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Some New Subclasses of Analytic Functions defined by Srivastava-Owa-Ruscheweyh Fractional Derivative Operator

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Abstract. In this article the Srivastava-Owa-Ruscheweyh fractional derivative operator $\mathcal{L}^{\alpha}_{a,\lambda}$ is applied for defining and studying some new subclasses of analytic functions in the unit disk E. Inclusion results, radius problem and other results related to Bernardi integral operator are also discussed. Some applications related to conic domains are given.

1. Introduction

Let $\mathcal{A}$ denote the class of all normalized functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \ z \in E, \tag{1.1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$.

Let $\mathcal{S}$, $\mathcal{C}$, $\mathcal{S}^*$, $\mathcal{C}(\beta)$ and $\mathcal{S}^*(\beta)$ denote the subclasses of $\mathcal{A}$ consisting of functions that are univalent, convex, starlike, convex of order β and starlike of order β in E respectively, see [10]. For the functions

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad \text{and} \quad g(z) = z + \sum_{j=2}^{\infty} b_j z^j, \ z \in E,$$

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the convolution (Hadamard product) is defined as

$$(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z), \; z \in E. \tag{1.2}$$

Let f and g be analytic functions in E. Then f is said to be subordinate to g written as $f \prec g$ and $f(z) \prec g(z), \; z \in E$, if there exits a Schwarz function ω analytic in E, with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in E$, such that$f(z) = g(\omega(z))$, $z \in E$.

Recently, the theory of fractional calculus has found interesting applications in the theory of analytic functions. The classical definitions of fractional operators and their generalizations have fruitfully been applied in obtaining, for example, the characterization properties, coefficient estimates, distortion inequalities and convolution structures for various subclasses of analytic functions.

The fractional derivative of order α, $0 \le \alpha < 1$ is defined in [23] as follows

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} \mathrm{d}t, \; 0 \le \alpha < 1, \; z \in E, \tag{1.3}$$

where the function $f(z)$ is analytic in a simply connected domain in the complex plane containing the origin and the multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $\log(z-t) \in \mathbb{R}$, whenever $(z-t) > 0$. The gamma function Γ is defined as

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} \mathrm{d}x, \quad \mathfrak{Re} z > 0.$$

Note that

$$D_z^0 f(z) = f(z).$$

Owa and Srivastava [24, 30] introduced the operator $\Omega^{\alpha} : \mathcal{A} \to \mathcal{A}$ as follows

$$\begin{aligned} \Omega^{\alpha} f(z) &= \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} f(z), \; 0 \le \alpha < 1, \\ &= z + \sum_{j=2}^{\infty} \frac{\Gamma(1+j)\Gamma(2-\alpha)}{\Gamma(1+j-\alpha)} a_j z^j, \; z \in E, \\ &= \phi(2, 2-\alpha; z) * f(z), \end{aligned} \tag{1.4}$$

where the incomplete beta function $\phi(a, c; z)$ is defined as follows

$$\phi(a, c; z) = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(c)}{\Gamma(c+j)\Gamma(a)} z^j,$$

where a, c are complex numbers different from $0, -1, -2, \ldots$. It can be seen that

$$\Omega^0 f(z) = f(z).$$

Mishra and Gochhayat [19] have studied some properties of the operator Ω^α and introduced new subclass of k-uniformly convex functions. In a recent paper Ibrahim and Darus [12] introduced the fractional differential subordination based on the operator Ω^α. Srivastava and Mishra [29] studied the applications of fractional calculus to the parabolic starlike and uniformly convex functions using the operator Ω^α.

Al-Oboudi [1, 2] defined the linear multiplier fractional differential operator of order 1 as follows

$$\begin{aligned} \mathfrak{D}_\lambda^\alpha f(z) &= (1-\lambda)\Omega^\alpha f(z) + \lambda z\left(\Omega^\alpha f(z)\right)' \\ &= z + \sum_{j=2}^{\infty} \frac{\Gamma(1+j)\Gamma(2-\alpha)}{\Gamma(1+j-\alpha)}(1+\lambda(j-1))a_j z^j, \\ &= \phi(2, 2-\alpha; z) * g_\lambda(z) * f(z),\ z \in E, \end{aligned} \tag{1.5}$$

where $0 \le \lambda \le 1$, $0 \le \alpha < 1$ and

$$g_\lambda(z) = \frac{z-(1-\lambda)z^2}{(1-z)^2},\ 0 \le \lambda \le 1. \tag{1.6}$$

Obviously

$$\mathfrak{D}_0^\alpha f(z) = \Omega^\alpha f(z),$$

where Ω^α was defined by (1.4) and

$$\mathfrak{D}_0^0 f(z) = \Omega^0 f(z) = f(z).$$

Bulut [5, 6, 7, 8] used the Al-Oboudi fractional differential operator of order n to define some new integral operators and obtain interesting results. Recently, Noor et al [22] used fractional derivative to define some new subclasses of analytic functions in the conic regions. Now for $a > 0$, let

$$\begin{aligned} k_a(z) &= \frac{z}{(1-z)^a} \\ &= z + \sum_{j=2}^{\infty} \frac{\Gamma(a+j-1)}{\Gamma(a)\Gamma(j)} z^j,\ z \in E. \end{aligned}$$

