KYUNGPOOK Math. J. 57(2017), 99-108
https://doi.org/10.5666/KMJ.2017.57.1.99
pISSN 1225-6951 eISSN 0454-8124

On the Construction of Polynomial β-algebras over a Field

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Abstract. In this paper we construct quadratic β-algebras on a field, and we discuss both linear-quadratic β-algebras and quadratic-linear β-algebras in a field. Moreover, we discuss some relations of binary operations in β-algebras.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras([3, 4]). We refer useful textbooks for BCK/BCI-algebra to [2, 7, 10]. J. Neggers and H. S. Kim([8]) introduced another class related to some of the previous ones, viz., B-algebras and studied some of its properties. They also introduced the notion of β-algebra([9]) where two operations are coupled in such a way as to reflect the natural coupling which exists between the usual group operation and its associated B-algebra which is naturally defined by it. P. J. Allen et al.([1]) gave another proof of the close relationship of B-algebras with groups using the observation that the zero adjoint mapping is surjective. H. S. Kim and H. G. Park([5]) showed that if X is a 0-commutative B-algebra, then $(x*a)*(y*b) = (b*a)*(y*x)$. Using this property they showed that the class of p-semisimple BCI-algebras is equivalent to the class of 0-commutative B-algebras. Y. H. Kim and K. S. So ([6]) investigated some properties of β-algebras and further relations with B-algebras. Especially, they showed that if $(X, -, +, 0)$ is a B^*-algebra, then $(X, +)$ is a semigroup with identity 0. They discussed some constructions of linear β-algebras in a field K.

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* Corresponding Author.
Received September 22, 2016; accepted November 10, 2016.
2010 Mathematics Subject Classification: 03G25, 06F35.
Key words and phrases: (linear, quadratic-linear, linear-quadratic) β-algebra, (left, right)-zero semigroup, BCK-algebra.

In this paper we construct quadratic β-algebras on a field, and we discuss both linear-quadratic β-algebras and quadratic-linear β-algebras in a field. Moreover, we discuss some relations of binary operations in β-algebras.

2. Preliminaries

A *β-algebra*([9]) is a non-empty set X with a constant 0 and two binary operations "+" and "−" satisfying the following axioms: for any $x, y, z \in X$,

(I) $x - 0 = x$,

(II) $(0 - x) + x = 0$,

(III) $(x - y) - z = x - (z + y)$.

Example 2.1.([9]) Let $X := \{0, 1, 2, 3\}$ be a set with the following tables:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

−	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(X, +, -, 0)$ is a β-algebra.

Given a β-algebra X, we denote $x^* := 0 - x$ for any $x \in X$.

Theorem 2.2.([6]) *Let $(K, +, \cdot, e)$ be a field (sufficiently large) and let $x, y \in K$. Then $(K, \oplus, \ominus, e)$ is a β-algebra, where $x \oplus y = x + y - e$ and $x \ominus y = x - y + e$ for any $x, y \in K$.*

We call such a β-algebra described in Theorem 2.2 a *linear β-algebra*.

If we let $\varphi : K \to K$ be a map defined by $\varphi(x) = e + bx$ for some $b \in K$. Then we have

$$\begin{aligned}\varphi(x + y) &= e + b(x + y)\\ &= (e + bx) + (e + by) - e\\ &= \varphi(x) \oplus \varphi(y)\end{aligned}$$

and

$$\begin{aligned}\varphi(x - y) &= e + b(x - y)\\ &= (e + bx) - (e + by) + e\\ &= \varphi(x) \ominus \varphi(y),\end{aligned}$$

so that $\varphi(0) = e$ implies $\varphi : (K, -, +, 0) \to (K, \ominus, \oplus, e)$ is a homomorphism of β-algebras, where "−" is the usual subtraction in the field K. If $b \neq 0$, then

