 0. \tag{4.8}$$

By combining (4.6) and (4.8), we obtain $C = C_1e_2 = C_1\beta$. From which we get $C = 0$. Thus, (4.5) implies E is constant. Hence M is a B-scroll.

The converse is given in Example 1. □

Now, we obtain the following proposition.

Proposition 4.3. *Let M be a null scroll in $\mathbb{E}^3_1$. Then, its Gauss map satisfies $\Box G = AG$ for a constant 3×3-matrix A if and only if M is a B-scroll.*

Proof. Suppose the the Gauss map G of M satisfies $\Box G = AG$ for a constant 3×3-matrix A. Then, we have

$$-2EE'e_2 + 2E^3G = AG, \tag{4.9}$$

from which, we get

$$-2EE'\widetilde{\nabla}_{e_2}e_2 + 2E^3\widetilde{\nabla}_{e_2}G = A\Big(\widetilde{\nabla}_{e_2}G\Big).$$

By using (4.1), we obtain

$$Ae_2 = 2E^3e_2 \tag{4.10}$$

from which we get $A\Big(\widetilde{\nabla}_{e_1}e_2\Big) = 2E^3\widetilde{\nabla}_{e_1}e_2$. From this equation and (4.1b) we obtain

$$-tE^2Ae_2 + EAG = E^3(-tE^2e_2 + EG).$$

By combining this equation with (4.9) and (4.10), we obtain $EE' = 0$ which implies E is constant. Hence, M is a B-scroll.

The converse is given in Example 1. □

By combining Proposition 4.2 and Proposition 4.3 with the result of [1], we obtain the following theorem.

Theorem 4.4. *Let M be a null scroll in $\mathbb{E}^3_1$. Then the following conditions are equivalent.*

(i) *M has $\Box$-pointwise 1-type Gauss map.*

(ii) *The Gauss map G of M satisfies $\Delta G = AG$ for a constant 3×3-matrix A.*

(iii) *The Gauss map G of M satisfies $\Box G = AG$ for a constant 3×3-matrix A.*

(iv) *M is a B-scroll.*

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