Ozkan [25] defined the convolution of f given by (1.1) and $k_a(z)$ such that

$$(k_a * f)(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(a+j-1)}{\Gamma(a)\Gamma(j)} a_j z^j,\ z \in E.$$

Now extending the concept of [4] and [25, 30], we define the Srivastava-Owa-

Ruscheweyh $\mathcal{L}^{\alpha}_{a,\lambda} : \mathcal{A} \to \mathcal{A}$ as follows

$$\begin{aligned} \mathcal{L}^{\alpha}_{a,\lambda} f(z) &= \frac{z}{(1-z)^a} * \mathfrak{D}^{\alpha}_{\lambda} f(z) \\ &= z + \sum_{j=2}^{\infty} \frac{\Gamma(a+j-1)\Gamma(2-\alpha)\Gamma(j+1)}{\Gamma(1+j-\alpha)\Gamma(a)\Gamma(j)} \{1+\lambda(j-1)\} a_j z^j \\ &= \frac{z}{(1-z)^a} * \phi(2, 2-\alpha; z) * g_{\lambda}(z) * f(z), \ z \in E, \end{aligned}$$

which is called the Srivastava-Owa-Ruscheweyh fractional derivative operator.

It can easily be seen that

(i) For $\alpha = 0, \ \lambda = 0.$

$$\mathcal{L}^{0}_{a,0} f(z) = (k_a * f)(z),$$

see [25].

(ii) For $a = n+1 > 0, \ \alpha = 0, \ \lambda = 0.$

$$\mathcal{L}^{0}_{n+1,0} f(z) = \frac{z}{(1-z)^{n+1}} * f(z),$$

see [28].

(iii) For $a = n+1 > 0, \ \alpha = 0.$

$$\mathcal{L}^{0}_{n+1,0} f(z) = \frac{z}{(1-z)^{n+1}} * \mathfrak{D}^{0}_{\lambda} f(z),$$

see [26].

From the definition of $\mathcal{L}^{\alpha}_{a,\lambda}$, we can establish the following identity as well

$$(a-1)(\mathcal{L}^{\alpha}_{a,\lambda} f(z)) + z(\mathcal{L}^{\alpha}_{a,\lambda} f(z))' = a(\mathcal{L}^{\alpha}_{a+1,\lambda} f(z)). \tag{1.7}$$

We assume that h is analytic, convex, univalent in E with $h(0) = 1$ and $\mathfrak{Re}\, h(z) > 0$, $z \in E$. Using the operator $\mathcal{L}^{\alpha}_{a,\lambda}$, we define the following

Definition 1.1. Let $\mathcal{P}^{\alpha}_{a,\lambda}(h,\delta)$ be the class of functions $f \in \mathcal{A}$ satisfying

$$\frac{z(\mathcal{L}^{\alpha}_{a,\lambda} f(z))' + \delta z^2 (\mathcal{L}^{\alpha}_{a,\lambda} f(z))''}{(1-\delta)(\mathcal{L}^{\alpha}_{a,\lambda} f(z)) + \delta z (\mathcal{L}^{\alpha}_{a,\lambda} f(z))'} \prec h(z),$$

for $a > 0$, $0 \le \delta \le 1$, $0 \le \lambda \le 1$, $0 \le \alpha < 1$ and for all $z \in E$.

We note that the class $\mathcal{P}^{0}_{a,0}(h,\delta) = \mathcal{P}_a(h,\delta)$, was studied by Ozkan in [25] and the class $\mathcal{P}^{0}_{a,0}(h,0) = \mathcal{S}_a(h)$, was studied by Padmanabhan and Parvatham in [26].

Obviously, for the special choices of function h and parameters α, a, λ, we have the following relationship.

$$\begin{aligned}
\mathcal{P}^0_{1,0}\left(\frac{1+z}{1-z},0\right) &= \mathcal{S}^*,\ \mathcal{P}^0_{2,0}\left(\frac{1+z}{1-z},0\right) = \mathcal{P}^0_{1,0}\left(\frac{1+z}{1-z},1\right) = \mathcal{C},\\
\mathcal{P}^0_{1,0}\left(\frac{1+(1-2\beta)\,z}{1-z},0\right) &= \mathcal{S}^*\left(\beta\right), \quad 0\le\beta<1,\\
\mathcal{P}^0_{2,0}\left(\frac{1+(1-2\beta)\,z}{1-z},0\right) &= \mathcal{P}^0_{1,0}\left(\frac{1+(1-2\beta)\,z}{1-z},1\right) = \mathcal{C}\left(\beta\right).
\end{aligned}$$

Definition 1.2. Let $\mathcal{T}^{\alpha}_{a,\lambda}(h,\gamma)$ denote the subclass of $\mathcal{A}$ consisting of the functions f which satisfies the following condition

$$(1-\gamma)\frac{(\mathcal{L}^{\alpha}_{a,\lambda}f(z))}{z}+\gamma(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'\prec h(z),$$

for some $a>0$, $0\le\lambda\le1$, $0\le\alpha<1$, $0\le\gamma\le1$ and for all $z\in E$. Note that

$$\mathcal{T}^{0}_{a,0}(h,\gamma)=\mathcal{T}_a(h,\gamma),$$

see [25].