$\psi : (K, \ominus, \oplus, e) \to (K, -, +, 0)$ defined by $\psi(x) := (x - e)/b$ is a homomorphism of β-algebras and the inverse mapping of the mapping φ, so that $(K, \ominus, \oplus, e)$ and $(K, -, +, 0)$ are isomorphic as β-algebras, i.e., there is only one isomorphism type in this case. We summarize:

Proposition 2.3.([6]) *The β-algebra $(K, \ominus, \oplus, e)$ discussed in Theorem 2.2 is unique up to isomorphism.*

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* ([3, 4]) if it satisfies the following conditions: for all $x, y, z \in X$,

(i) $(((x * y) * (x * z)) * (z * y) = 0)$,

(ii) $((x * (x * y)) * y = 0)$,

(iii) $(x * x = 0)$,

(iv) $(x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity: for all $x \in X$,

(v) $(0 * x = 0)$,

then X is called a *BCK-algebra* ([3, 4]).

3. Constructions of Polynomial β-algebras

Let $(K, +, \cdot, e)$ be a field (sufficiently large)and let $x, y \in K$. First, we consider the case of *quadratic-linear* β-algebras, i.e., $x \oplus y$ is the polynomial of x, y with degree 2, and $x \ominus y$ is the polynomial of x, y with degree 1. Define two binary operations "$\oplus, \ominus$" on K as follows:

$$x \oplus y := A + Bx + Cy + Dx^2 + Exy + Fy^2,$$

$$x \ominus y := \alpha + \beta x + \gamma y$$

where $\alpha, \beta, \gamma, A, B, C, D, E, F \in K$ (fixed). Assume that $(K, \oplus, \ominus, 0)$ is a β-algebra. It is necessary to find proper solutions for two equations. Since $x = x \ominus e = \alpha + \beta x + \gamma e$, we obtain $(\beta - 1)x + (\alpha + \gamma e) = 0$, and hence $\beta = 1$ and $\alpha = -\gamma e$. It follows that

$$x \ominus y = x + \gamma(y - e) \tag{3.1}$$

From (3.1) we obtain $e \ominus x = e + \gamma(x - e) = (1 - \gamma)e + \gamma x$. Since $(e \ominus x) \oplus x = e$, we obtain

$$\begin{aligned} e &= (e \ominus x) \oplus x \\ &= [(1-\gamma)e + \gamma x] \oplus x \\ &= A + B[(1-\gamma)e + \gamma x] + Cx + D[(1-\gamma)e + \gamma x]^2 \\ &\quad + E[(1-\gamma)e + \gamma x]x + Fx^2 \\ &= [A + B(1-\gamma)e + D(1-\gamma)^2 e^2] + [B\gamma + C + 2D\gamma(1-\gamma)e \\ &\quad + E(1-\gamma)e]x + [D\gamma^2 + E\gamma + F]x^2 \end{aligned}$$

If we assume that $|K| \geq 3$, then we obtain

$$\begin{gathered} D\gamma^2 + E\gamma + F = 0 \\ B\gamma + C + 2D\gamma(1-\gamma)e + E(1-\gamma)e = 0 \\ A + B(1-\gamma)e + D(1-\gamma)^2 e^2 = 0 \end{gathered} \tag{3.2}$$

Case(i). $\gamma \neq 0$. By formula (3.1) we obtain

$$\begin{aligned} (x \ominus y) \ominus z &= x \ominus y + \gamma(z - e) \\ &= x + \gamma(y - e) + \gamma(z - e) \\ &= x + \gamma(y + z - 2e) \end{aligned}$$

and

$$x \ominus (z \oplus y) = x + \gamma(z \oplus y - e)$$

By (III), we obtain $x + \gamma(y + z - 2e) = x + \gamma(z \oplus y - e)$. Since $\gamma \neq 0$, we have $z \oplus y - e = z + y - 2e$. This shows that $x \oplus y = x + y - e$ for all $x, y \in K$. In this case, we obtain the linear case as described in Theorem 2.2.

Case (ii). $\gamma = 0$. By (3.1), we obtain $x \ominus y = x$. Since $|K| \geq 3$, the formula (3.2) can be represented as

$$F = 0, C + Ee = 0, A + Be + De^2 = e$$

It follows that $F = 0, C = -Ee, A = e - Be - De^2$. This shows that

$$\begin{aligned} x \oplus y &= (e - Be - De^2) + Bx + (-Ee)y + Dx^2 + Exy \\ &= e + B(x - e) + D(x^2 - e^2) + E(x - e)y \\ &= e + [B + D(x + e) + Ey](x - e) \end{aligned}$$

which means that $x \oplus y$ is of the form:

$$x \oplus y = e + (a + bx + cy)(x - e).$$

It is a quadratic form (not a linear form) and so $(K, \oplus, \ominus, e)$ is a quadratic-linear β-algebra where $x \oplus y = e + (a + bx + cy)(x - e)$ and $x \ominus y = x$. We summarize:

Theorem 3.1. *Let $(K, +, \cdot, e)$ be a field (sufficiently large). If we define two binary operations "$\oplus, \ominus$" on K by*

$$\begin{aligned} x \oplus y &:= e + (a + bx + cy)(x - e) \\ x \ominus y &:= x \end{aligned}$$

for all $x, y \in K$, then $(K, \oplus, \ominus, e)$ is a quadratic-linear β-algebra.

Corollary 3.2. *Let $(K, +, \cdot, 0)$ be a field (sufficiently large) and let $a, b, c \in K$. If we define two binary operations "$\oplus, \ominus$" on K by*

$$\begin{aligned} x \oplus y &:= ax + bx^2 + cxy \\ x \ominus y &:= x \end{aligned}$$

for all $x, y \in K$, then $(K, \oplus, \ominus, 0)$ is a quadratic-linear β-algebra.

Proof. It follows immediately from Theorem 3.1 by letting $e := 0$. □

Example 3.3. Let $\mathbf{R}$ be the set of all real numbers. If we define $x \oplus y := x - x^2 - 2xy$ and $x \ominus y := x$ for all $x, y \in \mathbf{R}$, then $(\mathbf{R}, \oplus, \ominus, 0)$ is a quadratic-linear β-algebra.

Let $(K, +, \cdot, e)$ be a field (sufficiently large) and let $x, y \in K$. Next, we consider the case of *linear-quadratic* β-algebras, i.e., $x \oplus y$ is the polynomial of x, y with degree 1, and $x \ominus y$ is the polynomial of x, y with degree 2.

Theorem 3.4. *There is no linear-quadratic β-algebras over a field $(K, +, -, e)$.*

Proof. Let $(K, +, \cdot, e)$ be a field (sufficiently large). Define two binary operations "$\oplus, \ominus$" on K as follows:

$$x \oplus y := A + Bx + Cy,$$

$$x \ominus y := \alpha + \beta x + \gamma y + \delta x^2 + \varepsilon xy + \xi y^2$$

where $A, B, C, \alpha, \beta, \gamma, \delta, \varepsilon, \xi \in K$ (fixed). Assume that $(K, \oplus, \ominus, e)$ is a β-algebra and $|K| \geq 3$. Then

$$\begin{aligned} x &= x \ominus e \\ &= \alpha + \beta x + \gamma e + \delta x^2 + \varepsilon e x + \xi e^2 \\ &= [\alpha + \gamma e + \xi e^2] + [\beta + \varepsilon e]x + \delta x^2 \end{aligned}$$

It follows that

$$\begin{gathered} \alpha + \gamma e + \xi e^2 = 0 \\ \beta + \varepsilon e = 1 \\ \delta = 0 \end{gathered} \tag{3.3}$$