Definition 1.3. Let $\mathcal{M}^{\alpha}_{a,\lambda}(h,\gamma)$ be the class of functions $f\in\mathcal{A}$ which satisfies the following condition

$$(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'+\gamma z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))''\prec h(z),$$

for some $a>0$, $0\le\lambda\le1$, $0\le\alpha<1$, $0\le\gamma\le1$ and for all $z\in E$.

2. Preliminaries

To prove our main results, we need the following Lemmas.

Lemma 2.1.([18]) *Let h be analytic, univalent, convex in E, with $h(0)=1$ and $\mathfrak{Re}[\beta_1 h(z)+\gamma_1]>0$, $\beta_1,\gamma_1\in\mathbb{C}$, $z\in E$. If p is analytic in E, with $p(0)=h(0)$, then*

$$p(z)+\frac{zp'(z)}{\beta_1 p(z)+\gamma_1}\prec h(z),$$

implies

$$p(z)\prec h(z).$$

Lemma 2.2.([11]) *Let h be analytic, univalent, convex in E, with $h(0)=1$. Let p be analytic in E, with $p(0)=h(0)$. If $\gamma_2\neq0$, $\mathfrak{Re}\gamma_2\ge0$ and*

$$p(z)+\frac{zp'(z)}{\gamma_2}\prec h(z),\ \ \gamma_2\neq0,\ \ z\in E,$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma_2}{z^{\gamma_2}} \int_0^z t^{\gamma_2 - 1} h(t) \mathrm{d}t.$$

Lemma 2.3.([9]) *If* $\psi \in \mathcal{C}$, $g \in \mathcal{S}^*$ *and* F *is analytic function with* $\mathfrak{Re}(F(z)) > 0$ *for* $z \in E$, *then*

$$\mathfrak{Re}\left(\frac{(\psi * Fg)(z)}{(\psi * g)(z)}\right) > 0, \ z \in E.$$

Lemma 2.4.([21]) *Let* $p(z)$ *and* $q(z)$ *be analytic in* E, *with* $p(0) = q(0) = 1$ *and* $\mathfrak{Re}(q(z)) > \frac{1}{2}$ *for* $|z| < \rho$, $(0 < \rho \leq 1)$. *Then the image of* $E_\rho = \{z : |z| < \rho\}$ *under* $p * q$ *is a subset of closed convex hull of* $p(E)$.

3. Main Results

In this section, we will prove our main results.

Theorem 3.1. *Assume that* $a \geq 1$, $0 \leq \delta \leq 1$, $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$ *and*

$$\frac{1}{z}\left\{(1-\delta)(\mathcal{L}^{\alpha}_{a,\lambda} f(z)) + \delta z(\mathcal{L}^{\alpha}_{a,\lambda} f(z))''\right\} \neq 0 \quad z \in E. \tag{3.1}$$

Then

$$\mathcal{P}^{\alpha}_{a+1,\lambda}(h,\delta) \subset \mathcal{P}^{\alpha}_{a,\lambda}(h,\delta), \ for \ all \ z \in E.$$

Proof. We suppose that $f \in \mathcal{P}^{\alpha}_{a+1,\lambda}(h,\delta)$ and let

$$p(z) = \frac{z(\mathcal{L}^{\alpha}_{a,\lambda} f(z))' + \delta z^2 (\mathcal{L}^{\alpha}_{a,\lambda} f(z))''}{(1-\delta)(\mathcal{L}^{\alpha}_{a,\lambda} f(z)) + \delta z(\mathcal{L}^{\alpha}_{a,\lambda} f(z))'}. \tag{3.2}$$

Then by (3.1) the function $p(z)$ is analytic in E, with $p(0) = 1$. From (1.7) and (3.2), we obtain

$$p(z) + \frac{zp'(z)}{p(z) + (a-1)} = \frac{az(\mathcal{L}^{\alpha}_{a+1,\lambda} f(z))' + \delta a z^2 (\mathcal{L}^{\alpha}_{a+1,\lambda} f(z))''}{(1-\delta)a(\mathcal{L}^{\alpha}_{a+1,\lambda} f(z)) + \delta a z(\mathcal{L}^{\alpha}_{a+1,\lambda} f(z))'}. \tag{3.3}$$

Since $f \in \mathcal{P}^{\alpha}_{a+1,\lambda}(h,\delta)$, therefore by using (3.2), we have

$$p(z) + \frac{zp'(z)}{p(z) + (a-1)} \prec h(z) \text{ in } E.$$

Thus it follows from the Lemma 2.1 that

$$p(z) \prec h(z) \text{ for } a \geq 1 \text{ in } E.$$

Hence $f \in \mathcal{P}^{\alpha}_{a,\lambda}(h,\delta)$ for $a \geq 1$. □

As special cases of Theorem 3.1, we have the following results.

Corollary 3.2.([25]) *If* (3.1) *is satisfied for* $\alpha = 0$ *and* $\lambda = 0$, *then* $\mathcal{P}^{0}_{a+1,0}(h,\delta) \subset \mathcal{P}^{0}_{a,0}(h,\delta)$.