Case (i). $e = 0$. By formula (3.3) we obtain $\alpha = 0, \beta = 1, \delta = 0$. It follows that

$$x \ominus y = x + \gamma y + \varepsilon xy + \xi y^2 \tag{3.4}$$

It follows that $0 \ominus x = 0 + \gamma x + \varepsilon 0x + \xi x^2 = \gamma x + \xi x^2$ and hence

$$\begin{aligned} 0 &= (0 \ominus x) \oplus x \\ &= (\gamma x + \xi x^2) \oplus x \\ &= A + B(\gamma x + \xi x^2) + Cx \\ &= A + (B\gamma + C)x + B\xi x^2 \end{aligned}$$

for all $x \in K$. This shows that $A = 0, B\gamma + C = 0, B\xi = 0$.
Subcase (i-1). $B = 0$. Since $C = -B\gamma$, we obtain $C = 0$ and hence $x \oplus y = 0$ for all $x, y \in K$. This shows that $(0 \ominus x) \oplus x = 0$. Using formula (3.4) we obtain

$$\begin{aligned} (x \ominus y) \ominus z &= (x + \gamma y + \varepsilon xy + \xi y^2) \ominus z \\ &= (x + \gamma y + \varepsilon xy + \xi y^2) + \gamma z + \\ &\quad \varepsilon(x + \gamma y + \varepsilon xy + \xi y^2)z + \xi z^2 \\ &= x + \gamma y + \gamma z + \varepsilon xy + \xi(y^2 + z^2) \\ &\quad + \varepsilon(x + \gamma y + \varepsilon xy + \xi y^2)z \end{aligned}$$

and

$$x \ominus (z \oplus y) = x \ominus 0 = x$$

By (III), we obtain $\gamma = 0, \varepsilon = 0, \xi = 0$, proving that $x \ominus y = x$. Hence $x \oplus y = 0$ and $x \ominus y = x$ show that $(K, \oplus, \ominus, 0)$ is not a linear-quadratic β-algebra.
Subcase (i-2). $\xi = 0$. We apply this condition to the formula (3.4), and obtain $x \ominus y = x + \gamma y + \varepsilon xy$. It follows that $0 \ominus x = \gamma x$ and hence $0 = (0 \ominus x) \oplus x = \gamma x \oplus x = A + B(\gamma x) + Cx = A + (B\gamma + C)x$ for all $x \in K$. It follows that $A = 0, B\gamma + C = 0$. Hence $x \oplus y = B(x - \gamma y)$. It follows that

$$\begin{aligned} x \ominus (z \oplus y) &= x + \gamma(z \oplus y) + \varepsilon x(z \oplus y) \\ &= x + \gamma B(z - \gamma y) + \varepsilon Bx(z - \gamma y) \\ &= x + \gamma Bz - \gamma^2 By + \varepsilon Bxz - \varepsilon B\gamma xy \end{aligned}$$

and

$$\begin{aligned} (x \ominus y) \ominus z &= (x + \gamma y + \varepsilon xy) \ominus z \\ &= x + \gamma y + \varepsilon xy + \gamma z + \varepsilon(x + \gamma y + \varepsilon xy)z \end{aligned}$$

By (III), we obtain $\varepsilon = 0$ and $\gamma(1 + \gamma)B = 0$. If $\gamma = 0$, then $x \oplus y = Bx$ and $x \ominus y = x$. If $B = 0$, then it is the same case as subcase (i-1). If $\gamma = -1$, then $x \oplus y = B(x + y)$ and $x \ominus y = x - y$. This shows that $(K, \oplus, \ominus, 0)$ is not a linear-quadratic β-algebra.
Case (ii). $e \neq 0$. It follows from (3.3) that $\alpha = -\gamma e - \xi e^2, \beta = 1 - \varepsilon e, \delta = 0$. Hence

$$x \ominus y = (\gamma e - \xi e^2) + (1 - \varepsilon e)x + \gamma y + \varepsilon xy + \xi y^2 \tag{3.5}$$

Subcase (ii-1). $\xi \neq 0$. The formula (3.5) can be written as

$$x \ominus y = x + (\gamma + \varepsilon x)(y - e) + \xi(y^2 - e^2) \tag{3.6}$$

If $y := e$ in (3.6), then $x \ominus e = x + (\gamma + \varepsilon x)(e - e) + \xi(e^2 - e^2) = x$. If $x := e$ and $y := x$ in (3.6), then $e \ominus x = x + (\gamma + \varepsilon e)(x - e) + \xi(x^2 - e^2)$. It follows that

$$\begin{aligned} e &= (e \ominus x) \oplus x \\ &= A + B(e \ominus x) + Cx \\ &= A + B[e + (\gamma + \varepsilon e)(x - e) + \xi(x^2 - e^2)] + Cx \\ &= [A + Be - B(\gamma + \varepsilon e)e - B\xi e^2] + [B(\gamma + \varepsilon e) + C]x + B\xi x^2 \end{aligned}$$

for all $x \in K$, and hence we obtain

$$\begin{gathered} B\xi = 0 \\ B(\gamma + \varepsilon e) + C = 0 \\ A + Be - B(\gamma + \varepsilon e)e - B\xi e^2 = 0 \end{gathered}$$

Since $\xi \neq 0$, we have $B = 0$ and hence $C = 0, A = 0$, i.e., $x \oplus y = 0$ and $x \ominus y = x + (\gamma + \varepsilon x)(y - e) + \xi(y^2 - e^2)$ does not form a β-algebra, since $e \ominus (e \oplus e) = e - e(\gamma + \varepsilon e) - \xi e^2$ and $(e \ominus e) \ominus e = e$.
Subcase (ii-2). $\xi = 0$. The formula can be written as

$$x \ominus y = x + (\gamma + \varepsilon x)(y - e) \tag{3.7}$$

It follows that $x \ominus e = x + (\gamma + \varepsilon x)(e - e) = x$ and $e \ominus x = e + (\gamma + \varepsilon e)(x - e)$. By (II) we obtain the following.

$$\begin{aligned} e &= (e \ominus x) \oplus x \\ &= [e + (\gamma + \varepsilon e)(x - e)] \oplus x \\ &= A + B[e + (\gamma + \varepsilon e)(x - e)] + Cx \\ &= [A + Be - B(\gamma + \varepsilon e)e] + [B(\gamma + \varepsilon e) + C]x \end{aligned}$$

for all $x \in K$, and hence we obtain

$$\begin{gathered} A + Be - B(\gamma + \varepsilon e)e = e \\ B(\gamma + \varepsilon e) + C = 0 \end{gathered}$$

Hence $A = e + Be[\gamma + \varepsilon e - 1]$ and $C = -B(\gamma + \varepsilon e)$. It follows that

$$x \oplus y = [e + Be(\gamma + \varepsilon e - 1)] + Bx - B(\gamma + \varepsilon e)y \tag{3.8}$$

Using formulas (3.7) and (3.8), we obtain

$$\begin{aligned} x \ominus (z \oplus y) &= x + (\gamma + \varepsilon x)(z \oplus y - e) \\ &= x + (\gamma + \varepsilon x)[[e + Be(\gamma + \varepsilon e - 1)] + Bz - B(\gamma + \varepsilon e)y - e] \\ &= x + (\gamma + \varepsilon x)[[Be(\gamma + \varepsilon e - 1)] + Bz - B(\gamma + \varepsilon e)y] \\ &= x + B(\gamma + \varepsilon x)[e(\gamma + \varepsilon e - 1) + z - (\gamma + \varepsilon e)y] \end{aligned}$$

By formula (3.7) we obtain

$$\begin{aligned}(x \ominus y) \ominus z &= x \ominus y + (\gamma + \varepsilon(x \ominus y))(z - e) \\ &= x + (\gamma + \varepsilon x)(y - e) + [\gamma + \varepsilon(x + (\gamma + \varepsilon x)(y - e))](z - e) \\ &= x + (\gamma + \varepsilon x)[(y - e) + (1 + \varepsilon(y - e))(z - e)]\end{aligned}$$