Corollary 3.3. *If* (3.1) *is satisfied for* $\delta = 0$, $\alpha = 0$, $\lambda = 0$, $a = 1$ *and* $h(z) = \frac{1+z}{1-z}$, $\mathcal{P}^{0}_{2,0}\left(\frac{1+z}{1-z}, 0\right) \subset \mathcal{P}^{0}_{1,0}\left(\frac{1+z}{1-z}, 0\right)$. *That is* $\mathcal{C} \subset \mathcal{S}^*$, $z \in E$.

Theorem 3.4. *Assume that*

$$\frac{1}{z}\left\{(1-\delta)(\mathcal{L}^{\alpha}_{a,\lambda}F_c(z)) + \delta z(\mathcal{L}^{\alpha}_{a,\lambda}F_c(z))'\right\} \quad z \in E. \tag{3.4}$$

If $f \in \mathcal{P}^{\alpha}_{a,\lambda}(h,\delta)$ *for* $a \geq 1$, *then* $F_c \in \mathcal{P}^{\alpha}_{a,\lambda}(h,\delta)$, *where* F_c *is defined as*

$$F_c(z) = \frac{1+c}{z^c}\int_0^z t^{c-1} f(t)\mathrm{d}t, \; c > -1, \; z \in E. \tag{3.5}$$

Proof. Suppose that $f \in \mathcal{P}^{\alpha}_{a,\lambda}(h,\delta)$ and let

$$p(z) = \frac{z(\mathcal{L}^{\alpha}_{a,\lambda}F_c(z))' + \delta z^2(\mathcal{L}^{\alpha}_{a,\lambda}F_c(z))''}{(1-\delta)(\mathcal{L}^{\alpha}_{a,\lambda}F_c(z)) + \delta z(\mathcal{L}^{\alpha}_{a,\lambda}F_c(z))'}. \tag{3.6}$$

Then by (3.4) the function $p(z)$ is analytic in E, with $p(0) = 1$. From (3.3), we can write

$$z(F_c(z))' + cF_c(z) = (1+c)f(z).$$

This means that

$$z(\mathcal{L}^{\alpha}_{a,\lambda}F_c(z))' + c(\mathcal{L}^{\alpha}_{a,\lambda}F_c(z)) = (1+c)(\mathcal{L}^{\alpha}_{a,\lambda}f(z)). \tag{3.7}$$

From (3.4) and (3.5), we have

$$p(z) + \frac{zp'(z)}{p(z)+c} = \frac{z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))' + \delta z^2(\mathcal{L}^{\alpha}_{a,\lambda}f(z))''}{(1-\delta)(\mathcal{L}^{\alpha}_{a,\lambda}f(z)) + \delta z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'}. \tag{3.8}$$

Since $f \in \mathcal{P}^{\alpha}_{a,\lambda}(h,\delta)$, therefore from (3.6), it follows that

$$p(z) + \frac{zp'(z)}{p(z)+c} \prec h(z) \text{ in } E.$$

This implies by using Lemma 2.1 with $\beta_1 = 1$ and $\gamma_1 = c$ that $p(z) \prec h(z)$ in E. Thus $F_c \in \mathcal{P}^{\alpha}_{a,\lambda}(h,\delta)$ in E. □

Taking $a = 1$, $\alpha = 0$, $\lambda = 0$, $h(z) = \frac{1+z}{1-z}$, $z \in E$ in Theorem 3.4, we deduce Theorem 1 and Theorem 2 of Bernardi [3] with $\delta = 0$ and $\delta = 1$ respectively.

Theorem 3.5. *For $a > 0,\ 0 \leq \gamma \leq 1,\ 0 \leq \lambda \leq 1,\ 0 \leq \alpha < 1,$ we have*

$$\mathcal{T}^{\alpha}_{a+1,\lambda}(h,\gamma) \subset \mathcal{T}^{\alpha}_{a,\lambda}(h,\gamma).$$

Proof. Suppose that $f \in \mathcal{T}^{\alpha}_{a+1,\lambda}(h,\gamma)$ and let

$$p(z) = (1-\gamma)\frac{(\mathcal{L}^{\alpha}_{a,\lambda}f(z))}{z} + \gamma(\mathcal{L}^{\alpha}_{a,\lambda}f(z))', \tag{3.9}$$

where $p(z)$ is analytic in E, with $p(0) = 1$. Taking $\delta = 1$ in (1.7), we obtain

$$z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))' = a(\mathcal{L}^{\alpha}_{a+1,\lambda}f(z)) - (a-1)(\mathcal{L}^{\alpha}_{a,\lambda}f(z)). \tag{3.10}$$

Differentiating (3.8) and using (3.7), we have

$$p(z) + \frac{zp'(z)}{a} = (1-\gamma)\frac{(\mathcal{L}^{\alpha}_{a+1,\lambda}f(z))}{z} + \gamma(\mathcal{L}^{\alpha}_{a+1,\lambda}f(z))'. \tag{3.11}$$

Now by applying Lemma 2.2 and from (3.9), we can write $p \prec h$ in E. Thus $f \in \mathcal{T}^{\alpha}_{a,\lambda}(h,\gamma)$ in E. □

Theorem 3.6. *If $f \in \mathcal{T}^{\alpha}_{a,\lambda}(h,\gamma)$, then $F_c \in \mathcal{T}^{\alpha}_{a,\lambda}(h,\gamma)$, where F_c is defined by* (3.3).