By (III), we have

$$\begin{aligned}&B(\gamma + \varepsilon x)[e(\gamma + \varepsilon e - 1) + z - (\gamma + \varepsilon e)y] \\ &= (\gamma + \varepsilon x)[(y - e) + (1 + \varepsilon(y - e))(z - e)]\end{aligned}$$

Subcase (ii-2-a). $\gamma = \epsilon = 0$. We obtain $\beta = 1, \alpha = 0$ and hence $x \ominus y = x$, $x \oplus y = (e - Be) + Bx$, a linear β-algebra.
Subcase (ii-2-b). $\gamma + \varepsilon x \neq 0$ for all $x \in K$. We obtain the following formula.

$$\begin{aligned}&B[e(\gamma + \varepsilon e - 1) + z - (\gamma + \varepsilon e)y] \\ &= (y - e) + (1 + \varepsilon(y - e))(z - e)\end{aligned}$$

for all $y, z \in K$. This shows that $\varepsilon = 0, \gamma = -1, \delta = 0, \beta = 1, \alpha = e$ and $A = -e$ and $B = C = 1$. Hence $x \oplus y = x + y - e$ and $x \ominus y = x - y + e$, i.e., $(K, \oplus, \ominus, e)$ is a linear β-algebra which is not a quadratic-linear β-algebra. Hence there is no linear-quadratic β-algebra which is not a linear β-algebra. □

Problem. Construct a complete quadratic β-algebra over a field, i.e., $x \oplus y$ and $x \ominus y$ are both polynomials of x and y of degree 2.

4. Some Relations of Binary Operations in β-algebras

In this section, we discuss some relations of binary operations in β-algebras. For example, given a groupoid $(X, +)$, we want to know the structure of $(X, -)$ if $(X, +, -, 0)$ is a β-algebra. Given a non-empty set X, a groupoid $(X, *)$ is said to be a *left-zero semigroup* if $x * y = x$ for all $x, y \in X$. Similarly, a groupoid $(X, *)$ is said to be a *right-zero semigroup* if $x * y = y$ for all $x, y \in X$.

Proposition 4.1. *If $(X, \oplus)$ is a left-zero semigroup and if $(X, \oplus, \ominus, 0)$ is a β-algebra, then $(X, \ominus)$ is also a left-zero semigroup.*

Proof. Assume that $(X, \oplus, \ominus, 0)$ is a β-algebra. Then $x \ominus 0 = x$ for all $x \in X$, and $(x \ominus y) \ominus z = x \ominus (z \oplus y) = x \ominus z$, i.e., $(x \ominus y) \ominus z = x \ominus z$ for all $x, y, z \in X$. It follows that $x \ominus y = (x \ominus y) \ominus 0 = x \ominus 0 = x$, i.e., $x \ominus y = x$, for all $x, y \in X$, proving that $(X, \ominus)$ is a left-zero semigroup. □

Proposition 4.2. *Let $(X, \oplus, 0)$ be an algebra with $0 \oplus x = 0$ for all $x \in X$. If $(X, \ominus)$ is a left-zero semigroup, then $(X, \oplus, \ominus, 0)$ is a β-algebra.*

Proof. Since $(X, \ominus)$ is a left-zero semigroup, the conditions (I) and (III) hold. It follows from $0 \oplus x = 0$ for all $x \in X$ that $0 = 0 \oplus x = (0 \ominus x) \oplus x$, proving the proposition. □

Corollary 4.3. *Let $(X, *, 0)$ be a BCK-algebra. If $(X, \ominus)$ is a left-zero semigroup, then $(X, *, \ominus, 0)$ is a β-algebra.*

Proof. Straightforward. □

Proposition 4.4. *Let $(X, \oplus, 0)$ be an algebra with $0 \oplus x = 0$ for all $x \in X$. If $(X, \oplus, \ominus, 0)$ is a β-algebra, then $(X, \ominus)$ is a left-zero semigroup.*

Proof. Let $(X, \oplus, \ominus, 0)$ be a β-algebra. If we let $z := 0$ in (III), then $x \ominus y = (x \ominus y) \ominus 0 = x \ominus (0 \oplus y) = x \ominus 0 = x$ for all $x, y \in X$, proving the proposition. □

Proposition 4.5. *Let $(X, \oplus)$ be a right-zero semigroup. If $(X, \oplus, \ominus, 0)$ is a β-algebra, then $X = \{0\}$.*

Proof. Assume $(X, \oplus, \ominus, 0)$ is a β-algebra. Since $(X, \oplus)$ is a right-zero semigroup, by (II), we have $0 = (0 \ominus x) \oplus x = x$ for all $x \in X$, proving that $X = \{0\}$. □

Proposition 4.6. *Let $(X, \ominus)$ be a right-zero semigroup. If $(X, \oplus, \ominus, 0)$ is a β-algebra, then $X = \{0\}$.*

Proof. Assume $(X, \oplus, \ominus, 0)$ is a β-algebra. Since $(X, \ominus)$ is a right-zero semigroup, by (II), we have $0 = (0 \ominus x) \oplus x = x \oplus x$ for all $x \in X$. By (III), we have $z = (x \ominus y) \ominus z = x \ominus (z \oplus y) = z \oplus y$ for all $y, z \in X$. It follows that $x = x \oplus x = 0$ for all $x \in X$, proving the proposition. □

Theorem 4.7. *Let $(K, +, -, 0)$ be a field with $|K| \geq 4$ and let $e \in K$. Define a binary operation "$\oplus$" on K by $x \oplus y := p(x, y)$, i.e., a quadratic polynomial of x and y in K. If $e \oplus x = e$ for all $x \in K$, then $x \oplus y = e + (A + Bx + Cy)(x - e)$ for all $x, y \in K$ where $A, B, C \in K$.*

Proof. Assume $p(x, y) := A + Bx + Cy$ for all $x, y \in K$ where $A, B, C \in K$. Since $e \oplus x = e$, we have $e = e \oplus y = A + Be + Cy$ for all $y \in K$. It shows that $A = e(1 - B), C = 0$. Hence $x \oplus y = e(1 - B) + Bx = e + B(x - e)$. Assume $p(x, y) := A + Bx + Cy + Dx^2 + Exy + Fy^2$. Then

$$\begin{aligned} e &= e \oplus y \\ &= A + Be + Cy + De^2 + Eey + Fy^2 \\ &= (A + Be + De^2) + (C + Ee)y + Fy^2 \end{aligned}$$

It follows that $F = 0, C + Ee = 0, A + Be + De^2 = e$. Hence

$$\begin{aligned} p(x, y) &= (e - Be - De^2) + Bx - Eey + Dx^2 + Exy \\ &= e + [B + D(x + e) + Ey](x - e) \\ &= e + (B + De + Dx + Ey)(x - e), \end{aligned}$$

i.e., $p(x, y)$ is of the form $p(x, y) = e + (A + Bx + Cy)(x - e)$. This shows that $x \oplus y = e + q(x, y)(x - e)$ where $q(x, y)$ is a linear polynomial of degree ≤ 1. □

Using Theorem 4.7 and Proposition 4.2, we obtain the following.

Corollary 4.8. *Let $(K, +, -, 0)$ be a field with $|K| \geq 4$ and let $e \in K$. Define a binary operation "$\oplus$" on K by*

$$x \oplus y := e + q(x, y)(x - e)$$

where $q(x, y)$ is any polynomial of x and y in K. If $x \ominus y := x$ for all $x, y \in K$, then $(K, \oplus, \ominus, e)$ is a β-algebra.

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