Proof. We assume that $f \in \mathcal{T}^{\alpha}_{a,\lambda}(h,\gamma)$. Let

$$p(z) = (1-\gamma)\frac{(\mathcal{L}^{\alpha}_{a,\lambda}F_c(z))}{z} + \gamma(\mathcal{L}^{\alpha}_{a,\lambda}F_c(z))', \tag{3.12}$$

where $p(z)$ is analytic in E, with $p(0) = 1$. From (3.8) and (3.10), we obtain

$$p(z) + \frac{zp'(z)}{1+c} = (1-\gamma)\frac{(\mathcal{L}^{\alpha}_{a,\lambda}f(z))}{z} + \gamma(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'. \tag{3.13}$$

Using Lemma 2.2 with (3.11), we have $p \prec h$ in E and hence $F_c \in \mathcal{T}^{\alpha}_{a,\lambda}(h,\gamma)$. □

For $\alpha = 0$, $\lambda = 0$, we have the following result, see [25].

Corollary 3.7. *If $f \in \mathcal{T}^{0}_{a,0}(h,\gamma) = \mathcal{T}_a(h,\gamma)$, then $F_c \in \mathcal{T}_a(h,\gamma)$, where F_c is defined by* (3.3).

Theorem 3.8. *For $a > 0,\ 0 \leq \gamma \leq 1,\ 0 \leq \lambda \leq 1,\ 0 \leq \alpha < 1,$ we have*

$$\mathcal{M}^{\alpha}_{a+1,\lambda}(h,\gamma) \subset \mathcal{M}^{\alpha}_{a,\lambda}(h,\gamma).$$

Proof. Let $f \in \mathcal{M}^{\alpha}_{a+1,\lambda}(h,\gamma)$ and

$$p(z) = (\mathcal{L}^{\alpha}_{a,\lambda}f(z))' + \gamma z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'', \tag{3.14}$$

where $p(z)$ is analytic in E, with $p(0) = 1$.

Differentiating (3.8) and using (3.12), we obtain

$$p(z) + \frac{zp'(z)}{a} = (\mathcal{L}^{\alpha}_{a+1,\lambda} f(z))' + \gamma z(\mathcal{L}^{\alpha}_{a+1,\lambda} f(z))''. \tag{3.15}$$

Since $f \in \mathcal{M}^{\alpha}_{a+1,\lambda}(h,\gamma)$, therefore

$$(\mathcal{L}^{\alpha}_{a+1,\lambda} f(z))' + \gamma z(\mathcal{L}^{\alpha}_{a+1,\lambda} f(z))'' \prec h(z) \text{ for all } z \in E.$$

By using (3.13), it follows that

$$p(z) + \frac{zp'(z)}{a} \prec h(z) \text{ in } E.$$

Now by using Lemma 2.2, we obtain $p \prec h$ in E and hence $f \in \mathcal{M}^{\alpha}_{a,\lambda}(h,\gamma)$.

Corollary 3.9.([25]) *For* $\alpha = 0$, $\lambda = 0$, *we have*

$$\mathcal{M}^{0}_{a+1,0}(h,\gamma) \subset \mathcal{M}^{0}_{a,0}(h,\gamma).$$

Theorem 3.10.

(i) $f \in \mathcal{M}^{\alpha}_{a,\lambda}(h,\gamma) \Leftrightarrow zf' \in \mathcal{T}^{\alpha}_{a,\lambda}(h,\gamma)$,

(ii) $\mathcal{M}^{\alpha}_{a,\lambda}(h,\gamma) \subset \mathcal{T}^{\alpha}_{a,\lambda}(h,\gamma)$.

Proof. (i) Let $f \in \mathcal{M}^{\alpha}_{a,\lambda}(h,\gamma)$. Now using the fact

$$z(\mathcal{L}^{\alpha}_{a,\lambda} f(z))' = (\mathcal{L}^{\alpha}_{a,\lambda}(zf'(z)),$$

we can write

$$(1-\gamma)\frac{\mathcal{L}^{\alpha}_{a,\lambda}(zf'(z))}{z} + \gamma(\mathcal{L}^{\alpha}_{a,\lambda}(zf''(z)))' = (\mathcal{L}^{\alpha}_{a,\lambda} f(z))' + \gamma z((\mathcal{L}^{\alpha}_{a,\lambda} f(z))''. \tag{3.16}$$

Since $f \in \mathcal{M}^{\alpha}_{a,\lambda}(h,\gamma)$, therefore

$$(\mathcal{L}^{\alpha}_{a,\lambda} f(z))' + \gamma z(\mathcal{L}\zeta^{\alpha}_{a,\lambda} f(z))'' \prec h(z) \text{ in } E.$$

From (3.14), it follows that

$$(1-\gamma)\frac{\mathcal{L}^{\alpha}_{a,\lambda}(zf'(z))}{z} + \gamma(\mathcal{L}^{\alpha}_{a,\lambda}(zf'(z)))' \text{ in } E.$$

Hence $zf' \in \mathcal{T}^{\alpha}_{a,\lambda}(h,\gamma)$.
(ii) Let $f \in \mathcal{M}^{\alpha}_{a,\lambda}(h,\gamma)$ and let

$$p(z) = (1-\gamma)\frac{(\mathcal{L}^{\alpha}_{a,\lambda} f(z))}{z} + \gamma(\mathcal{L}^{\alpha}_{a,\lambda} f(z))', \tag{3.17}$$

where $p(z)$ is analytic in E, with $p(0) = 1$. From (3.15), we obtain

$$p(z) + zp'(z) = (\mathcal{L}^{\alpha}_{a,\lambda} f(z))' + \gamma z(\mathcal{L}^{\alpha}_{a,\lambda} f(z))''.$$

Thus from Lemma 2.2, we have $p \prec h$ in E. Hence $f \in \mathcal{T}^{\alpha}_{a,\lambda}(h, \gamma)$. □

For $\alpha = 0$, $\lambda = 0$, we have some well known results due to [25].

Corollary 3.11.

(i) $f \in \mathcal{M}^{0}_{a,0}(h, \gamma) \Leftrightarrow zf' \in \mathcal{T}^{0}_{a,0}(h, \gamma)$,

(ii) $\mathcal{M}^{0}_{a,0}(h, \gamma) \subset \mathcal{T}^{0}_{a,0}(h, \gamma)$.

Theorem 3.12. *For $\gamma > \eta \geq 0$, $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, we have*

(i) $\mathcal{T}^{\alpha}_{a,\lambda}(h, \gamma) \subset \mathcal{T}^{\alpha}_{a,\lambda}(h, \eta)$,

(ii) $\mathcal{M}^{\alpha}_{a,\lambda}(h, \gamma) \subset \mathcal{M}^{\alpha}_{a,\lambda}(h, \eta)$.

Proof. (i). The case $\eta = 0$ is trivial. Suppose that $\eta > 0$. Let $f \in \mathcal{T}^{\alpha}_{a,\lambda}(h, \gamma)$ and let z_1 be any arbitrary point in E. Then

$$(1-\gamma)\frac{\mathcal{L}^{\alpha}_{a,\lambda} f(z_1)}{z_1} + \gamma(\mathcal{L}^{\alpha}_{a,\lambda} f(z_1))' \in h(E).$$

Since $\frac{(\mathcal{L}^{\alpha}_{a,\lambda} f(z))}{z} \in h(E)$, we can write the following equality

$$\begin{aligned} (3.18) \qquad & (1-\eta)\frac{(\mathcal{L}^{\alpha}_{a,\lambda} f(z))}{z} + \eta(\mathcal{L}^{\alpha}_{a,\lambda} f(z))' \\ = \quad & (1-\frac{\eta}{\gamma})\frac{(\mathcal{L}^{\alpha}_{a,\lambda} f(z))}{z} + \frac{\eta}{\gamma}\left[(1-\gamma)\frac{(\mathcal{L}^{\alpha}_{a,\lambda} f(z))}{z} + \gamma(\mathcal{L}^{\alpha}_{a,\lambda} f(z))'\right]. \end{aligned}$$

Now as $\frac{\eta}{\gamma} < 1$ and $h(E)$ is convex. Therefore from (3.16), it follows that

$$(1-\eta)\frac{(\mathcal{L}^{\alpha}_{a,\lambda} f(z))}{z} + \eta(\mathcal{L}^{\alpha}_{a,\lambda} f(z))' \in h(E).$$

Thus $f \in \mathcal{T}^{\alpha}_{a,\lambda}(h, \eta)$.

The proof of part (ii) is similar to (i). □

Theorem 3.13. *If $\Psi \in \mathcal{C}$ and $f \in \mathcal{P}^{\alpha}_{a,\lambda}(h, \delta)$, then $\Psi * f \in \mathcal{P}^{\alpha}_{a,\lambda}(h, \delta)$, for $a \geq 1$.*

Proof. Let $\Psi \in \mathcal{C}$, $G(z) = (1-\delta)(\mathcal{L}^{\alpha}_{a,\lambda} f(z)) + \delta z(\mathcal{L}^{\alpha}_{a,\lambda} f(z))' \in \mathcal{S}^*(h) \subset \mathcal{S}^*$ and

$p(z) = \frac{zG'(z)}{G(z)}$. Consider

$$\begin{aligned}
&\frac{z\left[(1-\delta)\left\{\Psi * \mathcal{L}^{\alpha}_{a,\lambda}f(z)\right\}' + \delta\left\{z\left(\Psi * \mathcal{L}^{\alpha}_{a,\lambda}f(z)\right)'\right\}'\right]}{(1-\delta)\left\{\Psi * \mathcal{L}^{\alpha}_{a,\lambda}f(z)\right\} + \delta z\left\{\Psi * \mathcal{L}^{\alpha}_{a,\lambda}f(z)\right\}'} \\
&= \frac{\Psi * \left[(1-\delta)z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))' + \delta z\left\{(z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'\right\}'\right]}{\Psi * \left[(1-\delta)\mathcal{L}^{\alpha}_{a,\lambda}f(z) + \delta z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'\right]} \\
&= \frac{\Psi * zG'(z)}{\Psi * G(z)} \\
&= \frac{(\Psi * pG)(z)}{(\Psi * G)(z)}, \qquad (3.19)
\end{aligned}$$

where $p(z) = \frac{zG'(z)}{G(z)} \prec h(z)$ *in* E. We now apply Lemma 2.3 and using (3.17), we have $\Psi * f \in \mathcal{P}^{\alpha}_{a,\lambda}(h,\delta)$. □

Note that the above result also holds for the classes $\mathcal{T}^{\alpha}_{a,\lambda}(h,\gamma)$ and $\mathcal{M}^{\alpha}_{a,\lambda}(h,\gamma)$.

Theorem 3.14. *For* $\delta \geq 0$, $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, *we have*

$$\mathcal{P}^{\alpha}_{a,\lambda}(h,\delta) \subset \mathcal{P}^{\alpha}_{a,\lambda}(h,0).$$

Proof. The case $\delta = 0$ is trivial, so we suppose that $\delta > 0$. Assume that $f \in \mathcal{P}^{\alpha}_{a,\lambda}(h,\delta)$ and let

$$N(z) = (1-\delta)\left(\mathcal{L}^{\alpha}_{a,\lambda}f(z)\right) + \delta z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'.$$

Then

$$\frac{zN'(z)}{N(z)} = \frac{z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))' + \delta z^2(\mathcal{L}^{\alpha}_{a,\lambda}f(z))''}{(1-\delta)\left(\mathcal{L}^{\alpha}_{a,\lambda}f(z)\right) + \delta z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'}.$$

This means that $\frac{zN'(z)}{N(z)} \prec h(z)$ in E. That is $N \in \mathcal{S}^*(h)$. Let

$$p(z) = \frac{z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'}{(\mathcal{L}^{\alpha}_{a,\lambda}f(z))}, \qquad (3.20)$$

where $p(z)$ is analytic in E, with $p(0) = 1$. Now

$$\begin{aligned}\frac{zN'(z)}{N(z)} &= \frac{z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))' + \delta z^2(\mathcal{L}^{\alpha}_{a,\lambda}f(z))''}{(1-\delta)\mathcal{L}^{\alpha}_{a,\lambda}f(z) + \delta z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'} \\ &= \frac{z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))' + \delta z\{(z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'\}' - \delta z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))}{(1-\delta)\mathcal{L}^{\alpha}_{a,\lambda}f(z) + \delta z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'} \\ &= \frac{(1-\delta)\frac{z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'}{\mathcal{L}^{\alpha}_{a,\lambda}f(z)} + \delta z\frac{(z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))')'}{\mathcal{L}^{\alpha}_{a,\lambda}f(z)}}{(1-\delta) + \delta\frac{z(\mathcal{L}^{\alpha}_{a,\lambda}f(z))'}{\mathcal{L}^{\alpha}_{a,\lambda}f(z)}}.\end{aligned}$$

Therefore using (3.18), we obtain

$$\begin{aligned}\frac{zN'(z)}{N(z)} &= \frac{(1-\delta)p(z) + \delta(p^2(z) + zp'(z))}{(1-\delta) + \delta p(z)} \\ &= p(z) + \frac{zp'(z)}{p(z) + (1/\delta - 1)}.\end{aligned}$$

Since $\frac{zN'(z)}{N(z)} \prec h(z)$ in E. This implies that

$$p(z) + \frac{zp'(z)}{p(z) + (1/\delta - 1)} \prec h(z) \text{ in } E.$$

Now by using Lemma 2.2, we have $p(z) \prec h(z)$ in E. Thus $f \in \mathcal{P}^{\alpha}_{a,\lambda}(h,\delta)$. □

4. Applications

For different choices of analytic function h, we have some applications of the main results. For $k \in [0,\infty)$, Kanas [13, 14, 15] defined the conic domain Ω_k as follows

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\}.$$

For $k = 0$, this domain represents the whole right half plane. Also we obtain the right branch of hyperbola for $0 < k < 1$, the parabola for $k = 1$ and the ellipse for $k > 1$. The following functions $p_k(z)$ are univalent with $p_k(0) = 1$, $p_k'(0) > 0$ and plays the role of extremal functions mapping E onto the conic domains Ω_k

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0 \\ 1 + \frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2, & k = 1 \\ 1 + \frac{2}{1-k^2}\sinh^2\left(\frac{2}{\pi}\cos^{-1}\right)\tanh^{-1}\sqrt{z} & 0 < k < 1 \\ 1 + \frac{2}{k^2-1}\sin\left\{\frac{2}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{z}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\mathrm{d}x\right\} + \frac{1}{k^{2-1}}, & k > 1. \end{cases}$$

Here $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}, t \in (0,1)\,,\ z \in E$ and z is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R\,(t)$ is Legender's elliptic integral of the first kind and $R'\,(t)$ is the complementary integral of $R(t)$. For details, we refer to [13], [14], [15] and [20]. In [16] some linear operators associated with k-uniformly convex functions were considered. Now by choosing $h(z) = p_k(z)$ in Theorem 3.14, we can easily prove the following

Corollary 4.1. *For $\delta \geq 0,\ 0 \leq \lambda \leq 1,\ 0 \leq \alpha < 1,$ we have*

$$\mathcal{P}^{\alpha}_{a,\lambda}(p_k, \delta) \subset \mathcal{P}^{\alpha}_{a,\lambda}(q_k, 0),$$

where

$$q_k(z) = \left[\int_0^1 \left(\exp \int_t^{tz} \frac{p_k\,(u) - 1}{u} \mathrm{d}u\right) \mathrm{d}t\right]^{-1}.$$

Some of the special cases are given as follows.

(i) Let $k = 0$. Then $f \in \mathcal{P}^{\alpha}_{a,\lambda}\left(\frac{1+z}{1-z}, \delta\right) \Rightarrow f \in \mathcal{P}^{\alpha}_{a,\lambda}\left(\frac{1}{1-z}, 0\right)$. That is

$$\mathfrak{Re}\left(\frac{z(\mathcal{L}^{\alpha}_{a,\lambda} f(z))'}{\mathcal{L}^{\alpha}_{a,\lambda} f(z)}\right) > \frac{1}{2} \text{ in } E.$$

(ii) For $k = 1$, we have

$$\mathcal{P}^{\alpha}_{a,\lambda}\left(\left(1 + \frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2, \delta\right)\right) \subset \mathcal{P}^{\alpha}_{a,\lambda}\,(q_1, 0)$$

and

$$\mathfrak{Re}\left(\frac{z(\mathcal{L}^{\alpha}_{a,\lambda} f(z))'}{\mathcal{L}^{\alpha}_{a,\lambda} f(z)}\right) > q_1\,(-1) = \frac{1}{2}.$$

(iii) Let $k > 1$ and $f \in \mathcal{P}^{\alpha}_{a,\lambda}(p_k, \delta) \Rightarrow f \in \mathcal{P}^{\alpha}_{a,\lambda}\left(\frac{z}{(z-k)\log\left(1-\frac{z}{k}\right)}, 0\right)$. That is

$$\mathfrak{Re}\left(\frac{z(\mathcal{L}^{\alpha}_{a,\lambda} f(z))'}{\mathcal{L}^{\alpha}_{a,\lambda} f(z)}\right) > \frac{1}{(k+1)\log\left(1+\frac{1}{k}\right)}.$$

(iv) For the case $k = 2$, we note that $\mathcal{P}^{\alpha}_{a,\lambda}(p_2, \delta) \subset \mathcal{P}^{\alpha}_{a,\lambda}(q_2, 0)$. This gives us

$$\mathfrak{Re}\left(\frac{z(\mathcal{L}^{\alpha}_{a,\lambda} f(z))'}{\mathcal{L}^{\alpha}_{a,\lambda} f(z)}\right) > q_2\,(-1) = \frac{1}{3\log\frac{3}{2}} \approx 0.813.$$

Now we prove a radius result for the class $\mathcal{M}^{\alpha}_{a,\lambda}(h,\sigma)$.

Theorem 4.2. *Let* $f \in \mathcal{M}^{\alpha}_{a,\lambda}(h,0)$. *Then* $f \in \mathcal{M}^{\alpha}_{a,\lambda}(h,\sigma)$, *for* $|z| < r_\sigma$, *where*

$$r_\sigma = \left(1+\sigma^2\right)^{\frac{1}{2}} - \sigma. \tag{4.1}$$

Proof. Let

$$p(z) = \left(\mathcal{L}^{\alpha}_{a,\lambda} f(z)\right),$$

where $p(z)$ is analytic in E, with $p(0) = 1$. Then

$$p'(z) + \sigma z p''(z) = \left(\mathcal{L}^{\alpha}_{a,\lambda} f(z)\right)'' * g_\sigma(z), \tag{4.2}$$

where $g_\sigma(z) = \frac{z-(1-\sigma)z^2}{(1-z)^2}$. It is known [17] that $\mathfrak{Re}\left(\frac{g_\sigma(z)}{z}\right) > \frac{1}{2}$ in $|z| < r_\sigma$. This implies that $\frac{g_\sigma(z)}{z} \prec h_2(z)$ and $\mathfrak{Re}\left(h_2(z)\right) > \frac{1}{2}$ in $|z| < r_\sigma$. Now by using Lemma 2.4 and from (4.2), we obtain

$$p'(z) + \sigma z p''(z) \prec h_2(z) * h(z) \prec h(z) \text{ in } |z| < r_\sigma.$$

It follows that

$$p'(z) + \sigma z p''(z) = (\mathcal{L}^{\alpha}_{a,\lambda} f(z))' + \sigma z (\mathcal{L}^{\alpha}_{a,\lambda} f(z))'' \prec h(z) \textit{ in } |z| < r_\sigma.$$

Thus $f \in \mathcal{M}^{\alpha}_{a,\lambda}(h,\sigma)$ in $|z| < r_\sigma$, where r_σ is given by (4.1). □

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